The Linkage Between The Dirac's Approach And The Hamilton-Jacobi Approach For Higher-Order Hamiltonian Constrained Systems

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Abstract:
The linkage between the Dirac's approach and the Hamilton-Jacobi approach for higher order systems with constrained is studied. It is shown that the Dirac's approach his permanently agree with the Hamilton-Jacobi approach. The integrability conditions in the Hamilton-Jacobi approach hare equivalent of consistency conditions in the Dirac's approach.

1 - Introduction

The Lagrangian formula for systems with constrained it been has studied by Sundermeyer (1982),[28] Sudrshan and Mukunda (1974),[27] while the Hamiltonian formulation for individual systems is always done through two components developed by Dirac (1950,1964)[3,4]. And another strongest the is way to evelop it d and advanceHamilton-Jacobi for verification singular systems in the first order has been developed by (Rabei and Güler, 1992 Güler, 1992;, Rabei and Güler1995,)[7,22,23,25,26].

The formal generalize of Hamilton-Jacobi formula for singular systems with arbitrary higher-order Lagrangians was which he developed by Teixeira and Pimentel (1998)[19].

The linkage the two approachs for first order systems with constrained was developed by Rabei, 1996 [24].

The linkage the two approachs for second-order systems with constrained was developed by researcher [30].

An any physical system with N degrees of freedom The Lagrangian functions of are the functions of generalize coordinates $\overset{(k)}{q}_i$ and a parameter t i.e.

$$L \equiv (\overset{(k)}{q}_i, t) \qquad (1)$$

In Ostrogradski's formula the conjugates of momentum respectively to $q_{(k-1)i}$ and $q_{(m-1)i}$ ($m=1,\ldots,k$-1) write it follow: by Pimentel and Teixeira (1998) [19].

$$p_{(k-1)i} = \frac{\partial L}{\partial \overset{(k)}{q}_i} \qquad (2)$$

$$p_{(m-1)i} \equiv \frac{\partial L}{\partial \overset{(m)}{q}_i} - \dot{p}_{(m)i}\,; \quad (m=1,\ldots,k\text{-}1) \qquad (3)$$

The Hessian matrix is defined

$$W_{i\ell} \equiv \frac{\partial^2 L}{\partial \overset{(k)}{q}_i \, \partial \overset{(k)}{q}_\ell}, \qquad i,\ell = 1,\ldots,N \qquad (4)$$

If the rank of this matrix is N, systems which have this property are called regular and their treatments are found in standard mechanics books, systems which have the rank less than N are called singular systems.

In this research we want show up that the Hamilton-Jacobi approach is totally agree with Dirac's approach, In section (2) Hamilton-Jacobi approach, in section (3) Dirac's approach, in section (4) the linkage between the two smothed istudied been has t, and in section (5) an example of singular with thirdorder Lagrangian is constructed and solved by using the two methods.

2 -Hamilton-Jacobi approach

The Hamilton-Jacobi formulation for singular first order systems was developed by Güler [7] and developed by Pimentel (1996)[18] for second order systems and developed by Pimentel (1998)[19] for higher order systems obtained the equations of motion are written as total differential equations in many variables as follows:

$$dq_{(u)i} = \sum_{m=0}^{k-1} \frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}\, dt_{(m)\alpha} \qquad i=1,\ldots,N \qquad \text{and} \qquad u,m=0,\ldots,k-1, \qquad (5)$$

$$dp_{(u)c} = -\sum_{m=0}^{k-1} \frac{\partial H'_{(m)\alpha}}{\partial q_{(u)c}}\, dt_{(m)\alpha} \qquad (6)$$

Where, as before, $\alpha = 0,1,\ldots,R\,; u, m = 0,1, \ldots, k-1; \;\; c = 0,1,\ldots,N$

A set of Hamilton-Jacobi partial differential equations by Pimentel and Teixeira (1998)[19] given:

$$H'_{(m)\alpha} = P_{(m)\alpha} + H_{(m)\alpha}(t_{(u)\alpha}; q_{(u)a}; p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}) = 0, \qquad (7)$$

