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A comparison study between Bayesian robustness estimation and numerical methods for the scale parameter of the Rayleigh distribution

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Abstract

This paper uses robust statistical estimating techniques and the numerical methods to estimate and approximate the scale parameter of the Rayleigh distribution under complete data. Robust Bayes analysis depends on (unbalanced and balanced) loss functions based on (ML-II-ϵ-contaminated class and derived under the prior contaminant distribution of the Frechet distribution and the approximate values numerically are finding by using three numerical methods (Newton-Raphson method, false position method, and the secant method). The estimators' performances were contrasted according to simulation experiments for different cases and sample sizes depending on the value for the mean squared error.

Key word: Bayesian robustness, Rayleigh distribution, Frechet distribution, Newton-Raphson, False position, Secant method.

1. Introduction

The Rayleigh distribution (RD) is credited with naming to John Baron Rayleigh who introduced it in 1880 about the interference of random phase harmonic oscillations in a communication channel [11]. Furthermore, The Weibull-Rayleigh distribution, also known as the RD, is derived from the amplitude of sound that results from numerous significant sources. Life-testing experiments, reliability analysis, applied statistics, and clinical research are just a few of the many uses for the RD with the shape parameter equal to (2), this distribution is a special instance of the two-parameter Weibull distribution [6]. Klivans et al. (2018) developed robust learning algorithms that succeed on a data set contaminated with adversarial corrupted outliers [1]. Al-Ani B. G. et al. (2019) introduced a robust Bayesian method to estimate the reliability function When the shape parameter β is known under the quadratic loss function, they use the initial distribution of the parameter θ with class (ML-Ⅱ-∈-Contaminated) and discover that the data follows the two-parameter Weibull distribution so that the prior distribution of the scale parameter is a Frechet distribution (FD) [3]. Slob and Burgess (2020) studied Tukey's loss function combined with MM estimation to provide robustness against influential points [4]. Mahmoud et al. (2022) used estimation and numerical methods to find the unknown shape parameter of the Kumaraswamy distribution where the numerical approximate values are obtained by using Newton's method and the false position method [13]. Hussein et al. (2023) considered the exponential RD can be extracted mathematically by combining the exponential distribution's cumulative distribution function with the RD's cumulative distribution function [12]. Abraheem et al. (2024) introduced the methods for estimating the reversed hazard rate function of the inverse Kumaraswamy distribution; maximum likelihood estimation method as a non-Bayesian estimator, and the Bayesian estimators with two informative priors (Gamma and Exponential) under symmetric (squared error loss function) and asymmetric (entropy loss function). Also, a numerical method (Boubaker polynomials method) is used to find the approximate value for this function [19].

2. Basic Concepts

The RD is one of the most popular distributions, and it is a continuous probability family with a single scale parameter. The probability density function (p.d.f.) of RD with scale parameter θ as [15]:

$$f(t\,;\,\theta)_R = \frac{2}{\theta}\, t\, e^{\frac{-t^2}{\theta}} \quad ; \; t \geq 0\,, \theta > 0 \qquad (1)$$

and it equals zero otherwise.

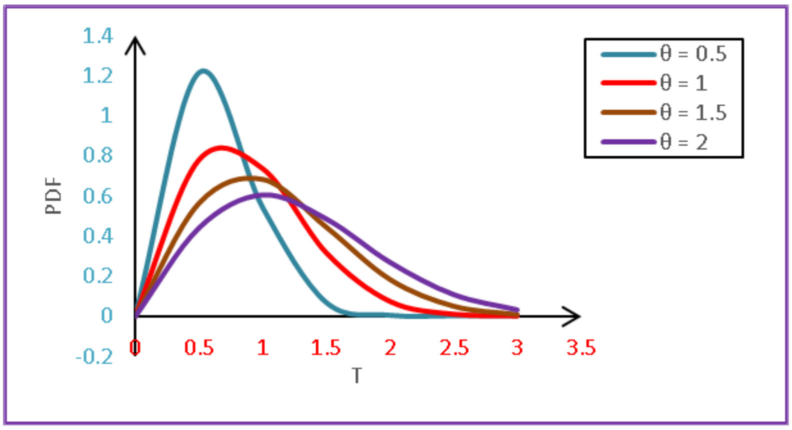

Figure 1: The p.d.f. for RD utilizing some values for θ.

The cumulative distribution function (CDF) for the RD is:

$$F(t\,;\,\theta)_R = 1 - \; e^{\frac{-t^2}{\theta}} \quad ; \; t \geq 0\,, \theta > 0 \qquad (2)$$

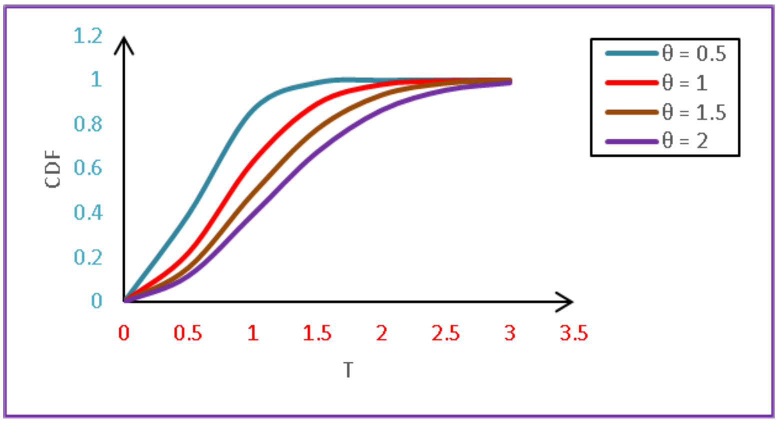

Figure 2: The CDF for the RD utilizing some values for θ.

