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Proposing Modified Successive Approximation Method for Treating Two Dimensional Volterra Integral Equations

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Abstract

The primary goal of this study is to investigate and enhance the successive approximation method (SAM) for addressing two dimensional Volterra integral equation of the second kind (TDVIE-2). By comparing numerical solutions with exact solutions, the aim is to achieve improved efficiency and simplicity. These iterative techniques are underpinned by several novel theorems concerning convergence, stability, and robustness. The newly proposed method is supported by favorable simulation outcomes and examples drawn from various case studies utilizing Matlab software.

Keywords: Successive Approximation Method; iterative approach; contractive mapping; and TDVIE-2.

1. General Introduction

Over the past 25 years, numerous integral equation problems have been developed from various scenarios in applied sciences [1, 5]. The TDVIE-2 appears in numerous phenomena within the realms of physics and engineering, as partial differential equations with two variables defined over a bounded area can be transformed into a two-dimensional Volterra integral equation [2, 4]. Many researchers have reformulated and utilized various methods and techniques to derive approximate solutions for integral equations, as many of these equations are often challenging to solve analytically. In several instances, it is necessary to obtain approximate solutions; for further information, see [3, 5, 7, 8]. Scientists have recently managed to tackle inquiries that are vital to our understanding of scientific phenomena, which were once much harder to explore. Numerous publications have emerged from researchers, and many precise approximate techniques for treating integral equations have been developed. Nevertheless, these equations exist within the realm of two-dimensional integral equations [6,7].

Integral equation solutions using numerical techniques have long been crucial to scientific research.since integral equation solutions have always been crucial in the applied sciences [9, 13].

Studying approximations to known or unknown functions is a crucial component of it. In these problems, we typically make these approximation of unknown function because we want to perform analytical operations or compute calculations involving these functions, like differentiation and integration for iteration techniques [14, 20].

More significantly,the application of numerical methods has expanded since the invention of computers and programming. Numerous mathematical problems, including differential equations, the optimal control issue, the Volterra integral equation, and the Fredholm integral equation,are resolved using the sequential SAM [10, 15]. A sequence of iterations that converge to the solution are how the approach operates. Since each new estimate is determined using the previous approximation as a basis, the iterations are recursive in nature.Involving a method for solving integral equations or initial value problems [14, 17].

The authors have explored various numerical methods for obtaining approximate solutions to TDVIE-2, employing different approaches. Marzabad in 2008 solved two-dimensional linear Volterra integral equations of the second kind. Banifatewmi in 2011 used Legendre wavelets method for solving two dimensional mixed Volterra-Fredholm integral equations. Also, Karzan in 2015 Solved Two-dimensional Linear Volterra-Fredholm Integral Equations of the Second Kind by Using Series Solution Methods. In 2018 Shaharuddin solved system of Volterra-Fredholm integral equations of the second kind. Talhat in 2019 solved two dimension linear mixed Volterra-Fredholm Integral equation. Talaat in 2022 treated system of Fredholm integral equation first kind. Chiman in 2024 studied the stability for mixed Volterra –Fredholm integral equations. Talaat and Chiman in 2025 solved Point Kinetics equation.

The primary objective of this work is to reframe and use the SAM and its enhancement for addressing these kinds of problems in order to obtain an approximate answer.

Definition 1. The general form of TDVIE-2:

$$\phi(x,t)u(x,t) = h(x,t) \;+\lambda\int_{a}^{b(t)}\int_{c}^{d(x)} k\big(x,t,y,z,u(y,z)\big)dydz, \quad (1)$$

where $(x,t),(y,z) \in [a,b] \times [c,d]$.

Here $k(x,t,y,z,u(y,z))$ and $h(x,t)$ are known functions, and λ is a complex-valued parameter, $u(x,t)$ represents the unknown function that needs to be found. [11, 19].

Theorem 1: The TDVIE-2 which satisfy these conditions.

1-$\|K(x,t,y,z)\| \le m$ **in** $\mathfrak{P} = \{(x,t,y,z): a \le y \le x \le b, 0 \le z \le t \le c_1\}$ **such that** $k(x,t,y,z) \neq 0$.

2- $f(x,t)$ **is real and continuous in** $\mathfrak{S} = \{(x,t): 0 \le t \le c_1,\ a \le x \le b\}$, $|f(x,t)| \le m$ **in** $\mathfrak{S}$ **and** $f(x,t) \neq 0$.