Where $\alpha = 0, 1, ..., R$. $u, m = (0, 1, ..., k-1)$

The equations of motion (5) and (6) can be written in the form

$$dq_{(u)i} = \frac{\partial H'_0}{\partial p_{(u)i}} dt + \sum_{m=0}^{k-1} \frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}} dt_{(m)\alpha} \qquad (8)$$

$$dp_{(u)i} = -\frac{\partial H'_0}{\partial q_{(u)i}} dt - \sum_{m=0}^{k-1} \frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}} dt_{(m)\alpha} \qquad (9)$$

These two equations (8) and (9) can be written using Poisson brackets as:

$$dq_{(m)i} = \{q_{(m)i}, H_0\}dt + \{q_{(m)i}, H'_{(m)\alpha}\}dt_{(m)\alpha} \qquad (10)$$

$$dp_{(m)i} = \{p_{(m)i}, H_0\}dt + \{p_{(m)i}, H'_{(m)\alpha}\}dt_{(m)\alpha} \qquad (11)$$

These total differential equations are integrable if, and only if, the corresponding system of partial differential equations. For this purpose, defining the linear operators χ_α as: by Güler [7]

$$\chi_{(m)\alpha} f(q_{(u)i}, p_{(u)i}, t) = \{f, H'_{(m)\alpha}\}$$
$$= \frac{\partial f}{\partial q_{(u)i}} \frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}} - \frac{\partial f}{\partial p_{(u)i}} \frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}} + \frac{\partial f}{\partial t} \frac{\partial H'_{(m)\alpha}}{\partial p_0} \qquad (12)$$

Where $u, m = 0, 1, 2, ..., k-1;$ and $i = 1, 2, ..., N$ the equations of motion are integrable if, and only if, the bracket relations:

$$\left[\chi_{(m)\alpha}, \chi_{(m)\beta}\right] f = \left(\chi_{(m)\alpha}\chi_{(m)\beta} - \chi_{(m)\beta}\chi_{(m)\alpha}\right) f = 0 \qquad (9) \qquad (13)$$

Is valid equation (13). If for a specific system relations (13) are not satisfied, then one should enlarge the original system, adding new operators in such a way that a complete system results. It should be noted that the integrability conditions (13) are equivalent to the Poisson bracket relations:

$$\{H'_{(m)\alpha}, H'_{(m)\beta}\} = 0 \quad , \quad \forall \ \alpha, \beta \qquad (14)$$

To prove this, let us show that if (14) is valid, (13) will be complete. Forming bracket

$$\left[\chi_{(m)\alpha}, \chi_{(m)\beta}\right] f = \left(\chi_{(m)\alpha}\chi_{(m)\beta} - \chi_{(m)\beta}\chi_{(m)\alpha}\right) f$$
$$= \chi_{(m)\alpha}\{f, H'_{(m)\beta}\} - \chi_{(m)\beta}\{f, H'_{(m)\alpha}\} \qquad (15)$$

One get:

$$\left[\chi_{(m)\alpha}, \chi_{(m)\beta}\right] f = \left\{\{f, H'_{(m)\beta}\}, H'_{(m)\alpha}\right\} - \left\{\{f, H'_{(m)\alpha}\}, H'_{(m)\beta}\right\} \qquad (16)$$

However, the Jacobi relation read:

$$\{\{ f, H'_{(m)\beta}\}, H'_{(m)\alpha}\} + \{\{H'_{(m)\beta}, H'_{(m)\alpha}\}, f\} + \{\{H'_{(m)\alpha}, f\}, H'_{(m)\beta}\} = 0 \qquad (17)$$

Thus

$$\left[\chi_{(m)\alpha}, \chi_{(m)\beta}\right] f = \{f, \{H'_{(m)\beta}, H'_{(m)\alpha}\}\} = 0 \qquad (18)$$

If the integrability conditions are satisfied, the solutions of the total differential equations of motion will be:

$$q_{(m)a} = \xi_a(t_{(m)\alpha}, u_b); \qquad (19)$$

$$p_{(m)c} = \eta_c(t_{(m)\alpha}, u_b); \qquad (20)$$

Where u_b is arbitrary parameter?