The reliability function for the RD as follows:

$$R(t)_R = 1 - F(t;\ \theta) = e^{\frac{-t^2}{\theta}}\ \ ;\ t \geq 0, \theta > 0 \qquad (3)$$

3. Loss Functions

The amount of loss resulting from a Bayesian decision around an unknown parameter is known as the loss function, and it is one of the metrics used to assess accuracy in the Bayesian estimation process. It measures the difference between this parameter's estimated and actual values $(\hat{\theta} - \theta)$. It ought to have a real, non-negative value, and typically symbolized by $L(\hat{\theta}, \theta)$ [5].

The two loss functions; unbalanced loss function (SELF) and balanced loss function (BQLF) as symmetric loss functions, Bayesian estimators are obtained [5].

- The SELF for θ is defined as [18]:

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \qquad (4)$$

Take the derivative of equation (4) for $\hat{\theta}$ with setting it to zero and under unbalanced loss function the Bayes estimator of θ symbolized by $\hat{\theta}_{BS}$ then:

$$\hat{\theta}_{BS} = E_\pi(\theta | \underline{t}) \qquad (5)$$

- The BQLF for θ in accordance with Zellner's formula [5]:

$$L_w(\hat{\theta}, \theta) = WL(\hat{\theta}, \theta_0) + (1 - W)\, L(\hat{\theta}, \theta) \qquad (6)$$

where

$L_w(\hat{\theta}, \theta)$: The balanced loss function.

W: weighted coefficient, $w \in (0,1)$.

θ_0: Primary estimator for the parameter (θ) depends on the observations.

$L(\hat{\theta}, \theta)$: The unbalanced loss function.

$L(\hat{\theta}, \theta_0)$: The unbalanced loss function for the likelihood function.

The Bayes estimators can be done by [5]:

$$\hat{\theta}_{BBS} = w\hat{\theta}_{ML} + (1-w)\, E_{\pi}(\,\theta|\,\underline{t}\,) \qquad (7)$$

4. Bayesian Robustness

The Bayesian robustness modeling shows how to turn an existing Bayesian model into a robust one. This approach uses the initial distribution with the class (ML-Π-ϵ-contaminated) of the parameter θ. The prior contaminant distribution is the FD, and the main distribution is RD under unbalanced and balanced loss functions [3].

The p.d.f. and the CDF for FD is given by [9][16]:

$$f(\,t;\,\alpha,\theta\,) = \alpha\theta t^{-(\alpha+1)}\exp(-\theta t^{-\alpha}\,) \quad ; \; t>0\,,\theta>0$$

(8) $F(\,t;\,\alpha,\theta\,) = e^{-\theta t^{-\alpha}} \quad ; \quad t>0$

(9)

where $\alpha > 0$ is the shape parameter, and $\theta > 0$ is the scale parameter for FD.

Bayesian robust analysis of the RD based on the ML-Π-ϵ-contaminated class of priors of the unknown scale parameter θ and known shape parameter α is considered under SELF and BQLF.

Some research has been done on Bayesian robust analysis under the ML-Π-ϵ-contaminated class of priors [8]. The ML-Π-ϵ-contaminated class of prior distribution for θ is:

$$\Gamma = \{\pi(\theta)\!:q(\theta) = \;(1-\epsilon)q_0(\theta) + \,\epsilon\, q(\,\theta\,)\,,q(\theta)\epsilon\, Q)\,\} \qquad (10)$$

where 0<ϵ<1 is pre-assigned and denotes the probability of error in the prior $q_0(\theta)$, elicitation, we take into consideration the base prior, a natural conjugate prior by [8]:

$$q_{F_0}(\,\theta|\sigma_0) = \frac{\sigma_0}{\theta^2}\exp\left(\frac{-\sigma_0}{\theta}\right) \quad ; \quad \sigma_0>0\,,\theta>0 \qquad (11)$$

where (σ_0,θ) represent the vector of hyper-parameters.

The class of all-natural conjugate priors with the vector of hyper-parameter (σ, θ) is known as the contamination class $q(\theta\,|\sigma)$, which is defined as:

$$q_F\,(\theta\,,\sigma\,) = \frac{\sigma}{\theta^2}\exp\left(\frac{-\sigma}{\theta}\right) \quad ; \quad \sigma>0\,,\theta>0 \qquad (12)$$

According to the prior prediction $q(\theta|\sigma)$, the predictive density is:

$$M_{Frechaet\ R}\,(\underline{t}|q) = \int_0^{\infty} L\,(\underline{t}|q)\; q(\theta\,|\sigma)\, d\theta \qquad (13)$$

and by [7]:

$$L\,(\underline{t}|q) = \mathrm{L}\,(\,\theta\,|\,\underline{t}\,)_{RD}^{MLE} = \left(\frac{2}{\theta}\right)^{n}\prod_{i=1}^{n}(\,t_i\,)\, e^{\frac{-\sum_{i=1}^{n}t_i^2}{\theta}}$$

Substituting equation (12) in equation (13) and simplifying, it obtains:

$$M_{Frechaet\ R}\,(\underline{t}|q) = 2^n\,(\prod_{i=1}^{n}t_i)\,\sigma\int_0^{\infty}\theta^{-n-2}\, e^{\frac{-(\sum_{i=1}^{n}t_i^2+\sigma)}{\theta}}\; d\theta$$

Using the transformation, $y = \frac{\sum_{i=1}^{n} t_i^2 + \sigma}{\theta}$ which implies that $\theta = \frac{\sum_{i=1}^{n} t_i^2 + \sigma}{y}$ and $d\theta = \frac{-(\sum_{i=1}^{n} t_i^2 + \sigma)}{y^2} \, dy$, gets:

$$\Rightarrow \quad -2^n \left(\prod_{i=1}^{n} t_i\right) \sigma \left(\sum_{i=1}^{n} t_i^2 + \sigma\right)^{-n-1} \int_0^\infty y^n \, e^{-y} \, dy$$

Then

$$M_{Frechaet\ R}\left(\underline{t}|q\right) = -2^n \left(\prod_{i=1}^{n} t_i\right) \Gamma(n+1)\, \sigma \left(\sum_{i=1}^{n} t_i^2 + \sigma\right)^{-n-1} \quad (14)$$