Then it has one and only one continuous solution $u(x,t)$ in $\mathfrak{S}$ [15].

Theorem 2. Let $u(x,t)$ be a smooth function and $u^n(x,t)$ is n^{th} approximation solution of $u(x,t)$ then $|u(x,t) - u_n(x,t)| \le \frac{\lambda E}{(2\pi)^{n-1}}$, such that n independent of E [21].

Theorem 3: Let $u(x,t)$ and $h(x,t)$ are defined in $\mathfrak{S} = \{(x,t): 0 \le t \le c_1,\ a \le x \le t\}$, $k(x,t,y,z)$ be a continuous function in $\mathfrak{P} = \{(x,t,y,z): a \le y \le x \le t, 0 \le z \le t \le c_1\}$ such that $|k(x,t,y,z)| \le M$ and $|h(x,t)| \le m$, then the sequence $\{u_n(x,t)\}$ defined by

$$u_n(x,t) = \lambda \int_0^x \int_a^t k(x,t,y,z) u_{n-1}(y,z)\,dydz,$$

Convergence absolutely and uniformly, such that $u_0(x,t) = h(x,t)$.

Proof : Since $u_0(x,t) = h(x,t)$**, then**

$$|u_0(x,t)| = |f(x,t)| \le m, \quad \textbf{and}$$

$$|u_1(x,t)| = \left|\lambda \int_a^x \int_a^t k(x,t,y,z) u_0(y,z)\,dydz\right|$$
$$\le |\lambda| mM(t-a)(x-a)$$

Also,

$$|u_2(x,t)| = \left|\lambda \int_a^x \int_a^t k(x,t,y,z)u_1(y,z)dydz\right|$$

$$\le |\lambda| \int_a^{tx} \int_a^t |k(x,t,y,z)| mM(y-a)(z-a)dydz$$

$$\le |\lambda|^2 mM^2 \left(\frac{1}{2*2}(t-a)^2(x-a)^2\right)$$

and

$$|u_3(x,t)| = \left|\lambda \int_a^x \int_a^t k(x,t,y,z)u_2(y,z)dydz\right|$$

$$\le |\lambda| \int_a^x \int_a^t |k(x,t,y,z)| \left(\frac{1}{2*2}(y-a)^2(z-a)^2\right))dydz$$

$$\le |\lambda|^3 mM^3 \left(\frac{1}{2^2 3^2}(y-a)^3(z-a)^3\right)$$

Carrying in a similar manner, we get

$$|u_n(x,t)| \le \frac{m}{2^2 3^2 4^2 \dots n^2} |\lambda|^n M^n (t-a)^n (x-a)^n]$$

As $n \to \infty$ **we have**

$$|u_\infty(x,t)| \le \lim_{k\to\infty} \frac{m}{2^2 3^2 4^2 \dots k^2} |\lambda|^k M^k (c_1-a)^k (c_2-a)^k$$

To demonstrate the convergence of the series mentioned above, we apply the Ratio test.

$$\lim_{k\to\infty} \frac{R_{k+1}}{R_k}$$

$$= \lim_{k\to\infty} \frac{\frac{1}{2^2 3^2 4^2 \dots k^2 (k+1)^2} |\lambda|^{k+1} M^{k+1} (c_1-a)^{k+1} (c_2-a)^{k+1}.}{\frac{1}{2^2 3^2 4^2 \dots k^2} |\lambda|^k M^k (c_1-a)^k (c_2-a)^k.}$$

$$= \lim_{k\to\infty} \frac{|\lambda|M((c_1-a)(c_2-a))}{(k+1)^2} = 0.$$

Therefore, this series converges for every value of $\lambda, M, m, (c_1-a)$ and (c_2-a).

