About One of the Generalizations of Extending Acts

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Abstract:
This work aims to present novel concept that generalizes strongly extending acts previously offered through author. More precisely, concept for strongly pseudo-extending acts was introduced. substantially pseudo-extending S-act M_S has been one at which each sub-act for M_S has been ∩-large at pseudo stable retract for M_S. For any strongly pseudo-extending right (left) act S, there exists monoid S that has been strongly pseudo-extending right (left) act. each sub-acts at this act have been ∩-large at pseudo stable retract for S-act M_S, as well as we investigate their characterizations. This concept has been shown using examples. idea for significantly pseudo-extending acts was defined at relation to other types for injectivity.
Taking things further, conditions under which sub-acts inherit property for strongly pseudo-extending acts have been investigated. concept for RP-act has been introduced. According to this concept, we obtained that strongly pseudo-extending acts have been equivalent to extending acts. Also, we illustrated at theorem (2.16) that direct sum for strongly pseudo-extending acts becomes again strongly pseudo-extending under certain conditions.
2020 Mathematics Subject Classification: 20M30, 20M99, 08B30.
Keywords: strongly pseudo-extending acts, completely pseudo stable sub-acts, RP-acts, strongly extending acts, Extending acts.

1. Introduction

The motivation for our research has been to provide generalizations for conditions at [13]. at particular, we study class for strongly pseudo-extending acts. Now, we assume that each action has been singular right action with value for zero unless stated otherwise. According to Lopez [10], unitary right act M_S with 0 has been non-empty collection with function f: $M\times S\rightarrow M_S$, defined through $f(m,s)\longmapsto ms$, to make subsequent features hold: (1) $m\cdot 1=m$ (2) $m(st)=(ms)t$, for each $m\in M$ as well as $s,t\in S$, in which 1 has been identity element for S. at case of non-empty subset N for S-act M_S has been such that $xs\in N$ holds for each $x\in N$ and $s\in S$, then N has been referred to as sub-act for M_S [8]. Sub-act N for M_S has been termed essential (or large) at M_S at case of as well as only at case of each homomorphism f: $M_S\rightarrow H_S$, in which H_S has been any S-act with constraint to N being one to one, implies that f has been also one to one [10]. Based on this scenario, we assert that M_S represents fundamental extension for N. nonzero sub-act N for M_S has been considered intersection essential at case of for each nonzero sub-act for M_S , intersection A⋂N has been not empty, as well as it will be denoted that N has been ⋂-large at M_S [10]. Based on [5], Feller and Gantos demonstrated that each substantial sub-act for M_S has been ⋂-substantial, reverse does not hold true at general. sub-act N for S-act M_S has been defined as closed at case of it lacks any suitable ⋂-large extension at M_S that serves as a sole solution for N ↪ (⋂l) $_L$≠↪. M_S has been N=L [12]. S-homomorphism f: $A_S \to B_S$ has been termed retraction (split) at case of there exists g: $A_S \to B_S$ such that fog=I_B. at this scenario, B_S has been referred to as retract for A_S [9, P.84]. certain S-act B_S qualifies as retract for S-act A_S at case of as well as only at case of there exists subact W for A_S as well as epimorphism f: $A_S \to W$ such that B_S has been isomorphic to W, as well as for each element w at W, condition f(w) = w holds true. [9, P.84]. S-act M_S has been termed semi-simple at case of as well as only at case of each sub-act for M_S has been either retract or union for simple sub-acts [13]. sub-act N for right S-act M_S has been termed completely invariant at case of f(N)⊆N for each endomorphism f for M_S , as well as M_S has been referred to as duo at case of each sub-act regarding M_S has been completely invariant [4]. Surpassing duo acts for M.S. Abbas as well as R.B. Hiba, concept for completely stable acts was introduced. sub-act N for S-act M_S has been termed stable at case of for each S-homomorphism f:N→M_S, it holds that f(N) has been subset for N. S-act M_S has been referred to as completely stable at case of each sub-act for M_S exhibits stability [6]. It has been demonstrated that each completely stable sub-act has been completely invariant sub-act. sub-act N for S-act M_S has been termed pseudo stable at case of f(N)⊆N for each S-monomorphism f:N→M_S. S-act M_S has been referred to as completely pseudo stable at case of each sub-act for M_S exhibits pseudo stability [2]. It has been evident that each stable sub-act has been pseudo stable, as well as therefore, each completely stable S-act has been completely pseudo stable. It has been demonstrated that each completely pseudo stable S-act exhibits property for being pseudo-PQ-injective.

