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Rings Whose Cyclic Modules Are ⊞-Supplemented (Adjunct) Modules

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Abstract

This paper introduces and systematically investigates the class of ⊞-adjunct modules, a generalization of adjunct modules defined by the existence of direct summand complements with a certain small intersection property. We establish sufficient conditions for direct summands and factor modules of a ⊞-adjunct modules to inherit this property. Our primary focus is the characterization of rings over which every cyclic right module is ⊞-adjunct modules. We demonstrate that such rings are necessarily quasi-flawless and provide a precise internal ring-theoretic characterization. Furthermore, we distinguish this new class by constructing examples showing it lies strictly between the rings whose cyclic modules are ascending and those whose cyclic modules are merely adjunct, and prove that this property is not left-right symmetric.

Keywords: ⊞-adjunct, adjunct modules, radical, ascending, quasi-flawless.

1. Introduction and Preliminaries

This paper is set in the context of associative rings with identity and their unital right modules. Let $\mathcal{R}$ be a ring and $\mathcal{M}_{\mathcal{R}}$ an $\mathcal{R}$-module. We adhere to the following notational conventions: $\mathcal{N} \sqsubseteq \mathcal{M}$ denotes that $\mathcal{N}$ is a submodule of $\mathcal{M}$. $\mathcal{N} \sqsubseteq_{ds} \mathcal{M}$ signifies that $\mathcal{N}$ is a direct summand (DS) of $\mathcal{M}$.

Definition 1.1. A submodule $\mathcal{S} \sqsubseteq \mathcal{M}$, if $\forall$ submodule $\mathcal{L} \sqsubseteq \mathcal{M}$, then $\mathcal{S} + \mathcal{L} \equiv \mathcal{M}$ insurance $\mathcal{L} \equiv \mathcal{M}$ then $\mathcal{S}$ is called small in $\mathcal{M}$ (denoted $\mathcal{S} \prec \mathcal{M}$). a radical of $\mathcal{M}$, express $\Psi(\mathcal{M})$, is defined as all maximal submodule's intersections of $\mathcal{M}$. Similarly, $\Psi(\mathcal{M})$ is the sum of each small submodule of $\mathcal{M}$. For foundational concepts and any undefined

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terms, we refer the reader to standard texts (Anderson & Fuller, 1992; Clark et al., 2006; Lam, 1998; Wisbauer, 1991).

The structure of a module can be analyzed through a hierarchy of supplementation properties, which we now define.

Definition 1.2. Let $\mathcal{U}, \mathcal{V} \sqsubseteq \mathcal{M}$.

- if $\mathcal{U} + \mathcal{V} \equiv \mathcal{M}$ and $\mathcal{U} \sqcap \mathcal{V} \prec \mathcal{V}$, then $\mathcal{V}$ is an adjunct of $\mathcal{U}$ in $\mathcal{M}$. This is equivalent to $\mathcal{V}$ being a minimum element belongs to a set $\{\mathcal{L} \sqsubseteq \mathcal{M} \mid \mathcal{U} + \mathcal{L} \equiv \mathcal{M}\}$.
- if every submodule has an adjunct in $\mathcal{M}$ then $\mathcal{M}$ is called adjunct.
- if for any submodules $\mathcal{A}, \mathcal{B} \sqsubseteq \mathcal{M}$ with $\mathcal{M} \equiv \mathcal{A} + \mathcal{B}, \exists$ an adjunct $\mathcal{P}$ in $\mathcal{A}$ provided $\mathcal{P} \sqsubseteq \mathcal{B}$ then $\mathcal{M}$ represent amply-adjunct.
- For each submodule $\mathcal{A} \sqsubseteq \mathcal{M}, \exists$ a DS $\mathcal{D} \sqsubseteq_{ds} \mathcal{M}$ then $\mathcal{M}$ is $\mathcal{H}$-adjunct provided that $\mathcal{A} + \mathcal{X} \equiv \mathcal{M}$ if and only if $\mathcal{D} + \mathcal{X} \equiv \mathcal{M} \forall \mathcal{X} \sqsubseteq \mathcal{M}$.
- if for every submodule $\mathcal{N} \sqsubseteq \mathcal{M}, \exists$ a DS $\mathcal{K} \sqsubseteq_{ds} \mathcal{M}$ then $\mathcal{M}$ is $\boxplus$-adjunct provided that $\mathcal{M} \equiv \mathcal{N} + \mathcal{K}$ and $\mathcal{N} \sqcap \mathcal{K} \prec \mathcal{K}$.