Hence it's absolutely and uniformly convergent $\forall(x,t) \in \mathfrak{S}$**.**

3. Solving TDVIE-2 by using SAM.

The main idea of this work, is to reformulate SAM for solving TDVIE-2. Suppose that the initial approximate solution for the problem be as the form

$$u_0(x,t) = 0 \tag{2}$$

Substuting it in TDVIE-2, we get

$$u_1(x,t) = h(x,t) + \lambda \int_a^x \int_a^t k(x,t,y,z)u_0(y,z)dydz \ = h(x,t) \tag{3}$$

Also substuting the results of $u^1(x,t)$ in TDVIE-2, we get

$$u_2(x,t) = h(x,t) + \lambda \int_a^x \int_a^t k(x,t,y,z)u_1(y,z)dydz$$

After n+1 iteration we obtain the general formula as

$$u_{n+1}(x,t) = h(x,t) + \lambda \int_a^x \int_a^t k(x,t,y,z)u_n(y,z)dydz \tag{4}$$

Then to determine $u_n(x,t)$ convergence to $u(x,t)$ for $n\to\infty$. It turns out that if $h(x,t)$ is continuous in $\mathfrak{S} = \{(x,t): 0 \le t \le c_1,\ a \le x \le b\}$, and $k(x,t,y,z)$ on $\mathfrak{P} = \{(x,t,y,z): a \le y \le x \le b, 0 \le z \le t \le c_1\}$, It can be demonstrated that the sequence will approach the precise solution of TDFIE-2. The verification of this outcome closely resembles the proof of theorem (3).

Theorem 4: Let $h(x,t)$ in $\mathfrak{S} = \{(x,t): 0 \le t \le c_1,\ a \le x \le t\}$, $k(x,t,y,z)$ in $\mathfrak{P} = \{(x,t,y,z): a \le y \le x \le t, 0 \le z \le t \le c_1\}$ are defined, such that $|k(x,t,y,z)| \le M$ and $|h(x,t)| \le m$ then the sequence $\{u_n(x,t)\}$ formulated by

$$u_n(x,t) = h(x,t) + \lambda \int_0^x \int_a^t k(x,t,y,z)u_{n-1}(y,z)dydz,$$

with $u_0(x,t) = 0$ **converges uniformly to a continuous function** $u(x,t)$.

Proof: Since $u_0(x,t) = 0$**, hence** $|u_0(x,t)| = 0 \le m$**,**

$$|u_1(x,t)| = \left|h(x,t) + \lambda \int_a^x \int_a^t k(x,t,y,z)u_0(y,z)dydz\right|$$

$$\le |h(x,t)| + \left|\lambda \int_a^x \int_a^t k(x,t,y,z)u_0(y,z)dydz\right| \le m + 0 = m,$$

and

$$|u_2(x,t)| = \left|h(x,t) + \lambda \int_a^x \int_a^t k(x,t,y,z)u_1(y,z)dydz\right|$$
$$\le |h(x,t)| + \left|\lambda \int_a^x \int_a^t k(x,t,y,z)u_1(y,z)dydz\right|$$

$$\le m + |\lambda| mM \int_a^x \int_a^t dydz$$

$$\le m + |\lambda| mM(t-a)(x-a)$$

Also,

$$|u_3(x,t)| = \left|h(x,t) + \lambda \int_a^x \int_a^t k(x,t,y,z)u_2(y,z)dydz\right|$$

$$\le |h(x,t)| + \left|\lambda \int_a^x \int_a^t k(x,t,y,z)u_2(y,z)dydz\right|$$

$$\le m + |\lambda| \int_a^x \int_a^t |k(x,t,y,z)||\lambda| mM(y-a)(z-a)dydz$$

$$\le m + |\lambda|^2 mM^2 \left(\frac{1}{2*2}(t-a)^2(x-a)^2\right)$$

and

$$|u_3(x,t)| = \left|h(x,t) + \lambda \int_a^x \int_a^t k(x,t,y,z)u_2(y,z)dydz\right|$$

$$\le |h(x,t)| + \left|\lambda \int_a^x \int_a^t k(x,t,y,z)u_2(y,z)dydz\right|$$

$$\le m + |\lambda| \int_a^x \int_a^t |k(x,t,y,z)||\lambda|^2 \left(\frac{1}{2*2}(y-a)^2(z-a)^2\right))dydz$$

$$\le m + |\lambda|^3 m M^3 \left(\frac{1}{2^2 3^2}(y-a)^3(z-a)^3\right)$$

Carrying in a similar manner, we get

$$|u_n(x,t)| \le m + \frac{m}{2^2 3^2 4^2 \dots n^2}|\lambda|^n M^n (t-a)^n (x-a)^n]$$

As $n \to \infty$ **we have**

$$|u_\infty(x,t)| \le m + \lim_{k\to\infty} \frac{m}{2^2 3^2 4^2 \dots k^2}|\lambda|^k M^k (c_1-a)^k (c_2-a)^k$$

$$= m(1 + \lim_{k\to\infty} \frac{1}{2^2 3^2 4^2 \dots k^2}|\lambda|^k M^k (c_1-a)^k (c_2-a)^k)$$