Recall that S-act M_S has been termed decomposable at case of there exist two sub-acts as well as B for M_S such that M_S = ∪ B as well as ∩ B = Θ. at this scenario, A∪B has been referred to as decomposition for M_S . Alternatively, M_S has been referred to as indecomposable ([9], p.65). Each

cyclic operation has been irreducible.
In [1], M.S. Abbas as well as A. Shaymaa presented groundbreaking idea: definition for reversible (⋂-reversible). non-zero S-act M_S over monoid S has been termed reversible (⋂-reversible) at case of each non-zero sub-act for M_S has been large (⋂-large). It has been evident that each non-zero reversible act qualifies as ⋂-reversible act; however, reverse has been not universally applicable, as well as they coincide when ψ_M=i. at [4], M. Ershad as well as M. Roueentan introduce multiplication operation characterized as follows: S-act M_S has been considered multiplication at case of each sub-act for M_S takes MI form, for specific right ideal I for S. This can be interpreted as stating that each principal sub-act takes on this particular structure. For instance, Zz represents multiplication operation, as well as due to this, each multiplication S-operation qualifies as duo, thus it will be classified as duo operation [4]. reverse has been not universally applicable. however, they have been equivalent at context for projective S-act [4].
Injective as well as quasi-injective acts hold significant importance at theory for acts. at 1967, P. Berthiaume presented notions for injective acts [3]. S-act M_S has been termed injective (H-injective) if, for any S-monomorphism $\alpha:N\rightarrow H_S$, in which N has been sub-act for H_S, each S-homomorphism $\beta:N\rightarrow M_S$ can be extended to S-homomorphism $\sigma:H_S\rightarrow M_S$. Quasi-injective S-acts have been examined through Lopez and Luedeman [10], in which S-act M_S has been defined as quasi-injective at case of as well as only at case of it has been M-injective. For additional details regarding acts, please consult references ([11],[16]).
Additionally, at [13], author presented ideas for extending as well as principally extending acts at following manner: S-act M_S has been known extending act (simply CS-act) at case of each sub-act for M_S has been ⋂-large at retract for M_S .
In this paper, we explore different generalization for extending act known as strongly -pseudo-extending. at same context, our primary finding has been to outline certain characterizations for strongly pseudo-extending acts. Similar to extending acts, that direct sum can be clarified for strongly pseudo-extending acts may not be strongly pseudo-extending. Nonetheless, we derive adequate criteria for direct sum for strongly pseudo-extending acts to qualify as strongly -pseudo-extending. Furthermore, property inherited for strongly pseudo-extending actions has been examined. We also explored whether strongly pseudo-extending acts have been equivalent to extending

acts. We arrived at conclusion that these actions would hold same value under specific circumstances.

2. strongly pseudo-extending acts:

This study will be based on following definitions as well as facts.

Definition (2.1): S-act M_S has been characterized as strongly pseudo-extending at case of each sub-act for M_S has been ∩-large at pseudo stable retract for M_S . monoid S has been called right (left) strongly pseudo-extending at case of S acts as strongly pseudo-extending right (left) entity. S-act M_S has been characterized as strongly pseudo-extending at case of each sub-act for M_S has been ∩-large at pseudo stable retract for M_S. monoid S has been described as right (left) strongly pseudo-extending at case of S functions as strongly pseudo-extending right (left) act.

Examples as well as Remarks (2.2):

1- Each strongly pseudo-extending act has been extending; however, a converse doesn't hold true at general. For instance: Z-act $M_S = Z_2 \dot{\cup} Z_2$ has been semi-simple act, thus it has been extending; however, it has been not regarded as strongly -pseudo-extending.

This has been because when $H = Z_2 \dot{\cup} \{\Theta\}$ has been sub-act for M_S, H has been closed within M_S not pseudo stable (For this, at case of $f: H \longrightarrow M_S$ has been monomorphism distinct through $f(2x, 0) = (0,2x)$, then $(0,2x) \notin H$ hence $f(H) \nsubseteq H$ as well as H has been not pseudo stable sub-act for M_S.