3- Dirac's Approach

The well-known method to investigate the Hamilton formulation of systems with constrained was initiated by Dirac [3,4]. Now, the usual Hamiltonian H_0 for any dynamical system is defined by Pimentel and Teixeira (1998)[19] as:

$$H_0 = \sum_{m=0}^{k-1} p_{(m)_i} \overset{(m+1)}{q_i} - L(q_i, ..., \overset{(k)}{q_i}) \quad (12) \quad \mathrm{m} = (0,....,\mathrm{k}-1)\ (m = 0,1,...,k\text{-}1) \qquad (21)$$

However, defined in this way, H_0 will not be uniquely determined, since we may add to the canonical Hamiltonian H_0 any linear combination of the primary constraints and define a new Hamiltonian, called total given by:

$$H_T = H_0 + u_\alpha \Phi_\alpha, \qquad (22)$$

Where u_α are arbitrary coefficients?

Making use of Poisson brackets, one can write the total time derivative of any function $g(q_{(m)}, p_{(m)})$ as:

$$\dot{g} = \frac{dg}{dt} = \{g, H_T\} = \{g, H_0\} + u_\alpha\{g, \Phi_\alpha\} \qquad (23)$$

Where Dirac's symbol $(\approx)$ for weak equality has been used in the sense that one cannot consider $\Phi_\alpha = 0$ identically before working out the Poisson bracket.

The constraints will produce consistency conditions because the must be valid at any time and consequently their time derivative must be weakly zero. The consistency conditions are gives by:

$$\dot{\Phi}_\alpha \approx \{\Phi_\alpha, H_T\} \approx 0 \qquad (24)$$

Replacing g by Φ_α in Equation (23), one obtains

$$\dot{\Phi}_\alpha \approx \{\Phi_\alpha, H_T\} \approx \{\Phi_\alpha, H_0\} + u_\alpha\{\Phi_\alpha, \Phi_\beta\} \approx 0; \ \alpha, \beta = 1, \ldots, R \qquad (25)$$

These conditions may be identically satisfied with the help of the primary constraints, either determine some of the arbitrary coefficients u , or generate new constraints that will be called secondary constraints. All these constraints are divided into two types: first-class constraints, which have vanishing Poisson brackets with all other constraints, and second-class constraints, which have non-vanishing Poisson brackets. As there is an even number of class Il constraints, this can be used for eliminate conjugate pair of $p_{(m)}$ and $q_{(m)}$ from the theory by expressing them a function of the remaining $p_{(m)}$ and $q_{(m)}$ Muslih 2002 [12]. The Dirac Hamiltonian for the remaining variables is then the canonical Hamiltonian plus all the independent first class constraints Ψ_λ. So that the total Hamiltonian is defined as

$$H_T = H_0 + V_\lambda \Psi_\lambda \qquad (26)$$

Where they Ψ_λ include all first - class constraints. V_λ unknwn is coefficients; which is called Lagrange's undetermined multiplier.

On the ,basis this equations of motion from equation (23) are written follows as:

$$\dot{q}_{(m)i} \approx \{q_{(m)_i}, H_0\} + V_\lambda\{q_{(m)_i}, \Psi_\lambda\} \qquad (27)$$

$$\dot{p}_{(s)_i} \approx \{q_{(m)_i}, H_0\} + V_\lambda\{p_{(m)_i}, \Psi_\lambda\} \qquad (28)$$

4 - The Linkage Between the Two Approach

In the Hamilton-Jacobi approach the equations of motion (8) and (9)given:

$$dq_{(u)i} = \frac{\partial H'_0}{\partial p_{(u)i}} dt + \sum_{m=0}^{k-1} \frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}} dt_{(m)\alpha}$$

$$dp_{(u)i} = -\frac{\partial H'_0}{\partial q_{(u)i}} dt - \sum_{m=0}^{k-1} \frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}} dt_{(m)\alpha}$$