Further structural analysis is provided by the following decomposition conditions.

Definition 1.3. Let $\mathcal{M}$ be a module. We say that $\mathcal{M}$ satisfies:

- $(\Omega 1) \forall$ submodule $\mathcal{A} \sqsubseteq \mathcal{M}, \exists$ a decomposition $\mathcal{M} \equiv \mathcal{M}_1 \boxplus \mathcal{M}_2$ with $\mathcal{M}_1 \sqsubseteq \mathcal{A}$ and $\mathcal{A} \sqcap \mathcal{M}_2 \prec \mathcal{M}_2$.
- ($\Omega 2$) if whenever $\mathcal{A} \sqsubseteq \mathcal{M}$ and $\mathcal{M}/\mathcal{A}$ is isomorphic to a DS of $\mathcal{M}$, then $\mathcal{A} \sqsubseteq_{ds} \mathcal{M}$.
- ($\Omega 3$) if whenever $\mathcal{M}_1$ and $\mathcal{M}_2$ are DS of $\mathcal{M}$ with $\mathcal{M}_1 + \mathcal{M}_2 \equiv \mathcal{M}$, then $\mathcal{M}_1 \sqcap \mathcal{M}_2 \sqsubseteq_{ds} \mathcal{M}$.

These conditions give rise to important classes of modules.

Definition 1.4. we called $\mathcal{R}$-module $\mathcal{M}$:

- an Ascending Module (or an $\Omega 1$-module) if the condition $(\Omega 1)$ is verified.
- a Discrete Module if it satisfies both $(\Omega 1)$ and $(\Omega 2)$.
- a Quasi-Discrete Module if both $(\Omega 1)$ and $(\Omega 3)$ are satisified.

The connection between $\boxplus$-adjunct-adjunction and the ($\Omega 1$) condition is captured by the following known result.

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Proposition 1.5. A module $\mathcal{M}$ is $\boxplus$-adjunct if and only if for every submodule $\mathcal{N} \sqsubseteq \mathcal{M}, \exists$ a DS $\mathcal{K} \sqsubseteq_{ds} \mathcal{M}$ such that $\mathcal{M} \equiv \mathcal{N} + \mathcal{K}$ and $\mathcal{N} \sqcap \mathcal{K} \prec \mathcal{K}$.

As a generalization of adjunct modules (Adj. Mod.), $\boxplus$-Adj. Mod. play a key role in ring theory.

Theorem 1.6. A ring $\mathcal{R}$ is quasi-flawless if and only if the right regular module $\mathcal{R}_{\mathcal{R}}$ is $\boxplus$-adjunct.

A concept related to the ($\Omega 1$) condition is defined as follows.

Definition 1.7. we call $\mathcal{M}$ Radical Ascending (RA) if $\forall$ submodule $\mathcal{A} \sqsubseteq \mathcal{M}$ with $\Psi(\mathcal{M}) \sqsubseteq \mathcal{A}, \exists$ a disintegration $\mathcal{M} \equiv \mathcal{M}_1 \boxplus \mathcal{M}_2$ with $\mathcal{M}_1 \sqsubseteq \mathcal{A}$ and $\mathcal{A} \sqcap \mathcal{M}_2 \prec \mathcal{M}_2$.

Recent research, (Nguyen & Zhou, 2018) has studied rings at each cyclic right $\mathcal{R}$ module is ascending. This line of inquiry motivates the present work. This paper aims to systematically introduce and study the class of rings for which every cyclic right $\mathcal{R}$-module is $\boxplus$-adjunct.

2. Some Concepts on a $\boxplus$-adjunct Module

To characterize the varieties of $\boxplus$-Adj. Mod., we recall the following established hierarchy:

ascending $\Rightarrow \boxplus$ -adjunct $\Rightarrow$ amply-adjunct $\Rightarrow \mathcal{H}$-adjunct $\Rightarrow$ adjunct

The following examples illustrate the distinctions within this hierarchy.

Example 2.1.

(1) The $\mathbb{I}$-module $\mathcal{M} \equiv \mathbb{I}/2\mathbb{I} \boxplus \mathbb{I}/8\mathbb{I}$ is ascending, and hence it is $\boxplus$-adjunct (Harmanci et al., 1999; Kosan & Keskin, 2007).