2- each ∩-reversible act has been strongly pseudo-extending converse has been not true at general for example Z_6 as Z-act with multiplication has been strongly pseudo-extending act yet has been not ∩-reversible.

3- Suppose M_S be non-decomposable S-act. S-act M_S has become ∩-reversible when M_S has been strongly -pseudo-extending merely.

Proof: ⟹) Consider N to be sub-structure for M_S. Through robust pseudo-extending property for M_S, N has been ∩-large within pseudo stable retract for M_S. Given that M_S has been indecomposable, it follows that Θ as well as M_S have been sole retracts for M_S. Thus, N has become ∩-large at M_S. Therefore, M_S constitutes ∩-reversible S-act.

In next point, we introduce notion for indecomposable acts for equivalence for strongly extending as well as extending acts.

4- Soppose M_S be indecomposable act. S-act M_S stands for strongly pseudo-extending at case for M_S has been extending.

Proof: Consider that M_S has become strongly -pseudo-extending, thus through (1) M_S has been extending act. On other hand, suppose M_S represent extending act. Since each indecomposable act as well as extending act has become ∩-reversible (if it has been sub-act for M_S through extending property for M_S, it has been ∩-large at retract for M_S. Given that M_S has been indecomposable, it follows that Θ as well as M_S have been sole retracts for M_S. Therefore, represents ∩-large at M_S. Consequently, M_S has become ∩-reversible S-act, thus through (3), M_S constitutes strongly pseudo-extending act.

As stated at [13], S-act M_S has been deemed extending when each closed sub-act for M_S serves as retract for M_S. In similar manner, one can characterize strongly pseudo-extending acts as follows:

Proposition (2.3): S-act M_S can be said to be strongly pseudo-extending at case of as well as only at case of each closed sub-act for M_S serves as pseudo stable retract for M_S.

Proof: ⟹) suppose M_S be strongly pseudo-extending act, as well as suppose be closed sub-act for M_S. Based on strongly pseudo-extending property for M_S, there exists pseudo stable retract B for M_S such that has been ∩-large at B. However, constitutes closed sub-act for B, which implies that equals B. Consequently, serves as pseudo stable retract for M_S.

⟸) suppose us consider that has been subset for M_S. Based on Zorn's lemma, closed sub-act B for M_S exists in which has been ∩-large at B. Since B has been closed, as well as under assumption that B has been pseudo stable retract for M_S, it can be concluded that has been ∩-large at pseudo stable retract for M_S. Consequently, M_S has been robustly pseudo-extending action.

Here's alternative method to define strongly pseudo-extending acts, as demonstrated through following theorem:

Theorem (2.4): For S-act M_S, following statements have been equivalent

1- M_S has been strongly pseudo-extending act;

2- Each closed sub-act for M_S stands for pseudo stable retract;

3- As has been retract for $E(M_S)$, $A \cap M_S$ will be pseudo stable retract for M_S.

Proof: $(1 \longrightarrow 2)$ According to proposition (2.3).

(2⟶3) suppose $E(M_S) = \dot{\cup}$ B, in which B has become subset for $E(M_S)$. suppose us consider that $A \cap M_S$ has been ∩-large within H, in which H functions as sub-act for M_S, as well as subsequently for $E(M_S)$. Furthermore, suppose h be element for H. Consequently, h represents element for $E(M_S)$, which indicates that h belongs to or h belongs to B (this signifies that h equals or h equals b). Now, consider h not at as well as h equals b not equal to Theta. Given that M_S has been ∩-large at $E(M_S)$, it follows that there exists Θ≠s∈S such that ks=bs∈ M_S . Furthermore, given that Θ has been not equal to b at set B as well as that bs has been element for B. Therefore, bs has been element for intersection for M_S as well as B. Conversely, it has been established that A∩M_S has been ∩-large at H, while B has been ∩-large at B. Consequently, A∩M_S∩B has been ∩-significant at H∩B. Nonetheless, we also recognize that M_S ∩A∩B=Θ, which leads us to conclude that H∩B=Θ. Therefore, bs=Θ, leading to contradiction. Consequently, A∩M_S corresponds to closed subset for M_S, as well as thus, according to (2), it qualifies as pseudo stable retract for M_S.