Making use of Poisson brackets, we can write these two equations as:

$$dq_{(m)i} = \{q_{(m)i}, H_0\}dt + \{q_{(m)i}, H'_{(m)\alpha}\}dt_{(m)\alpha} \qquad (29)$$

$$dp_{(m)i} = \{p_{(m)i}, H_0\}dt + \{p_{(m)i}, H'_{(m)\alpha}\}dt_{(m)\alpha} \qquad (30)$$

Where

$$\frac{\partial H'_0}{\partial p_{(m)i}} = \frac{\partial H_0}{\partial p_{(m)i}} = \{q_{(m)i}, H_0\} \qquad (31)$$

$$\frac{\partial H'_{(m)\alpha}}{\partial p_{(m)i}} = \{q_{(m)i}, H'_{(m)\alpha}\} \qquad (32)$$

$$-\frac{\partial H'_{(m)\alpha}}{\partial q_{(m)i}} = \{p_{(m)i}, H'_{(m)\alpha}\} \qquad (33)$$

In addition, the partial differential equation (12) can be written in the form:

$$\chi_0 f = \frac{\partial f}{\partial q_{(u)i}} \frac{\partial H'_0}{\partial p_{(u)i}} - \frac{\partial f}{\partial p_{(u)i}} \frac{\partial H'_0}{\partial q_{(u)i}} + \frac{\partial f}{\partial t} = 0;$$

$$\chi_{(m)\alpha} f = \frac{\partial f}{\partial q_{(u)i}} \frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}} - \frac{\partial f}{\partial p_{(u)i}} \frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}} = 0, \qquad (34)$$

say can we, the integral conditions (13) we can write in briefly :

$$\left[\chi_0, \chi_{(m)\alpha}\right] f = \left(\chi_0 \chi_{(m)\alpha} - \chi \chi_0 = 0;\right)$$

$$\left[\chi_{(m)\alpha}, \chi_{(m)\beta}\right] f = \left(\chi_{(m)\alpha} \chi_{(m)\beta} - \chi_{(m)\beta} \chi_{(m)\alpha}\right) f = 0 \qquad (35)$$

We substituting equation (34) in to equation (35) we get:

$$[\chi_0,\chi_{(m)\alpha}]f=\frac{\partial}{\partial p_{(u)i}}\left(\frac{\partial H'_0}{\partial p_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}-\frac{\partial H'_0}{\partial q_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}+\frac{\partial H'_{(m)\alpha}}{\partial t}\right)\frac{\partial f}{\partial q_{(u)i}}$$
$$-\frac{\partial}{\partial q_{(u)i}}\left(\frac{\partial H'_0}{\partial p_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}-\frac{\partial H'_0}{\partial q_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}+\frac{\partial H'_{(m)\alpha}}{\partial t}\right)\frac{\partial f}{\partial p_{(u)i}} \quad (36)$$

$$[\chi_{(m)\alpha},\chi_{(m)\beta}]f=\frac{\partial}{\partial p_{(u)i}}\left(\frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial q_{(u)i}}-\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial p_{(u)i}}+\right)\frac{\partial f}{\partial q_{(u)i}}$$
$$-\frac{\partial}{\partial q_{(u)i}}\left(\frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial q_{(u)i}}-\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial p_{(u)i}}+\right)\frac{\partial f}{\partial p_{(u)i}} \quad (37)$$

Equating these bracket relations to zero leads to the following conditions:

$$\frac{\partial H'_0}{\partial p_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}-\frac{\partial H'_0}{\partial q_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}=0; \quad (38)$$

$$\frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial q_{(u)i}}-\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial p_{(u)i}}=0; \quad (39)$$

$$\frac{\partial H'_{(m)\alpha}}{\partial t}=0. \quad (40)$$

These terms (38-40) nuance are functions to $H'_{(m)\alpha}(q_{(m)i},p_{(m)i})$ equal to zero; i.e.