Now maximized $M_{Frechaet\ R}\left(\underline{t}|q\right)$, substitute's σ at its ML estimator, which is provided by:

$$\frac{dM_{Frechaet\ R}\left(\underline{t}|q\right)}{d\sigma} = -2^n \left(\prod_{i=1}^{n} t_i\right) \Gamma(n+1) \left(\sum_{i=1}^{n} t_i^2 + \sigma\right)^{-(n+2)} \left[\sigma(-n-1) + \left(\sum_{i=1}^{n} t_i^2 + \sigma\right)\right] \quad (15)$$

The following formula is obtained by equating the partial derivative to zero:

$$\hat{\sigma} = \frac{\sum_{i=1}^{n} {t_i}^2}{n} = \hat{\theta}_{ML} \quad (16)$$

Put equation (16) in equation (12). Then we have:

$$q_{FR}\left(\theta|\hat{\sigma}\right) = \begin{cases} \frac{\sum_{i=1}^{n} {t_i}^2}{n\,\theta^2} \exp\left(\frac{-\sum_{i=1}^{n} {t_i}^2}{n\theta}\right) & if\ \sigma_0 < \hat{\sigma} \\ q_{FR_0}\left(\theta|\sigma_0\right) & if\ \sigma_0 \geq \hat{\sigma} \end{cases}$$

Thus, ML- II posterior density is found by:

$$\hat{\pi}_{FR}(\theta) = (1-\epsilon)\, q_{FR_0}\left(\theta|\sigma_0\right) + \epsilon\, q_{FR}\left(\theta|\hat{\sigma}\right) \quad (17)$$

From [8], the ML- II posterior of θ is obtained as:

$$\hat{\pi}_{FR}^*(\theta) = \hat{\lambda}\, q_{FR_0}^*\left(\theta|\sigma_0\right) + \left(1-\hat{\lambda}\right) q_{FR}^*\left(\theta|\hat{\sigma}\right) \quad ; \quad 0 < \theta < \infty$$

and the ML- II posterior mean of θ is given:

$$E\left(\hat{\pi}_{FR}^*(\theta)\right) = \hat{\lambda} E\left(q_{FR_0}^*(\theta|\sigma_0)\right) + \left(1-\hat{\lambda}\right) E\left(q_{FR}^*(\theta|\hat{\sigma})\right); \quad 0 < \theta < \infty \quad (18)$$

where

$$\hat{\lambda} = \frac{(1-\epsilon)\, M_{Frechaet\ R}\left(\underline{t}|q_0(\theta)\right)}{(1-\epsilon)\, M_{Frechaet\ R}\left(\underline{t}|q_0(\theta)\right) + \epsilon\, M_{Frechaet\ R}\left(\underline{t}|\hat{q}(\theta)\right)}$$

$$\hat{\lambda} = \begin{cases} \left[1 + \dfrac{\epsilon\, M_{Frechaet\ R}\left(\underline{t}\middle|\hat{q}\,(\theta)\right)}{(1-\epsilon)\, M_{Frechaet\ R}\left(\underline{t}\middle|q_0(\theta)\right)}\right]^{-1} & if\ \sigma_0 < \hat{\sigma} \\ (1-\epsilon) & if\ \sigma_0 \geq \hat{\sigma} \end{cases}$$

$$= \begin{cases} \left[1 + \dfrac{\epsilon\,(-2^n)\,(\prod_{i=1}^{n} t_i)\Gamma(n+1)\,\hat{\sigma}\,(\sum_{i=1}^{n} {t_i}^2 + \hat{\sigma}\,)^{-(n+1)}}{(1-\epsilon)(-2^n)\,(\prod_{i=1}^{n} t_i)\Gamma(n+1)\,\sigma_0\,(\sum_{i=1}^{n} {t_i}^2 + \sigma_0)^{-(n+1)}}\right]^{-1} & if\ \sigma_0 < \hat{\sigma} \\ (1-\epsilon) & if\ \sigma_0 \geq \hat{\sigma} \end{cases}$$

$$\Rightarrow \qquad \hat{\lambda} = \begin{cases} \left[1+\frac{K\epsilon}{(1-\epsilon)}\ \left(\frac{Kn+1}{K(n+1)}\right)^{(n+1)}\right]^{-1} & if\ \sigma_0 < \hat{\sigma} \\ (1-\epsilon) & if\ \sigma_0 \geq \hat{\sigma} \end{cases}$$

(19)

where $K=\frac{\hat{\sigma}}{\sigma_0}$ and $\hat{\sigma} = \frac{\sum_{i=1}^{n} {t_i}^2}{n}$.

Now, find

$$q^*_{FR_0}(\theta|\sigma_0) = \frac{L(\theta)\ q_0(\theta|\sigma_0)}{M_{Frechaet\ R}\left(\underline{t}\ |q_0(\theta\ )\right)}$$

$$= \frac{\frac{2^n}{\theta^n}\ \prod_{i=1}^{n} t_i\ e^{-\frac{\sum_{i=1}^{n} {t_i}^2}{\theta}}\ \ \frac{\sigma_0}{\theta^2}\ \ e^{-\frac{\sigma_0}{\theta}}}{-2^n\ \ (\prod_{i=1}^{n} t_i)\,\Gamma(n+1)\ \sigma_0\,(\sum_{i=1}^{n} {t_i}^2 +\ \sigma_0\ )^{-(n+1)}}$$

$$q^*_{FR_0}(\theta|\sigma_0) = \frac{(\sum_{i=1}^{n} {t_i}^2 + \sigma_0\ )^{(n+1)}\ \theta^{-(n+2)}\ \ e^{-\frac{(\sum_{i=1}^{n} {t_i}^2+\sigma_0)}{\theta}}}{-\,\Gamma(n+1)}$$

(20)

The probability function in equation (20) is similar to invers gamma distribution IG $(\alpha_1\ ,\beta_1)$, where $\alpha_1 = (n+1)$ and $\beta_1 = (\sum_{i=1}^{n} {t_i}^2 + \sigma_0)$.