(3⟶1) suppose be subset for M_S as well as B be complement for with respect to M_S. Based on lemma (2.6) at [13], A∪̇B has been ∩-large sub-act for M_S. Given that M_S has been ∩-large at E(M_S), according to lemma (3.1) at [7], A∪̇B has been also ∩-large at E(M_S). Consequently, we have E(A)∪̇ E(B)=E(A(∪̇)B)=E(M_S). Moreover, given that E(A) serves as retract for E(M_S), it follows from (3) that E(A)∩ M_S represents pseudo stable retract for M_S. Since has been ∩-large at E(A) as well as M_S has been ∩-large at M_S, it follows that A=A∩M_S has been ∩-large at E(A)∩ M_S. As result, M_S represents act that strongly extends at pseudo manner.

Definition (2.6): [15] sub-act for S-act M_S has been considered to be lie under retract for M_S at case of there has been direct decomposing $M_S = M_1 \dot{\cup} M_2$ with $N \subseteq M_1$ as well as N has been ∩-large at M_S

According to this definition as well as concept for strongly -pseudo-extending, we can conclude that S-act M_S has been strongly pseudo-extending at case of each sub-act for M_S fits within pseudo stable retract for M_S.

Theorem (2.7): S-act M_S can be seen as strongly pseudo-extending at case of as well as only at case of for each sub-act N for M_S, there exists direct decomposition $M_S = M_1 \dot{\cup} M_2$ such that $N \subseteq M_1$, in which M_1 constitutes pseudo stable sub-act for M_S, as well as $N \dot{\cup} M_2$ has been ∩-large at M_S, with M_2 being any sub-act for M_S.

Proof: ⟹) Suppose that M_S has been strongly pseudo-extending act as well as has been sub-act for M_S. Based on strongly pseudo-extending property, N has been ∩-large at pseudo stable retract at $M_1$of M_S . This means that M_S can be expressed as $M_S = M_1 \dot{\cup} M_2$, in which M_2 has been another sub-act for M_S. Furthermore, since has been ∩-large at M_1 as well as M_2 has been ∩-large at M_2, it follows from remark (2.5) at

[15] that $A\dot{\cup} M_2$ has been ⋂-large at $M_1\dot{\cup}M_2 = M_S$. Therefore, $A\dot{\cup} M_2$ has been ⋂-large sub-act for M_S .
⟸) suppose us consider that has been subset for M_S. According to provided hypothesis, direct decomposition exists such that $M_S = M_1 \dot{\cup} M_2$. It has been subset for $A\subseteq M_1$, as well as M_1 functions as pseudo stable sub-act for M_S, while union for as well as M_2 has been ⋂-large at M_S. We assert that has been also ⋂-large at M_1. suppose B be non-zero sub-act for M_1 as well as consequently for M_S. It can be concluded that $(A\dot{\cup}M_2)\cap B$ cannot be equal to empty set. Therefore, there has been x (not equal to Θ) that belongs to intersection for (A union M_2) as well as B. at case of x has been element for intersection for M_2as well as B, then x equals Θ, given that B has been subset for M_1. Thus, x belongs to intersection for sets as well as B, as well as intersection for as well as B has been not empty. This demonstrates that has been certainly significant at M_1. Consequently, we can deduce that M_S constitutes strongly pseudo-extending act.

Lemma (2.8): each completely invariant retracts for S-act have been regarded as pseudo stable.

Proof: Assume that has been completely invariant retract for S-act M_S as well as $f: N \longrightarrow M_S$ has been any S-homomorphism. Because has been retract for M_S. As result, there has been projection map $\pi: M_S \longrightarrow A$. So $(f\circ\pi): M_S \longrightarrow M_S$. Since has been completely invariant, therefore, $(f\circ\pi)(A) \subseteq A$,and $f(A) = f\big(\pi(N)\big) = (f\circ\pi)(A) \subseteq A$. Consequently, has been pseudo stable retract for M_S.

According to lemma (2.8) other characterizations for strongly pseudo-extending acts have been given below:

Proposition (2.9): S-act M_S has been considered as strongly pseudo-extending at case for each sub-act for M_S has been ⋂-large at completely invariant retract for M_S.