(2) suppose $\mathcal{R}$ is discrete valuation ring with field of fractions $\mathbb{K}$, let $\mathcal{P}$ be the unique maximal ideal of $\mathcal{R}$, and let $\mathcal{P} \equiv \mathcal{R}\alpha$ for some $\alpha \imath \mathcal{P}$. An $\mathcal{R}$-module $\mathcal{M} \equiv (\mathbb{K}/\mathcal{R}) \boxplus (\mathcal{R}/\mathcal{P})$ satisfies $\Psi(\mathcal{M}) \equiv \mathbb{K}/\mathcal{R} \boxplus 0$. This module is not $\boxplus$-adjunct, but it is $\mathcal{H}$-adjunct (Harmanci et al., 1999; Kosan & Keskin, 2007).

(iii) The $\mathbb{I}$-module $\mathbb{F}$ satisfies $\Psi(\mathbb{F}) \equiv \mathbb{F}$. However, $\mathbb{F}$ is not adjunct, and hence not $\boxplus$-adjunct (Clark et al., 2006).

A $\boxplus$-Adj. Mod. $\mathcal{M}$ with $\Psi(\mathcal{M}) \equiv 0$ is an ascending module. However, a $\boxplus$-Adj. Mod. $\mathcal{M}$ with $\Psi(\mathcal{M}) \prec \mathcal{M}$ may not ascending. We recognize the next weaker result.

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Proposition 2.2. Let $\mathcal{M}$ be a $\boxplus$-adjunct module with $\Psi(\mathcal{M}) \prec \mathcal{M}$. So, $\mathcal{M}$ is $\mathcal{H}$-adjunct.

Proof. Let $\mathcal{U}$ be a submodule of $\mathcal{M}$. Since $\mathcal{M}$ is $\boxplus$-adjunct, $\exists$ a DS $\mathcal{D}$ such that $\mathcal{M} \equiv \mathcal{U} + \mathcal{D}$ and $\mathcal{U} \sqcap \mathcal{D} \prec \mathcal{D}$. This $\mathcal{D}$ is the required $\mathcal{H}$-adjunct for $\mathcal{U}$ (Wang & Wu, 2012).

Whenever a DS of an ascending module is ascending (Mohamed & Müller, 1990), the analogous property for $\boxplus$-Adj. Mod. requires additional conditions.

Proposition 2.3. If the condition ($\Omega 3$) is verified by $\mathcal{M}$, then each DS of $\mathcal{M}$ is $\boxplus$-adjunct where $\mathcal{M}$ is a $\boxplus$-Adj. Mod.

Proof. assume $\mathcal{M} \equiv \mathcal{M}_1 \boxplus \mathcal{M}_2$. Let $\mathcal{Y} \sqsubseteq \mathcal{M}_2$. Since $\mathcal{M}$ is $\boxplus$-adjunct, $\exists$ a DS $\mathcal{D}$ such that $\mathcal{M} \equiv (\mathcal{Y} \boxplus \mathcal{M}_1) + \mathcal{D}$ and $(\mathcal{Y} \boxplus \mathcal{M}_1) \sqcap \mathcal{D} \prec \mathcal{D}$. Since $\mathcal{Y} \boxplus \mathcal{M}_1 + \mathcal{M}_2 \equiv \mathcal{M}$, we have $\mathcal{D} + \mathcal{M}_2 \equiv \mathcal{M}$. Using the ($\Omega 3$) condition, $\mathcal{D} \sqcap \mathcal{M}_2$ is a DS of $\mathcal{M}_2$, so $\mathcal{M}_2 \equiv (\mathcal{D} \sqcap \mathcal{M}_2) \boxplus \mathcal{K}$ for some $\mathcal{K} \sqsubseteq \mathcal{M}_2$. One can then verify that $\mathcal{K}$ is an adjunct for $\mathcal{Y}$ in $\mathcal{M}_2$.

Recall the following definitions: $\mathcal{M}$ represent duo module if all of its submodules are fully invariant (FI); it has the summand intersection property (SIP) if an intersection of any two DS is a DS; it has the summand sum property (SSP) if the sum of any two DS is a DS; a submodule $\mathcal{N}$ of $\mathcal{M}$ is FI if $\psi(\mathcal{N}) \sqsubseteq \mathcal{N}$ for all $\psi \iota \mathrm{Hom}_{\mathcal{R}}(\mathcal{M})$.