Similarly, we get:

$$q^*_{FR}(\theta\,|\hat{\sigma}) = \frac{(\sum_{i=1}^{n} {t_i}^2 + \hat{\sigma}\ )^{n+1}\ \theta^{-(n+2)}\ \ e^{-\frac{(\sum_{i=1}^{n} {t_i}^2+\hat{\sigma})}{\theta}}}{-\,\Gamma(n+1)}$$

(21)

The probability function in equation (21) is similar to invers gamma distribution IG $(\alpha_2\ ,\beta_2)$, where $\alpha_2 = (n+1)$ and $\beta_2 = (\sum_{i=1}^{n} {t_i}^2 +\ \hat{\sigma})$.

Under SELF using equation (4) and simplifying, the ML-II estimator θ is given by:

$$E_{\mathrm{SELF}}(q^*_{FR_0}(\theta|\sigma_0)) = \frac{(\sum_{i=1}^{n} {t_i}^2 + \sigma_0)^{(n+1)}}{-\ \Gamma(n+1)} \int_0^\infty \theta^{-(n+1)} e^{-\frac{(\sum_{i=1}^{n} {t_i}^2 + \sigma_0)}{\theta}} d\theta$$

Using the transformation $y = \frac{\sum_{i=1}^{n} {t_i}^2 + \sigma_0}{\theta}$ which implies that $\theta = \frac{\sum {t_i}^2 + \sigma_0}{y}$ and $d\theta = \frac{-(\sum {t_i}^2 + \sigma_0)}{y^2}$ dy, gets:

$$E_{\mathrm{SELF}}(q^*_{FR_0}(\theta|\sigma_0)) = \frac{(\sum_{i=1}^{n} {t_i}^2 +\ \sigma_0)}{\Gamma(n+1)} \int_0^\infty y^{n-1}\, e^{-y}\, dy$$

$$= \frac{(\sum_{i=1}^{n} {t_i}^2 +\ \sigma_0)}{\Gamma(n+1)}\ \Gamma(n)$$

That is,

$$E_{\mathrm{SELF}}(q^*_{FR_0}(\theta|\ \sigma_0)) = \frac{\sum_{i=1}^{n} {t_i}^2 + \sigma_0}{n} \qquad (22)$$

Similarly,

$$E_{\mathrm{SELF}}(q^*_{FR}(\theta|\ \hat{\sigma})) = \frac{(n+1)\ \sum_{i=1}^{n} {t_i}^2}{n^2} \qquad (23)$$

Put equations (22) and (23) in (18), the robust Bayes estimators of θ for the mixed posterior distribution under SELF is given by:

$$\hat{\theta}_{\mathrm{RBSE}} = \mathrm{E}\left(\hat{\pi}^*_{\mathrm{RBSE}}(\theta)\right) = \begin{cases} \frac{\hat{\lambda}}{n}\ (\sum_{i=1}^{n} {t_i}^2 + \sigma_0) + \frac{(1-\hat{\lambda})(n+1)\sum_{i=1}^{n} {t_i}^2}{n^2} & if\ \sigma_0 < \hat{\sigma} \\ \frac{1}{n}(\sum_{i=1}^{n} {t_i}^2 + \sigma_0) & if\ \sigma_0 \geq \hat{\sigma} \end{cases}$$

where $\hat{\lambda}$ yield equation (19).

Similarly, the ML-II estimator of θ under BQLF, given as:

$$E_{\mathrm{BQLF}}(q^*_{FR_0}(\ \theta|\ \sigma_0)) = W\ \hat{\theta}_{ML} + \frac{(1-w)}{n}\ \left(\ \sum_{i=1}^{n} t_i^2 +\ \sigma_0\ \right) \qquad (24)$$

And

$$E_{\mathrm{BQLF}}(q^*_{FR}(\ \theta\ |\ \hat{\sigma}\ )) = W\ \hat{\theta}_{ML} + \frac{(1-w)\ \ (n+1)}{n^2} \sum_{i=1}^{n} t_i^2 \qquad (25)$$

where $\hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} t_i^2}{n}$.

Put equations (24) and (25) in (18), robust Bayes estimators for θ of a mixed posterior distribution under BQLF are given by:

$\hat{\theta}_{RBBQ} = E\left(\hat{\pi}^*_{RBBQ}(\theta)\right) =$
$$\begin{cases} \hat{\lambda}\left[W\,\hat{\theta}_{ML}+\frac{(1-W)}{n}\left(\sum_{i=1}^{n} t_i^2+\sigma_0\right)\right]+(1-\hat{\lambda})\left[W\hat{\theta}_{ML}+\frac{(1-W)(n+1)}{n^2}\sum_{i=1}^{n} t_i^2\right] & if\ \sigma_0<\hat{\sigma} \\ W\,\hat{\theta}_{ML}+\frac{(1-W)}{n}\left(\sum_{i=1}^{n} t_i^2+\sigma_0\right) & if\ \sigma_0 \geq \hat{\sigma} \end{cases}$$
(26)

5. Numerical Methods

In this section, we will illustrate how to find the approximate values for the scale parameter θ of RD numerically by using three numerical methods (Newton-Raphson method, false position method, and the secant method), which will be introduced as follows.

The first method; the Newton-Raphson method (NRM) that will be used to find the approximate values for the scale parameter θ of RD, and it is also called Newton's method (or the Newton iteration). It is one of the most remarkable and notable numerical approaches to find the roots of an equation Z(t) = 0 [17][21][10].

There are two good derivations of NRM; one is geometric, and the other is analytic. We will discuss both, beginning with the geometric. We wish to find a root of y = Z(t), given an "initial guess" of t_0. How do we improve upon this initial guess to get a better approximation?