Remember that S-act M_S has been said to be quasi-injective at case of for each homomorphism from sub-act N for M_S into M_S can be extended to S-endomorphism for M_S[10].

it has been familiar that each quasi-injective act has been extending act, as well as each strongly pseudo-extending act has been extending. Then it implies that each strongly pseudo-extending act has been quasi-injective, converse has not be true generally. at next proposition, link between quasi-injective acts as well as strongly pseudo-extending acts has been considered. at fact, they have been independent concepts. Z-act Z has been strongly pseudo-extending (since it has been ⋂-reversible) it has been not quasi-injective. On other hand, vector space for dimension two over filed F has been quasi-injective act, while it has been not strongly pseudo-extending F-act.

In subsequent findings, we examine conditions that determine when quasi-injective action has been strongly pseudo-extending.

Proposition (2.10): each multiplication quasi-injective (extending) act has been strongly -pseudo-extending.

Proof: Suppose that M_S has been multiplication quasi-injective act as well as has been closed sub-act for M_S. Because M_S has been quasi-injective, as well as based on [13], we may conclude has been retract for M_S. Here, it has been sufficient for showing that has been completely invariant for M_S. Assume that $\beta: M_S \longrightarrow M_S$ has been any S-endomorphism for M_S. Based on M_S has been multiplication act, thereby $A = MI$ for several ideal I for S. After that, $\beta(A) = \beta(MI) = \beta(M)I \subseteq MI = A$. Therefore, lemma (2.8) implies that has been pseudo stable as well as M_S has been strongly pseudo-extending act.

In [1], it was clarified that each cyclic operation over commutative monoid has been multiplication operation. Based on preceding proposition, we derive subsequent corollaries.

Corollary (2.11): each cyclic quasi-injective act over commutative monoid exhibits strong pseudo-extension properties.

Remember that at case of each S-homomorphism from closed sub-act for M_S into M_S can be extended to S-endomorphism for M_S, then S-act M_S has been C-quasi-injective [14]. category for strongly pseudo-extending acts has been considered to be encompassed within category for C-quasi-injective acts, as stated at following proposition.

Proposition (2.12): Each strongly pseudo-extending act has been C-quasi-injective.

Proof: proof has been removed since it has been usual.

The findings presented here offer some insights into inquiry regarding conditions under which strongly pseudo-extending property has been passed down to sub-acts.

Proposition (2.13): closed sub-act (and hence retract) for strongly pseudo-extending act has been strongly -pseudo-extending.

Proof: suppose N be closed sub-act for strongly pseudo-extending act. M_S as well as form closed sub-act for N, which subsequently qualifies as closed sub-act for M_S, as demonstrated through lemma (2.4) at [13]. Since M_S exhibits strong pseudo-extension properties. Consequently, N serves as pseudo stable retract for M_S. at case of has been subset for N, then serves as retract for N. We assert that has been pseudo stable sub-act for N. suppose f: → N be S-monomorphism as well as i: N → M_S be inclusion map. Now, (i∘f): A⟶M_S as well as since has been pseudo stable sub-act for M_S, then (i∘f)(A)⊆A, therefore, f(A)⊆A as well as has been pseudo stable retract for N. Consequently, N exhibits strong pseudo-extension properties.

The following proposition gives us another answer for above question:

Proposition (2.14): each sub-act N for strongly pseudo-extending S-act M_S possesses characteristic that intersection for N with any pseudo stable retract for M_S constitutes pseudo stable retract for N. Then, N exhibits strong pseudo-extension properties.

Proof: suppose be subset for N, which has been also subset for M_S . Since M_S exhibits strong pseudo-extension properties, it follows that there exists pseudo stable retract B for M_S at which has been ⋂-large within B. Since has been subset for intersection for B as well as N, it follows from lemma (3.1) at [4] that has been significantly large within intersection for B as well as N. According to proposed hypothesis, B⋂N serves as pseudo stable retract for N. Consequently, N exhibits strong pseudo-extension properties.

Proposition (2.15): suppose M_S be S-act represented as $M_S = M_1 \dot{\cup} M_2$, in which both M_1 as well as M_2 have been strongly pseudo-extending acts. Then, M_S has been strongly pseudo-extending at case of as well as only at case of each closed sub-act for M_S with $A \cap M_1 = \Theta$ or $A \cap M_2 = \Theta$ has been pseudo stable retract.