$$dH'_{(m)\alpha}=\frac{\partial H'_{(m)\alpha}}{\partial q_{(m)i}}dq_{(m)i}+\frac{\partial H'_{(m)\alpha}}{\partial p_{(m)i}}dp_{(m)i}=0 \quad (41)$$

If we substitute for $dq_{(m)i}$ and $dp_{(m)i}$ their expressions from equation (41) in equations (8) and (9) become:

$$dH'_{(m)\alpha}=\left(\frac{\partial H'_0}{\partial p_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}-\frac{\partial H'_0}{\partial q_{(u)i}}\frac{\partial H'_{(m)\alpha}}{\partial p_{(u)i}}\right)dt$$
$$+\left(\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial p_{(u)i}}-\frac{\partial H'_{(m)\alpha}}{\partial q_{(u)i}}\frac{\partial H'_{(m)\beta}}{\partial p_{(u)i}}\right)dt_{(m)\alpha} \quad (42)$$

$$dH'_{(m)\alpha}=\{H'_{(m)\alpha},H_0\}dt+\{H'_{(m)\alpha},H'_{(m)\alpha}\}dt_{(m)\alpha}=0 \quad (43)$$

This equation, which results from integrability conditions, is equivalent to the consistency conditions (27-28) in Dirac's approach.
Thus, we conclude that in the Hamilton-Jacobi approach is always in exact agreement with the Dirac's approach.

5 -An Example with Third-Order Singular Lagrangian:

The.5.1 Hamilton-Jacobi approach .

$$L=\frac{1}{2}(\ddot{q}_1^{\,2}+\ddot{q}_2^{\,2})+\ddot{q}_3\ddot{q}_3 \tag{44}$$

Equations (2) and (3) yield the generalized momenta:

$$p_1=0=-H_1^p \tag{45}$$

$$p_2=0=-H_2^p \tag{46}$$

$$p_3=-\dddot{q}_3 \quad (47) \qquad \pi_1=-\dddot{q}_1 \quad (48)$$

$$\pi_2=-\dddot{q}_2 \quad (49) \qquad \pi_3=0=-H_3^{\pi} \quad (50)$$

$$\phi_1=\ddot{q}_1 \quad (51) \qquad \phi_2=\ddot{q}_2 \quad (52)$$

$$\phi_3=\ddot{q}_3=-H_3^{\chi} \quad (53)$$

The Hamiltonian H_0 is defined as:

$$H_0=p_1\dot{q}_1+p_2\dot{q}_2+p_3\dot{q}_3+\pi_1\ddot{q}_1+\pi_2\ddot{q}_2 \tag{54}$$

$$+\pi_3\ddot{q}_3+\phi_1\dddot{q}_1+\phi_2\dddot{q}_2+\phi_3\dddot{q}_3-L$$

or $$H_0=p_3\dot{q}_3+\pi_1\ddot{q}_1+\pi_2\ddot{q}_2+\frac{1}{2}(\phi_1^{\,2}+\phi_2^{\,2}) \tag{55}$$

The corresponding set of Hamilton-Jacobi partial differential equations are, according to (16)

$$H_0'=p_0+H_0=0; \tag{56}$$

$$H_1'^{p}=p_1+H_1^p=p_1=0; \tag{57}$$

$$H_2'^{p}=p_2+H_2^p=p_2=0; \tag{58}$$

$$H_3'^{\phi}=\pi_3+H_3^{\phi}=\pi_3=0; \tag{59}$$

$$H_3'^{\phi}=\phi_3+H_3^{\phi}=\phi_3-\ddot{q}_3=0. \tag{60}$$

This set equations leads to the total differential equation:

$$\begin{aligned} dq_1&=dq_1\\ dq_2&=dq_2\\ dq_3&=\dot{q}_3 dq_3 \end{aligned} \tag{61}$$

$$d\dot{q}_1 = \ddot{q}_1 dt$$
$$d\dot{q}_2 = \ddot{q}_2 dt \qquad (62)$$
$$d\dot{q}_3 = d\dot{q}_3$$