To demonstrate the convergence of the series mentioned above, we apply the Ratio test.

$$\lim_{k\to\infty} \frac{R_{k+1}}{R_k}$$

$$= \lim_{k\to\infty} \frac{m\left(1 + \frac{1}{2^2 3^2 4^2 \dots k^2 (k+1)^2}|\lambda|^{k+1} M^{k+1} (c_1-a)^{k+1} (c_2-a)^{k+1}\right)}{m\left(1 + \frac{1}{2^2 3^2 4^2 \dots k^2}|\lambda|^k M^k (c_1-a)^k (c_2-a)^k\right)}$$

since $|u_n(x,t)| \le \frac{m}{2^2 3^2 4^2 \dots n^2}|\lambda|^n M^n (t-a)^n (x-a)^n]$

in SAM is convergent, { by using theorem (3)}

Since $m > 0$**, then we get,**

$$\begin{aligned} |u_n(x,t)| &\le \frac{m}{2^2 3^2 4^2 \dots n^2}|\lambda|^n M^n (t-a)^n (x-a)^n] \\ &\le m + \frac{m}{2^2 3^2 4^2 \dots n^2}|\lambda|^n M^n (t-a)^n (x-a)^n] \end{aligned}$$

Then it is converges for every value of $\lambda, M, m, (c_1 - a)$and $(c_2 - a)$.

Therefore, in SAM is also uniformaly covergence.

Theorem 5: Supposes that $u(x,t)$ be the exact solution of TDVIE-2, $h(x,t)$ in $\mathfrak{S} = \{(x,t); a \le x \le t, a \le t \le k_1\}$ and $K(x,t,y,z)$ in $\mathfrak{P} = \{(x,t,y,z); a \le y \le x \le t, a \le z \le t \le k_1\}$ are bounded functions, then a mapping $L: M \to N$ is contraction mapping.

Proof: The nth approximate solution has the form,

$$u_n(x,t) = h(x,t) + \lambda \int_a^t \int_a^x k(x,t,y,z) u_{n-1}(y,z) dydz$$

Next, we wish to demonstrate that contraction mapping L is applicable for sufficiently large.

$$\|L(u(x,t)) - L(u_n(x,t))\| = \left\| h(x,t) + \lambda \int_a^t \int_a^x K(x,t,y,z) u(y,z) dydz - [f(x,t) + \lambda \int_a^t \int_a^x K(x,t,y,z) u_{n-1}(y,z) dydz] \right\|$$

$= \left\| \lambda \int_a^t \int_a^x K(y,t,y,z)[(u(y,z) - u_{n-1}(y,z)] dydz \right\|$, since by suppose $|K(y,t,y,z)| \le d$

$$= \left\| \lambda \int_a^t \int_a^x d[u(y,z) - u_{n-1}(y,z)] dydz \right\| \le \lambda d \int_a^t \int_a^x \|u(y,z) - u_{n-1}(y,z)\| dydz$$

By using theorem (2), we have $\|u(x,t) - u_{n-1}(x,t)\| \le \dfrac{\lambda E}{(2\pi)^{n-1}}$

$$\le \lambda d \int_a^t \int_a^x \frac{\lambda E}{(2\pi)^{n-1}} dvdw \le \lambda d \frac{\lambda E}{(2\pi)^{n-1}} \int_a^t \int_a^x dydz$$

$$\|L(u(x,t)) - L(u_n(x,t))\| \le \frac{\lambda^2 EM}{(2\pi)^{n-1}} (t-a)(x-a)$$

Then for $n \to \infty$, obtain $\dfrac{\lambda^2 EM}{(2\pi)^{n-1}}(t-a)(x-a) \to 0$, Therefor

$$\|L(u(x,t)) - L(u_n(x,t))\| \to 0$$

Hence, L is contraction mapping.

Algorithm of the technique SAM:

Input: a, t, n, x, k_1, Er

Step 1: let the initial solution of TDVIE-2 be $u_0(x,t) = 0$

Step 2: for $i = 1$ to n

Step 3: equation (2)'s relation is used to determine the $u_i(x,t)$.

Step 4: computing the absolute error by using $r^n = \left| \sum_{i=0}^{n} u_i(x,t) - u(x,t) \right|$.