The fundamental idea in Newton's method is to use the tangent line approximation to the function f at the point (t_0, $Z(t_0)$). The point-slope formula for the equation of the straight line gives us:

$$\frac{y-y_0}{t-t_0} = Z'(t_0)$$

Therefore, we have a straight line with equation

$y = Z(t_0) + Z'(t_0)(t - t_0)$

To find where this crosses the axis, we set y = 0 and solve for t:

$$t = t_0 - \frac{Z(t_0)}{Z'(t_0)}$$

Call this new approximate value t_1

Now continue the process with another straight line to get:

$$t_2 = t_1 - \frac{Z(t_1)}{Z'(t_1)}$$

Or generally:

$$t_{n+1} = t_n - \frac{Z(t_n)}{Z'(t_n)}$$

This is NRM; it is based on a fundamental, simple idea, and it is an iterative method in numerical methods. The second derivation of NRM depends on our analytical Taylor's theorem. We expand Z in a Taylor series about t_n:

$Z(t) = Z(t_n) + (t - t_n) Z'(t_n) + (1/2)(t - t_n)^2 Z''(\xi_n)$

where ξ_n is between t and t_n, to get a useful algorithm out of this, we set $Z(t) = 0$ and solve for t in terms of $Z(t_n)$, $Z'(t_n)$, and the remainder:

$$t = t_n - \frac{Z(t_n)}{Z'(t_n)} - \frac{1}{2}(t - t_n)^2 \frac{Z''(\xi_n)}{Z'(t_n)}$$

Now, define t_{n+1} to be the quantity that results from the above equation when we drop the remainder term:

$$t_{n+1} = t_n - \frac{Z(t_n)}{Z'(t_n)} \tag{27}$$

To find the approximate value of scale parameter θ of RD using NRM, we must find ($Z(\theta) = 0$). From equation (3) obtain it as:

$$Z(\theta) = R(t) - e^{\frac{-t^2}{\theta}} = 0 \tag{28}$$

In NRM, start one initial value θ_0 and find $\acute{Z}(\theta)$ with respect to θ as:

$$\acute{Z}(\theta) = e^{\frac{-t^2}{\theta}} \left(\frac{t}{\theta}\right)^2 \tag{29}$$

Then, by Newton's iteration formula as in equation (27), get:

$$\theta_1 = \theta_0 - \frac{Z(\theta_0)}{\acute{Z}(\theta_0)}$$

Repeating this process, we have the approximate value of θ called $\hat{\theta}_{NR}$, if:

$|\theta_n - \theta_{n-1}| < \epsilon_{NR}$ where ϵ_{NR} is very small

Algorithm 1 describes the NRM as follows:

Algorithm 1: Newton-Raphson Method NRM

To find a solution to $Z(\theta) = 0$ given an initial θ_0, ϵ_{NR} (EPS)

Set $N_i = 1$.

Set $\theta_1 = \theta_0 - Z(\theta_0)/\acute{Z}(\theta_0)$

While $|\theta_1 - \theta_0| > \epsilon_{NR}$

Set $N_i = N_i + 1$

Set $\theta_0 = \theta_1$

Set $\theta_1 = \theta_0 - Z(\theta_0)/\acute{Z}(\theta_0)$

end while

Compute $\hat{\theta}_{NR} = \theta_1$

end

A potential problem in implementing the Newton-Raphson method is finding the derivative. Although this is not inconvenient for polynomials and many other functions, there are certain functions whose derivatives may be extremely difficult or inconvenient to evaluate. Therefore, the second numerical method (false position method (FPM), also called regular falsi) will be used to compute the real roots of an algebraic equation without the need to calculate the derivative. The method can be presented as follows [17][21]:

We assume two numbers a < b such that:

$Z \in C\,[a, b]$, $Z(a)\,Z(b) < 0$

Generating a sequence of nested intervals $[a_n, b_n]$, n=1, 2, … with $a_1=a$, $b_1=b$, such that $Z(a_n)\,Z(b_n) < 0$. The solution at $t = t_n$ of the linear equation

$p_1(Z; a_n, b_n; t) = 0$

where $p_1(Z; a_n, b_n; .)$ is the linear interpolant of Z at a_n and b_n. If Z is a linear function, we obtain the root in one-step rather than in an infinite number of steps. More explicitly, the method proceeds as follows:

Choose two initial values t_0, t_1 and define $a_1=t_0$ and $b_1=t_1$ then:

$$t_n = a_n - Z(a_n)\frac{a_n - b_n}{Z(a_n) - Z(b_n)} \quad \text{for } n=1, 2, \dots \qquad (30)$$

If $Z(t_n)\,.\,Z(a_n) > 0$ then $a_{n+1}=t_n$, $b_{n+1}=b_n$ else $a_{n+1}=a_n$, $b_{n+1}=t_n$.

To find the approximate value of scale parameter θ of RD using FPM. Let two initial values $a_1 = \theta_0$ and $b_1 = \theta_1$ and apply equations (28) and (30) to find θ_2 from 1st iteration. Additional iterations can be performed to get the approximate value of θ called $\hat{\theta}_{FP}$.

In this method, we stop and find the approximate value of θ if:

$|\theta_n - \theta_{n-1}| < \epsilon_{FP}$ where ϵ_{FP} is very small

Algorithm 2 describes the FPM as follows:

The secant method (SEM) is the third method, it's a recursive method for finding the root of a polynomial by successive approximation. In this method, the roots are approximated by a secant line (A line that intersects the curve at two distinct points) to the function Z(t). It is similar to the FPM. Both use two initial approximate values to compute an approximation of the tangent slope of the function, which is used to project to the x-axis for a new approximate value of the root. However, a critical difference between the methods is how one of the initial values is replaced by the new approximate value, and doesn't need to check $Z(t_1)\,Z(t_2) < 0$ again and again after every approximation. In the FPM, the latest approximate value of the root replaces whichever of the original values yielded a function value with the same sign

as Z (t_n). Consequently, the two approximate values always bracket the root. Therefore, the method converges because the root is kept within the bracket; in contrast, the secant method replaces the values in strict sequence, with the new value t_{i+1} replacing t_i and t_i replacing t_{i-1}. As a result, the two values can sometimes lie on the same side of the root. In some cases, this can lead to divergence. Also, the difference between SEM and NRM is that there is no need for a derivative for the given function Z(t) [2][14].
The method proceeds can presented as follows:
Choose two initial values t_0, t_1 and the signs of the values Z(t_0) and Z(t_1) do not matter in the following formula:

$$t_{n+1} = t_n - Z(t_n)\frac{t_n - t_{n-1}}{Z(t_n) - Z(t_{n-1})} \quad \text{for} \quad n=1, 2, \ldots \qquad (31)$$