$$d\ddot{q}_1 = \phi_1 dt$$
$$d\ddot{q}_2 = \phi_2 dt \qquad (63)$$
$$d\ddot{q}_3 = d\ddot{q}_3$$

$$dp_1 = 0$$
$$dp_2 = 0 \qquad (64)$$
$$dp_3 = 0$$

$$d\pi_1 = 0$$
$$d\pi_2 = 0 \qquad (65)$$
$$d\pi_3 = -p_3 dt$$

$$d\phi_1 = -\pi_1 dt$$
$$d\phi_2 = -\pi_2 dt \qquad (66)$$
$$d\phi_3 = d\ddot{q}_3$$

The integrall condition of the constraints H_0' , $H_1'^{p}$, $H_2'^{p}$, $H_3'^{\pi}$ and $H_3'^{\phi}$ must equal zero. In fact, the variation of $H_3'^{\pi}$ leads to a new constraint $H_3'^{p}$, such that:

$$dH_3'^{\pi} = d\pi_3 = -p_3 dt; \qquad (67)$$

And

$$H_3'^{p} = p_3 = 0. \qquad (68)$$

Making use of (68), then, one can rewrite the set equations (61 -66) in the form:

$$dq_1 = dq_1$$
$$dq_2 = dq_2 \qquad (69)$$
$$dq_3 = \dot{q}_3 dq_3$$

$$d\dot{q}_1 = \ddot{q}_1 dt$$
$$d\dot{q}_2 = \ddot{q}_2 dt \qquad (70)$$
$$d\dot{q}_3 = d\dot{q}_3$$

$$d\ddot{q}_1 = \phi_1 dt$$
$$d\ddot{q}_2 = \phi_2 dt \qquad (71)$$
$$d\ddot{q}_3 = d\ddot{q}_3$$

$$dp_1 = 0$$
$$dp_2 = 0 \qquad (72)$$
$$dp_3 = 0$$

$$d\pi_1 = 0$$
$$d\pi_2 = 0 \qquad (73)$$
$$d\pi_3 = 0$$

$$d\phi_1 = -\pi_1 dt$$
$$d\phi_2 = -\pi_2 dt \qquad (74)$$
$$d\phi_3 = \mathrm{d}\ddot{\mathrm{q}}_3$$

The solutions of these equations are:

$$\pi_3 = -c_1 t + c_2$$
$$\phi_1 = -c_3 t + c_4$$
$$\phi_2 = -c_5 t + c_6$$
$$\ddot{q}_1 = -c_3' t^2 + c_4 t + c_7 \qquad (75)$$
$$\ddot{q}_2 = -c_5' t^2 + c_6 t + c_8$$
$$\dot{q}_1 = -c_3'' t^3 + c_4' t^2 + c_7 t + c_9$$
$$\dot{q}_2 = -c_5'' t^3 + c_6' t^2 + c_8 t + c_{10}$$

$$\phi_3 = \ddot{q}_3 + c \qquad (76)$$

The constraint (72) (73) and (76) implies that $c = 0$ and $p_1, p_2, p_3, \pi_1, \pi_2$ and π_3 are constants.

5.2. Dirac's approach

Following Dirac, the total Hamiltonian reads

$$H_T = H_0 + V_1 H_1'^p + V_2 H_2'^p + \eta_3 H_3'^\pi + \rho_3 H_3'^\phi \qquad (77)$$

or

$$H_T = p_3\dot{q}_3 + \pi_1\ddot{q}_1 + \pi_2\ddot{q}_2 + \frac{1}{2}(\phi_1^2 + \phi_1^2) + V_1 p_1 + V_2 p_2 + \eta_3\pi_3 + \rho_3(\phi_3 - \ddot{q}_3) \qquad (78)$$