If $r^n < Er$, visit the output.

Step 6: containing the TDVIE-2 approximation solution in this method.

Output: the approximate solution's outcomes and r^i.

4. Solving Examples.

This section uses numerical examples to explain and illustrate the SAM for solving TDVEI-2.

Example 1. Solve TDVIE-2

$u(x,t) = xt - \frac{\left(t^5 x^4\right)}{9} + \int_0^x \int_0^t (xt^2 yz) u(y,z)\,dydz$, such that $u(x,t) = xt$ is exact solution.

.

Solution: Making use of SAM after three iterations, the answer is:

First, we can choose, for the zeroth approximation,

$u_0(x,t) = 0$, so the first approximation can be obtained as $u_1(x,t) = xt - \frac{\left(t^5 x^4\right)}{9}$

The second approximate is,

$$u_2(x,t) = xt - \frac{42\left(t^5 x^4\right) + \left(t^5 x^4 \left(t^3 x^4 - 42\right)\right)}{378}$$

Finally, $u_3(x,t) = xt - \frac{\left(t^{12} x^{11}\right)}{37800}$

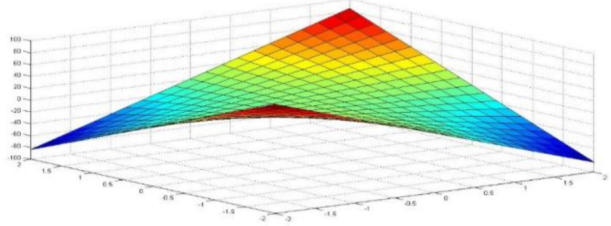

Figure 1: Plot of $u_3(x,t)$ approximation by SAM.

Table 1: shows a comparison at $(x,t) = (0.1, 0.2)$ and the exact solution 0.02000000.

Number of iterations	Approximate solution by SAM	Absolute error for SAM
0	0.000000000000000	0.0200000000000
1	0.019999996444444	0.0011111364731
2	0.020000000000000	0.0000000000000
3	0.020000000000000	0.0000000000000

Example 2. Solve TDVIE-2

$$u(x,t)=x^2+t^2-\frac{(tx(t+x)^2(3t^2-tx+3x^2)}{12}+\int_0^x\int_0^t(x+t)(y+z)u(y,z)dydz$$

Such that $u(x,t)=x^2+t^2$ is exact solution.

Solution: Making use of SAM after three iterations, the answer is:

First, we can choose, for the zeroth approximation, select, $u_0(x,t)=0$, so the first approximation can be obtained as

$$u_1(x,t)=x^2+t^2-\frac{\left(tx\left(t+x\right)^2\left(3t^2-tx+3x^2\right)\right)}{12}$$

The second approximate is,

$$u_2(x,t)=x^2+t^2-\frac{[1260\left(tx\left(t+x\right)^2\left(3t^2-tx+3x^2\right)\right)+\left(tx\left(t+x\right)^2\left(270t^5x+290t^4x^2+277t^3x^3+290t^2x^4-3780t^2+270tx^5+1260tx-3780x^2\right)\right)]}{15120}$$

Finally,

$$u_3(x,t)=x^2+t^2-\frac{\left(tx\left(t+x\right)^2\left(3249t^8x^2+7760t^7x^3+9853t^6x^4+9527t^5x^5+9853t^4x^6+77603t^3x^7+3240t^2x^8-1360800t^2+453600tx-1360800x^2\right)\right)}{5443200}$$

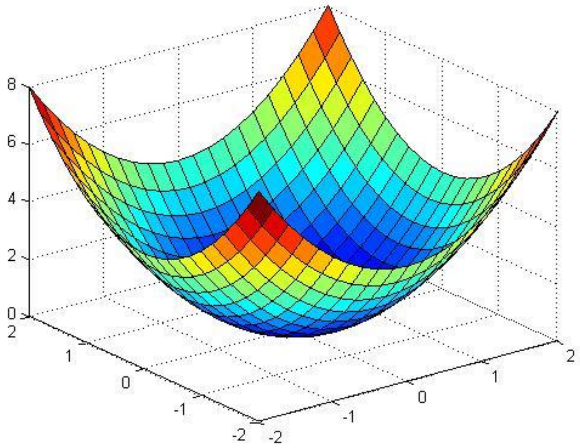

Figure 2: Plot of $u_3(x,t)$ approximation by SAM.