To find the approximate value of scale parameter θ of RD using SEM. Let two initial values $t_0 = \theta_0$ and $t_1 = \theta_1$ and the signs of the values Z(θ_0) and Z(θ_1) do not matter, then apply equations (28) and (31) to find θ_2 from 1st iteration and then using θ_1 and θ_2 to find a new approximate value θ_3 from 2nd iteration and so on. Additional iterations can be performed to get the approximate value of θ called $\hat{\theta}_{SE}$.
In this method, we stop and find the approximate value of θ if:
$|\theta_n - \theta_{n-1}| < \epsilon_{SE}$ where ϵ_{SE} is very small
Algorithm 3 describes the SEM as follows:

Algorithm 3: Secant Method SEM

To find a solution of Z(θ)=0 given the continuous function Z on the interval [θ_0, θ_1] and the signs of the values Z(θ_0) and Z(θ_1) do not matter:

Let an initial θ_0, θ_1 ϵ_{SE} (EPS)

Set S_i=1

Set $\theta_{SEapp} = \theta_1 - (Z(\theta_1)(\theta_1 - \theta_0) / (Z(\theta_1) - Z(\theta_0))$

While $|\theta_{SEapp} - \theta_1| > \epsilon_{SE}$

Set $S_i = S_i + 1$

Set $\theta_0 = \theta_1$

Set $\theta_1 = \theta_{SEapp}$

Set $\theta_{SEapp} = \theta_1 - (Z(\theta_1)(\theta_1 - \theta_0) / (Z(\theta_1) - Z(\theta_0))$

end while

6. The Simulation Experiments Study

In this section; a Monte Carlo simulation uses to show the performances robust Bayesian estimators and the approximate values using numerical methods for the unknown scale parameter θ of RD using MSE. The simulation experiments can be summarized as follows:

Stage 1: The constant and parameter values imposed in simulation experiments are defined in Table 1.

Table 1: Assumed values for parameters and constants in simulation experiments

Simple size	**n**	**10 , 15, 25, 50, 100**
Scale parameter	$\boldsymbol{\theta}$	**1, 1.5, 2.5**
Weighted coefficient	**w**	**(0, 1)**
Probability of error	$\boldsymbol{\epsilon}$	**(0, 1)**
Hyper parameter of natural conjugate prior Frechet distribution	$\boldsymbol{\sigma_0}$	**0.05, 0.1, 0.5**
Number of Sample Replicate	**L**	**1000**

- The scale parameter of RD which are varied into three cases: $\boldsymbol{\theta}$ = 1, 1.5, and 2.5.
- Selected the two different values of weighted coefficient (w) on the interval (0, 1) is (w = 0.2, and 0.6), and two different values of probability of error (ϵ) on the interval (0, 1) is (ϵ = 0.00001, and 0.9).
- The hyper-parameters (σ_0) of the natural conjugate prior Frechet distribution are taken as (σ_0= 0.05, 0.1, and 0.5).
- The process is repeated 1000 times to obtain 1000 independent samples of size n.

Stage 2: Generating n observations from the random number U_r where (r = 1, 2, …, n) according to a continuous uniform distribution on the unit interval (0, 1) by using the inverse transformation method which is based on finding the inverse of cumulative distribution function as follows:

$$U = F(t;\theta) \qquad (32)$$

$$t = F^{-1}(U) \qquad (33)$$

Substituting equation (2) in equation (32), gets:

$$U_r = (1 - e^{\frac{-t_r^2}{\theta}}) \; ; \quad t \geq 0,\ \theta > 0 \quad r = 1, 2, \ldots, n \qquad (34)$$

Simplifying equation (34), have the following:

$$t_r = \left[\ln\left(\frac{1}{1-U_r}\right)^{\theta}\right]^{\frac{1}{2}}; \quad t \geq 0,\ \theta > 0 \quad r = 1, 2, \ldots, n \qquad (35)$$

Stage 3: Calculate the robust Bayesian estimators and approximate values of the unknown scale parameter θ for RD according to the formulas which obtained in the previous sections, and compare these estimators by using MSE.

Stage 4: The best estimator is the estimator which has the smallest value of MSE, where MSE is given as [20]:

$$\text{MSE}\left(\hat{\theta}\right) = \frac{1}{\text{L}}\sum_{k=1}^{\text{L}}(\hat{\theta}_k - \theta)^2 \qquad (36)$$

where

L: is the replicated number of samples.

$\hat{\theta}_k$: is the estimate of θ at the k^{th}-replicate.

7. Results of the Simulation

The simulation results depend on the Monte Carlo simulation to estimate the unknown scale parameter of RD with MSE values are presented in tables (2 and 3) using robust Bayes and numerical methods with different values of weighted coefficient (w = 0.2, 0.6), hyper-parameter of natural conjugate prior Frechet distribution (σ_0= 0.05, 0.1, 0.5), and different simple sizes (n =10, 15, 25, 50, 100).

- From table (3) case I with w = 0.2, and σ_0= 0.05, we observe:

When (θ = 1, 1.5, 2.5) and (ϵ = 0.00001) the MSE values associated with robust Bayes based on an unbalanced loss function are less than the MSE values of robust Bayes based on the balanced loss function for all sample sizes except (θ = 2.5, n =10, 15).

- From table (3) case II with w = 0.2, and σ_0= 0.1, we observe:

When (θ = 1, 1.5, 2.5) and (ϵ = 0.00001) the MSE values associated with robust Bayes based on an unbalanced loss function are less than the MSE values of robust Bayes based on the balanced loss function for all sample sizes except (n =10, 15).