Where $H_1'^p$, $H_2'^p$, $H_3'^\pi$, and $H_3'^\phi$ are the four primary constraints defined in -60) (57. These primary constraints satisfy the consistency condition (9) identically zero:

$$\dot{H}_1'^p = \{H_1'^p, H_T\} \equiv 0;$$
$$\dot{H}_2'^p = \{H_2'^p, H_T\} \equiv 0; \qquad (79)$$
$$\dot{H}_3'^\phi = \{H_3'^\phi, H_T\} \equiv 0.$$

But it gives a new secondary constraint for $H_3'^\pi$:

$$\dot{H}_3'^\pi = \{H_3'^\pi, H_T\} = p_3 = H_3'^p \approx 0. \qquad (80)$$

Imposing the condition $\dot{H}_3'^p \approx 0$.

Making the equations of motion (11-12) we have

$$\dot{q}_1 = V_1$$
$$\dot{q}_2 = V_2 \qquad (81)$$
$$\dot{q}_3 = \dot{q}_3$$

$$\dot{p}_1 = 0$$
$$\dot{p}_2 = 0 \qquad (82)$$
$$\dot{p}_3 = 0$$

$$\dot{\pi}_1 = 0$$
$$\dot{\pi}_2 = 0 \qquad (83)$$
$$\dot{\pi}_3 = p_3 = 0$$

$$\ddot{q}_1 = \ddot{q}_1$$
$$\ddot{q}_2 = \ddot{q}_2 \qquad (84)$$
$$\ddot{q}_3 = \eta_3$$

$$\dot{\phi}_1 = -\pi_1$$
$$\dot{\phi}_2 = -\pi_2 \qquad (85)$$
$$\dot{\phi}_3 = \rho_3$$

By direct comparison, it is again evident that these equations are in exact agreement with these obtained by in the Hamilton-Jacobi approach (69-74). equations

In the Hamilton-Jacobi approach the equations of motion are written in terms of $\dot{q}_1, \ddot{q}_2, \ddot{q}_3$ and $\dot{\phi}_3$, whereas, in the Dirac approachterms they which are following the by expressed is of V_1, V_2, η_3 and ρ_3 respectively. However the two approaches are equivalent since

$$\dot{q}_1 = V_1$$
$$\dot{q}_2 = V_2$$
$$\ddot{q}_3 = \eta_3 \qquad (86)$$
$$\phi_3 = \rho_3 = \eta_3 = \ddot{q}_3$$

Conclusion

The most common method for investigating the Hamiltonian treatment of constrained systems with first order was initiated by Dirac [3,4].

What distinguishes this method is the consideration of initial limitations are obtained using consistency conditions has been which he studied by Pimentel for Higher-Order Singular of systems Hamiltonian [19].

Pimentel developed the second approach, with second-order singular systems and this was treated which is Hamilton-Jacobi approach [18].

The equations of motion are written as total differential equations in many variables. The coordinates corresponding to dependent momenta considered as parameters.

This leads us to the fact that Dirac's approach completely consistent with Hamilton-Jacobi approach. The eqs. of motion (24-25) are equivalence to eqs. (27-28). The integral conditions (35) are valid if, and only if, the total differential of $H'_{(m)\alpha}(q_{(m)i}, p_{(m)i})$ must equal zero. In other form consistency conditions leads to the integral conditions.

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الربط بين طريقة ديراك وطريقة هاميلتون- جاكوبي للأنظمة الهاملتونية المقيدة للرتب العالية

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مستخلص البحث

تمت دراسة العلاقة بين طريقة ديراك وطريقة هاملتون-جاكوبي للأنظمة المقيدة للرتب العالية يظهر أن طريقة ديراك تتفق تماما مع طريقة هاملتون- جاكوبي. شروط التكامل في طريقة هاملتون-جاكوبي تعادل شروط الاتساق في طريقة ديراك. وهنا سوف نبين بأنَّ طريقةَ ديراك دائماً تتفق معَ طريقةِ هاملتون – جاكوبي. وأنَّ الشروط القابلة للتكامل في طريقةِ هاملتون – جاكوبي تكافئُ الشروطُ التناسقية في طريقةِ ديراك.