Table 2: shows a comparison at $(x,t)=(0.1,0.2)$ and the exact solution 0.05000000.

Number of iterations	Approximate solution by SAM	Absolute error for SAM
0	0.00000000000000	0.050000000000000
1	0.04998050000000	0.000019500000000
2	0.04999999795274	0.000000001946943
3	0.04999999999988	0.000000000000012

Table 3: Shown LSE and RT for example (1).

		First iteration	Third iteration	Fifth iteration	Seventh iteration
SAM	LSE	1.3557×10^{-4}	7.7654×10^{-6}	5.6632×10^{-9}	4.3431×10^{-12}
	RT	0: 0: 1.8077	0: 0: 3.5770	0: 0: 6.9685	0: 0: 7.4235

Table 4: Shown LSE and RT for example (2).

		First iteration	Third iteration	Fifth iteration	Seventh iteration
SAM	LSE	4.2267×10^{-5}	6.3422×10^{-7}	4.7659×10^{-9}	6.2341×10^{-11}
	RT	0: 0: 1.9021	0: 0: 2.8873	0: 0: 3.5416	0: 0: 5.7632

5. Modification of SAM for solving TDVIE-2.

Assume

$$h(x,t) = h_1(x,t) + h_2(x,t).$$

(5)

For accelerating the convergences of SAM. Put $u_0(x,t) = h_1(x,t)$ and $u_2(x,t) = h_2(x,t) + \tau \int_0^t \int_\Omega G(x,t,y,z) u_0(y,z)\,dydz,$ and

$$u_{k+1}(x,t) = h_2(x,t) + \tau \int_0^t \int_\Omega G(x,t,y,z) u_k(y,z)\,dydz, \qquad (6)$$

If the opposite terms appear from $u^0(x,t)$ and $u^1(x,t)$, then delete them, therefore remaining terms in $u_0(x,t)$ may be given the exact solution for this problem.

Example 3: Solve example one by MSAM.

Solution: Applying MSAM. Let $h(x,t) = xt - \frac{(t^5x^4)}{9}$

Then $h(x,t) = h_1(x,t) + h_2(x,t)$ put $h_1(x,t) = xt$ and $h_2(x,t) = -\frac{(t^5x^4)}{9}$, then $u_0(x,t) = h_1(x,t) = xt$

Computing $u_1(x,t)$ by, $u_1(x,t) = h_2(x,t) + \int_0^x\int_0^t (xt^2yz) u_0(y,z)\,dydz$

$$u_1(x,t) = -\frac{\left(t^5x^4\right)}{9} + \int_0^x\int_0^t (xt^2yz)(yz)dydz = 0$$

Since $u_1(x,t)=0$ then all the next components equal to zero.
Hence, we obtain the precise result by calculating these two elements.

Example 4: Solve example two by MSAM.

Solution: Applying MSAM, since $h(x,t) = x^2 + t^2 - \frac{(tx(t+x)^2(3t^2-tx+3x^2)}{12}$

Then $h(x,t) = h_1(x,t) + h_2(x,t)$ put $h_1(x,t) = x^2 + t^2$ and

$$h_2(x,t) = -\frac{(tx(t+x)^2(3t^2-tx+3x^2)}{12}$$

, then $u_0(x,t) = h_1(x,t) = x^2 + t^2$. Then obtaining the first approximate by,

$$u_1(x,t) = h_2(x,t) + \int_0^x\int_0^t (xt^2yz)u_0(y,z)dydz$$

$$u(x,t) = -\frac{(tx(t+x)^2(3t^2-tx+3x^2)}{12} + \int_0^x\int_0^t (x+t)(y+z)(y^2+z^2)dydz = 0$$

Since $u_1(x,t)=0$ then all the next components equal to zero.
Hence, we obtain the precise result by calculating these two elements.

6. Conclusions.

Proposing SAM and MSAM for solving TDVIE-2nd, suggested suitable algorithm and got results via Matlab software program. Comparing the results in tables (3) and (4) by using LSE and RT to show facilate a discussion. Lastly, we can deduce that:

1. The MSAM performs faster than SAM for solving problem.
2. The MSAM needs only one iteration for getting the exact solution for the problem.
3. The SAM shown their effectiveness and producing precise answers for treating TDVIE-2nd.
4. The LSE changes oppositely with number of iterations.

7. Acknowledgment

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