Proof: requirement can be readily demonstrated through proposition (2.3). On other hand, suppose us consider scenario in which each closed sub-act N for M_S, with $N \cap M_1 = \Theta$ or $N \cap M_2 = \Theta$, has been identified as pseudo stable retract for M_S . suppose us consider that has been closed sub-act for M_S. Then, there exists complement B at in which $A \cap M_2$ has been ⋂-large at B, as well as since has been closed for M_S, B has been also closed for M_S through lemma (2.4) at [13]. Since $(A \cap M_2) \cap M_1$ has been ⋂-large at $B \cap M_1$, it follows that M_1 has been ⋂-large at M_1. Consequently, $B \cap M_1 = \Theta$ (since $A \cap (M_1 \cap M_2) = A \cap \Theta = \Theta$, indicating that Θ has been ⋂-large at Θ. Then, through hypothesis, $M = B \dot{\cup} B'$ for some B' sub-act for M_S as well as B has been pseudo stable retract for M_S. Now, equals intersect M_S equals intersect (B union dot B complement) equals B union dot (A intersect B complement). Therefore, $A \cap B'$ has been closed at M_S (as $A \cap B'$ has been closed at A). Additionally, $(A \cap B') \cap M_2 = \Theta$, thus according to hypothesis, $A \cap B'$ serves as pseudo stable retract for M_S as well as consequently for $B^/$ (given that $A\cap B^/\subseteq B^/$). Therefore, $B^/$has been equal to intersection for as well as $B^/$combined with union for N, in which N has been sub-act for $B^/$. Now, M_S= B $\dot{\cup}$ $B^/$=B $\dot{\cup}$ ((A $\cap$ $B^/$) $\dot{\cup}$ N)=(B$\dot{\cup}$ (A$\cap$ $B^/$)) $\dot{\cup}$ N= A$\dot{\cup}$ N. It can be concluded that has been retract for M_S . suppose g:A⟶M_S be any S-monomorphism, as well as i_1:B⟶B $\dot{\cup}$ (A $\cap$ B^/), i_2: (A $\cap$ $B^/$) ⟶ B $\dot{\cup}$ (A $\cap$ $B^/$) be inclusion mappings. suppose π_1: M_S⟶ M_1 as well as π_2: M_S ⟶ M_2 represent projection functions. suppose h_1 be defined as π_1 composed with g as well as i_1, as well as suppose h_2 be defined as π_2 composed with g as well as i_2. Hence, h equals union for h_1 as well as h_2, as well as g equals union for g_1 as well as g_2. Consequently, g(A) = g(B $\dot{\cup}$ (A $\cap$ $B^/$)) = g_1(B) $\dot{\cup}$ g_2(A $\cap$ $B^/$) ⊆ B $\dot{\cup}$ (A $\cap$ $B^/$) = A. This indicates that g(A) has been subset for as well as serves as pseudo stable retract for M_S. Therefore, M_S is exhibits strong pseudo-extension properties.

A direct sum for strongly pseudo-extending acts can revert to being strongly pseudo-extending under specific conditions. subsequent theorem elucidates these scenarios.

Theorem (2.16): Assume that $M_S = \dot{\bigcup}_{i\in I} M_i$ has been S-act, in which M_i have been sub-act for M_S for each i ∈ I , in which i has been finite index set. Consequently, subsequent assertions hold same value:

1- M_S has been strongly -pseudo-extending;

2- Each M_i has been strongly pseudo as well as each closed sub-act for M_S has been completely invariant;

3- Each M_i has been extending as well as each closed sub-act for M_S has been completely invariant.

Proof: (1 ⟶ 2) Based on proposition (2.14) together with proposition (2.3), prove has been clear.

(2 ⟶ 3) It has been easy to prove, so it has been omitted.