- From table (3) case III with w = 0.2, and σ_0= 0.5, we observe:

When (θ = 1, 1.5, 2.5) and (ϵ = 0.00001) the MSE values associated with robust Bayes based on the balanced loss function are less than the MSE values of robust Bayes based on the unbalanced loss function for all sample sizes.

- From table (3) case IV with w = 0.6, and σ_0= 0.05, we observe:

When (θ = 1, 1.5, 2.5) and (ϵ = 0.00001) the MSE values associated with robust Bayes based on an unbalanced loss function are less than the MSE values of robust Bayes based on the balanced loss function for all sample sizes except (θ = 2.5, n = 15).

- From table (3) case V with w = 0.6, and σ_0= 0.1, we observe:

When (θ = 1, 1.5, 2.5) and (ϵ = 0.00001) the MSE values associated with robust Bayes based on an unbalanced loss function are less than the MSE values of robust Bayes based on the balanced loss function for all sample sizes except (θ = 1, 1.5, n = 10), and (θ = 2.5, n = 10, 15).

- From table (3) case VI with w = 0.6, and σ_0= 0.5, we observe:

When (θ = 1, 1.5, 2.5) and (ϵ = 0.00001) the MSE values associated with robust Bayes based on an balanced loss function are less than the MSE values of robust Bayes based on the unbalanced loss function for all sample sizes except (θ = 1.5, n = 25).

- From table (2), we observe:

MSE values associated with the Newton-Raphson method are less than other methods (false position and the secant method) for all sample sizes.

- From tables (2 and 3), we observe:

I. MSE values associated with numerical methods (Newton-Raphson, False Position, and Secant Method) are less than robust Bayes estimates with different cases and for all sample sizes.

II. MSE values associated with the Newton-Raphson method are less than all the numerical methods, and the robust statistical method for all cases and sample sizes.

Table 2: Values of MSE to the scale parameter estimators θ using numerical methods for Rayleigh distribution

θ	n	$\hat{\theta}_{NR}$	$\hat{\theta}_{FP}$	$\hat{\theta}_{SE}$
1	10	0.94782 x10^{-10}	0.39599 x10^{-4}	0.14182 x10^{-7}
	15	0.11582 x10^{-9}	0.43206 x10^{-4}	0.11500 x10^{-7}
	25	0.14424 x10^{-9}	0.47637 x10^{-4}	0.81470 x10^{-8}
	50	0.17694 x10^{-9}	0.53222 x10^{-4}	0.87985 x10^{-8}
	100	0.20242 x10^{-9}	0.57041 x10^{-4}	0.11209 x10^{-7}
1.5	10	0.28334 x10^{-10}	0.22209 x10^{-3}	0.31493 x10^{-8}
	15	0.70910 x10^{-11}	0.25702 x10^{-3}	0.76301 x10^{-9}
	25	0.53490 x10^{-11}	0.28879 x10^{-3}	0.61091 x10^{-9}
	50	0.73580 x10^{-11}	0.32639 x10^{-3}	0.72680 x10^{-9}
	100	0.91770 x10^{-11}	0.35190 x10^{-3}	0.80802 x10^{-9}
2.5	10	0.31939 x10^{-10}	0.83010 x10^{-3}	0.14549 x10^{-7}
	15	0.14364 x10^{-10}	0.92510 x10^{-3}	0.11077 x10^{-7}
	25	0.31250 x10^{-11}	0.10318 x10^{-2}	0.39801 x10^{-8}
	50	0.10690 x10^{-11}	0.10900 x10^{-2}	0.32866 x10^{-9}
	100	0. 14560 x10^{-11}	0.11001 x10^{-2}	0.39010 x10^{-11}

Table 3: Values of MSE to the scale parameter estimators θ using robust Bayes for Rayleigh distribution

Case I: w = 0.2, σ_0 = 0.05

θ	n	$\hat{\theta}_{RBSE}$		$\hat{\theta}_{RBBQ}$	
		ϵ		ϵ	
		10^{-5}	0.9	10^{-5}	0.9
1	10	0.09799	0.12658	0.09798	0.11921
	15	0.056595	0.06794	0.056591	0.06496
	25	0.039111	0.04346	0.039113	0.04235
	50	0.019774	0.02078	0.019776	0.02051
	100	0.009533	0.009779	0.009534	0.009716
1.5	10	0.22539	0.29234	0.22538	0.27515
	15	0.141968	0.17005	0.141965	0.16281
	25	0.09209	0.10142	0.0921	0.09901
	50	0.041083	0.04342	0.041084	0.04281
	100	0.023045	0.02365	0.023046	0.023497
2.5	10	0.55758	0.73138	0.55757	0.68603
	15	0.40515	0.48639	0.40514	0.46553
	25	0.24593	0.27328	0.24595	0.26621
	50	0.123481	0.13032	0.123484	0.12856
	100	0.058831	0.06049	0.058832	0.06007

Case II: w = 0.2, σ_0 = 0.1.

θ	n	$\hat{\theta}_{RBSE}$		$\hat{\theta}_{RBBQ}$	
		ϵ		ϵ	
		10^{-5}	0.9	10^{-5}	0.9
1	10	0.09799	0.12658	0.09798	0.11921
	15	0.056595	0.06794	0.056591	0.06496
	25	0.039111	0.04346	0.039113	0.04235
	50	0.019774	0.02078	0.019776	0.02051
	100	0.009533	0.009779	0.009534	0.009716
1.5	10	0.22539	0.29234	0.22538	0.27515
	15	0.141968	0.17005	0.141965	0.16281
	25	0.09209	0.10142	0.0921	0.09901
	50	0.041083	0.04342	0.041084	0.04281
	100	0.023045	0.02365	0.023046	0.023497
2.5	10	0.55758	0.73138	0.55757	0.68603
	15	0.40515	0.48639	0.40514	0.46553
	25	0.24593	0.27328	0.24595	0.26621
	50	0.123481	0.13032	0.123484	0.12856
	100	0.058831	0.06049	0.058832	0.06007

Case III: w = 0.2, σ_0 = 0.5.