(3 ⟶ 1) suppose N be closed sub-act for M_S , as well as for each i at I, suppose π_i: M_S⟶M_i denote projection map onto M_i. suppose us consider that x belongs to set for natural numbers, thus π_i (x) equals m_i, in which m_i has been element for M_i. Since N has been closed sub-act for M_S, it follows from (3) that N has been completely invariant, as well as therefore $\pi_i(N)\subseteq N\cap M_i$. Consequently, π_i (x)=m_i∈N$\cap$ M_i as well as x∈$\dot{\bigcup}_{i\in I}$ (N$\cap$ M_i). Consequently, N has been subset for union over index set I for intersection for N as well as M_i. For other direction, we have y ∈ $\dot{\bigcup}_{i\in I}$ (N$\cap$ M_i), then there exists j∈I={1,2,…,n} such that y∈N$\cap$M, which implies y∈N as well as

y∈ M_i for some j∈I. Thus, $\dot{\bigcup}_{i\in I}$ $(N\cap M_i)\subseteq N$, as well as this implies that $N=\dot{\bigcup}_{i\in I}$ $(N\cap M_i)$. Given that $N\cap M_i$ serves as retract for N, it follows that $N\cap M_i$ has been closed within N. However, N has been closed at M, therefore through lemma (2.4) at [13], $N\cap M_i$ has been closed at M_S . Since $N\cap M_i$ has been subset for M_i as well as M_i has been subset for M_S , it follows that $N\cap M_i$ has been closed at M_i. through extending property for M_i, we find that $N\cap M_i$ serves as retract for M_i. Therefore, N has been union for intersections for N as well as M_i for each i at I, as well as it serves as retract for M_S , which has been union for M_i for each i at I. Then N has been completely invariant retract for M_S , as well as thus, through lemma (2.8), N has been pseudo stable retract for M_S. Thus, according to proposition (2.3), M_S has been strongly pseudo-extending act.

3. RP-Acts

Recall that sub-act for S-act M_S has been regarded as pseudo-stable at case of $f(A)\subseteq A$ for each S-monomorphism $f:A\longrightarrow M_S$. M_S has been considered completely pseudo stable act at case of each sub-act for M_S exhibits pseudo-stability. monoid S has been termed completely pseudo-stable at case of it qualifies as completely pseudo-stable S-act. [2]. It has been clear that each stable sub-act has been pseudo stable, as well as therefore, each completely stable S-act has been completely pseudo-stable.

It has been important to observe that ideas for extending acts as well as strongly pseudo-extending have been equivalent when condition "every retract has been pseudo stable" holds true. This prompted us to present as well as examine this condition as suitable extension for completely pseudo stable actions as demonstrated at following:

Definition (3.1): S-act M_S has been regarded as RP-act at case of each retract sub-act for M_S exhibits pseudo stability. monoid S has been referred to as RP-monoid at case of S functions as RP-act at context for S-act.

Examples as well as Remarks (3.2):

1. each $\cap$-reversible act has been RP-act.

2. Q as Z-act has been RP-act, not completely pseudo stable.

3. each completely pseudo stable act has been RP-act, converse has been not true. For example, Z-act Z has been RP-act, not fully-pseudo stable.

4. From lemma (2.8) above, at case of each retract for S-act M_S has been fully-invariant, then M_S has been RP-act.

5. each duo act has been RP-act.

6. each indecomposable act has been RP-act.

7. each strongly pseudo-extending act has been RP-act.

By providing notion for RP-act, one can establish equivalence for strongly pseudo-extending acts with extending acts as at next theorem:

Theorem (3.3): **At case of S**-act M_S has been RP-act, then M_S has been strongly pseudo-extending at case of as well as only at case of M_S has been Extending.

Proof: proof has been straightforward.

Corollary (3.4): S-act M_S has been strongly pseudo-extending at case of as well as only at case of M_S has been extending as well as RP-act.

Before present next corollary, we need following definition:

Recall that S-act M_S has been called Θ-simple at case of it contains no sub-acts other than M_S as well as one element sub-act [12].

Corollary (3.5): Suppose that M_S has been Θ-simple S-act. M_S has been strongly pseudo-extending act at case of as well as only at case of M_S has been RP-act.

The following proposition gives characterization for RP-act at class for extending acts.

Proposition (3.6): **Suppose** M_S has been extending act. M_S has been RP-act at case of as well as only at case of each closed sub-act for M_S has been pseudo stable.

Proof: $\Longrightarrow$)According to corollary (3.4), M_S has been strongly pseudo-extending act as well as through proposition (3.3) each closed sub-act for M_S has been pseudo stable. required has been completed

$\Longleftarrow$)It has been obvious.