θ	n	$\hat{\theta}_{RBSE}$		$\hat{\theta}_{RBBQ}$	
		ϵ		ϵ	
		10^{-5}	0.9	10^{-5}	0.9
1	10	0.09987	0.12487	0.0991	0.1179
	15	0.05741	0.06729	0.05707	0.06447
	25	0.03935	0.04323	0.03924	0.04216
	50	0.01979	0.02072	0.01977	0.02047
	100	0.00954	0.00977	0.009535	0.009705
1.5	10	0.2273	0.28941	0.22652	0.27301
	15	0.14275	0.16882	0.14242	0.16191
	25	0.09202	0.10099	0.09199	0.09869
	50	0.041126	0.04331	0.041103	0.04274
	100	0.023048	0.02363	0.023045	0.02348
2.5	10	0.55934	0.72598	0.55859	0.68224
	15	0.40595	0.48394	0.40561	0.46381
	25	0.24591	0.27247	0.24587	0.26565
	50	0.12348	0.13012	0.12347	0.12842
	100	0.058834	0.06045	0.058831	0.06003

Case IV: w = 0.6, σ_0 = 0.05.

θ	n	$\hat{\theta}_{RBSE}$		$\hat{\theta}_{RBBQ}$	
		ϵ		ϵ	
		10^{-5}	0.9	10^{-5}	0.9
1	10	0.09799	0.12687	0.09801	0.10703
	15	0.05659	0.06806	0.0566	0.0601
	25	0.03911	0.04351	0.03912	0.04049
	50	0.01978	0.02079	0.019785	0.02009
	100	0.0095349	0.00978	0.009536	0.00961
1.5	10	0.22538	0.29279	0.22539	0.24657
	15	0.14197	0.17025	0.14198	0.1508
	25	0.09213	0.10148	0.09216	0.09504
	50	0.041086	0.04343	0.041089	0.04181
	100	0.023047	0.02366	0.023048	0.02324
2.5	10	0.557605	0.73214	0.557606	0.61136
	15	0.405155	0.48673	0.405153	0.43079
	25	0.24597	0.27339	0.24599	0.25452
	50	0.123491	0.13035	0.123496	0.12564
	100	0.058834	0.0605	0.058835	0.05935

Case V: w = 0.6, σ_0 = 0.1.

θ	n	$\hat{\theta}_{RBSE}$		$\hat{\theta}_{RBBQ}$	
		ϵ		ϵ	
		10^{-5}	0.9	10^{-5}	0.9
1	10	0.097997	0.12658	0.097992	0.10696
	15	0.0565948	0.06794	0.0565949	0.06008
	25	0.03911	0.04346	0.03912	0.04349
	50	0.019774	0.02078	0.019782	0.02009
	100	0.009533	0.00978	0.0095353	0.00961
1.5	10	0.22539	0.29234	0.22537	0.24647
	15	0.14196	0.17005	0.14197	0.15076
	25	0.09209	0.10142	0.09214	0.09503
	50	0.041083	0.04342	0.041087	0.04181
	100	0.0230453	0.02365	0.023048	0.02324
2.5	10	0.55758	0.73138	0.55757	0.61122
	15	0.405147	0.48639	0.405141	0.43073
	25	0.24593	0.27328	0.24598	0.25451
	50	0.12348	0.13032	0.12349	0.12563
	100	0.058832	0.06049	0.058834	0.05935

Case VI: w = 0.6, σ_0 = 0.5.

θ	n	$\hat{\theta}_{RBSE}$		$\hat{\theta}_{RBBQ}$	
		ϵ		ϵ	
		10^{-5}	0.9	10^{-5}	0.9
1	10	0.09987	0.12487	0.09816	0.10637
	15	0.05741	0.06729	0.05666	0.05986
	25	0.03935	0.04323	0.03912	0.04041
	50	0.01979	0.02072	0.019766	0.02007
	100	0.00954	0.009766	0.009532	0.009606
1.5	10	0.2273	0.28941	0.22556	0.2456
	15	0.14275	0.16882	0.14203	0.1504
	25	0.092016	0.10099	0.092019	0.0949
	50	0.04113	0.04331	0.041081	0.04178
	100	0.023048	0.02363	0.023043	0.02323
2.5	10	0.55934	0.72598	0.55771	0.60987
	15	0.40595	0.48394	0.40521	0.4301
	25	0.27247	0.24591	0.2543	0.24588
	50	0.12348	0.13012	0.123468	0.12558
	100	0.058834	0.06045	0.0588292	0.05934

8. Conclusions

This section provides the most essential results depending on simulation experiments using robust statistical estimations and numerical methods were used to find estimates and numerical values of the scale parameter θ for RD. These methods are compared depending on MSE to show which is best and all the computations were performed in (MATLAB 2015). The perfect results are presented and a comparison is done as follows:

- From table (2), the Newton-Raphson method is better than the false position and the secant methods for approximating the scale parameter θ for RD for all sample sizes.
- From table (3), the smaller ϵ and approximate to the zero, the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) are the best from the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) when ϵ approximate to the one.
- From tables (2 and 3):

I. For all sample sizes and all cases, the MSE values associated with numerical methods (Newton-Raphson, False Position, and Secant Method) are less than robust Bayes estimates.

II. The Newton-Raphson method is better than numerical methods, and the robust statistical method for all cases and sample sizes.

III.When the value of sample sizes increases, the MSE values of the robust statistical method used decrease and approximate each other.

9. Recommendations

For future studies, the following points are recommended:

• Use distributions other than Rayleigh distribution and compare them with what the researcher reached in this paper.

•Use loss functions other than squared error (balanced and unbalanced) loss functions to find out the behavior of the robust Bayesian estimation in the presence of these functions.

•Comparison of robust Bayesian estimates with classical robust estimates such as robust maximum likelihood and robust least squares methods.

• Using the same estimation methods of Rayleigh distribution based on other prior distributions.

•Use other numerical methods to find the numerical value of the scale parameter of Rayleigh distribution.

• Use the same numerical methods with other distributions.

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