Proposition (3.7): **At case of each** cyclic sub-act for S-act M_S has been pseudo stable, then M_S has been fully-pseudo stable.

Proposition (3.8): Suppose that M_S has been S-act in which End(M) has been commutative. Then, M_S has been RP-act.

Proof: Suppose that has been retract for M_S as well as $f: A \longrightarrow M_S$ has been any S-monomorphism. There has been sub-act B for M_S, in which $M_S = A\dot{\cup}B$. Then, f can be extended to S-homomorphism $g: M_S \longrightarrow M_S$ as well as defined through $g(b) = \Theta$ for each b at B. Define $h: M_S \longrightarrow M_S$ through $h(a,b) = a$ for each $a \in A, b \in B$. suppose $f(a) = (y,z)$ for some $y \in A, z \in B$. $(h \circ g)(w) = (h \circ g)(a,b) = h\big(f(a)\big) = h(y,z) = y$ as well as $(g \circ h)(w) = (g \circ h)(a,b) = g(a) = (y,z)$

. Because End(M) has been commutative, then $h \circ g = g \circ h$ as well as hence

$z = \Theta$. Therefore, $f(a) \in A$ as well as $f(A) \subseteq A$. As result, we have M_S has been RP-act.

In next Proposition, we prove that class for multiplication acts has been contained at class for RP-acts.

Proposition (3.9): **Each** multiplication S-act has been RP-act.

Proof: **Suppose** N be retract for multiplication act M_S, as well as suppose $f: N \longrightarrow M_S$ be any S- monomorphism. Since M_S has been multiplication, then $N = IM$ for some ideal I for S. N has been retract for M_S.Thus f can be extended to S-homomorphism $g: M_S \longrightarrow M_S$.Then, $f(N) = g(N) = g(IM) \subseteq IM = N$. Therefore, N has been pseudo stable for M_S as well as M_S has been RP-act.

The converse for proposition (3.9) has been not true at general. For example, Z- act Q has been RP-act, not multiplication.

Next proposition clarifies that retract for RP-act inherit property.

Proposition (3.10): **Each** retract for RP-act has been RP-act.

Proof: Suppose that M_S has been RP-act as well as N has been retract for M_S. suppose $f: A \longrightarrow N$ be any S-monomorphism. Because N has been retract for M_S .Then has been retract for M_S. Furthermore, as M_S has been RP-act. Then, has been pseudo stable sub-act for M_S. Thereby, $i \circ f: A \longrightarrow M_S$, in which $i: N \longrightarrow M_S$ has been inclusion map as well as so $(i \circ f)(A) \subseteq A$. This means $f(A) \subseteq A$. Thus, has been pseudo stable for N. As result, N has been RP-act.

4. Conclusions

Extending acts as well as their generalizations at algebra has been significant as it enables us to create new structures, explore properties, as well as better understand algebraic objects. This extension also enables us to study connection between various acts, as has been done at our study. at this paper, notions for strongly pseudo-extending acts as well as RP-acts were introduced in which we were focused on certain characterizations for strongly pseudo-extending acts as at proposition (2.3) as well as theorem (2.4). at addition, link between multiplication quasi-injective acts with strongly pseudo-extending acts was studied as well as clarified at proposition (2.10) as well as corollary (2.11). According to proposition (2.12), we noticed that class for strongly pseudo-extending acts has been said to be included at class for C-quasi-injective acts. Interesting results were reached at proposition (2.15) as well as theorem (2.16) such that proposition (2.15) demonstrates that union for two strongly pseudo-extending acts will be strongly pseudo-extending under certain conditions as well as theorem (2.16) represents generalization for this proposition for finite index. We have utilized RP-act to prove equivalence between strongly pseudo-injective acts with extending acts. This has been appeared at theorem (3.3) as well as corollary (3.4). Besides, we explain relationship between multiplication acts with RP-act. Also, inheriting property for RP-act was investigated as well as illustrated at case of sub-act has been retracted, it will be RP-act. For our future work, we aim to discover new generalizations for extending acts

Data Availability

This study was conducted without use for any data.

Conflicts for Interest

The author states that there have been no conflicts for interest related to publication for this paper.

ACKNOWLEDGMENT

The author would like to thank Mustansiriyah University (https://uomustansiriyah.edu.iq), Baghdad, Iraq, for its support at present work.

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