Proposition 2.4. suppose $\mathcal{M}$ be a $\boxplus$-Adj. Mod. with the SIP. Then every DS of $\mathcal{M}$ is $\boxplus$-Adj. Mod.

Proof. The proof is similar to that of the previous proposition.

Proposition 2.5. Assume that $\mathcal{M}$ is a $\boxplus$-Adj. Mod. suppose $\mathcal{M}$ have SSP. Hence, each DS belongs to $\mathcal{M}$ represent a $\boxplus$-Adj. Mod.

Proof. assume $\mathcal{M} \equiv \mathcal{N} \boxplus \mathcal{N}'$. We prove that $\mathcal{M}/\mathcal{N}'$ is $\boxplus$-adjunct. suppose $\mathcal{T}/\mathcal{N}' \sqsubseteq \mathcal{M}/\mathcal{N}'$. Since $\mathcal{M}$ is $\boxplus$-adjunct, $\exists$ a DS $\mathcal{D}$ of $\mathcal{M}$ such that $\mathcal{M} \equiv \mathcal{T} + \mathcal{D}$ and $\mathcal{T} \sqcap \mathcal{D} \prec \mathcal{D}$. By the SSP, $\mathcal{D} + \mathcal{N}'$ is a DS of $\mathcal{M}$, so $\mathcal{M} \equiv (\mathcal{D} + \mathcal{N}') \boxplus \mathcal{K}$ for some $\mathcal{K} \sqsubseteq \mathcal{M}$. Then $\mathcal{M}/\mathcal{N}' \equiv (\mathcal{D} + \mathcal{N}')/\mathcal{N}' \boxplus (\mathcal{K} + \mathcal{N}')/\mathcal{N}'$. It can be verified that $(\mathcal{D} + \mathcal{N}')/\mathcal{N}'$ is a DS adjunct of $\mathcal{T}/\mathcal{N}'$ in $\mathcal{M}/\mathcal{N}'$. □

We now investigate when factor modules of $\boxplus$-Adj. Mod. are themselves $\boxplus$-adjunct.

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Proposition 2.6. suppose $\mathcal{M}$ is a $\boxplus$-adjunct module and $\mathcal{N} \sqsubseteq \mathcal{M}$. If $\forall$ DS $\mathcal{K}$ of $\mathcal{M}$, the image $(\mathcal{K}+\mathcal{N})/\mathcal{N}$ is a DS of $\mathcal{M}/\mathcal{N}$, then $\mathcal{M}/\mathcal{N}$ is $\boxplus$-adjunct.

Proof. Let $\mathcal{T}/\mathcal{N} \sqsubseteq \mathcal{M}/\mathcal{N}$. Since $\mathcal{M}$ is $\boxplus$-adjunct, $\exists$ a DS $\mathcal{D}$ of $\mathcal{M}$ provided that $\mathcal{M} \equiv \mathcal{T}+\mathcal{D}$ and $\mathcal{T} \sqcap \mathcal{D} \prec \mathcal{D}$. By hypothesis, $(\mathcal{D}+\mathcal{N})/\mathcal{N}$ is a DS of $\mathcal{M}/\mathcal{N}$. This image serves as the required adjunct for $\mathcal{T}/\mathcal{N}$. □

Proposition 2.7. If $\mathcal{M}$ is a distributive and $\boxplus$-adjunct module, then $\mathcal{M}/\mathcal{N}$ is $\boxplus$-adjunct for every submodule $\mathcal{N} \sqsubseteq \mathcal{M}$.

Proof. Let $\mathcal{N} \sqsubseteq \mathcal{M}$ and $\mathcal{D}$ a DS of $\mathcal{M}$ with $\mathcal{M} \equiv \mathcal{D} \boxplus \mathcal{D}'$. Then $\mathcal{M}/\mathcal{N} \equiv (\mathcal{D}+\mathcal{N})/\mathcal{N} + (\mathcal{D}'+\mathcal{N})/\mathcal{N}$. By distributivity, $(\mathcal{D}+\mathcal{N}) \sqcap (\mathcal{D}'+\mathcal{N}) \equiv \mathcal{N}$, so $\mathcal{M}/\mathcal{N} \equiv (\mathcal{D}+\mathcal{N})/\mathcal{N} \boxplus (\mathcal{D}'+\mathcal{N})/\mathcal{N}$. The result follows from Proposition 2.6. □

Proposition2.8. suppose $\mathcal{N}$ is a submodule of $a \boxplus -$ Adj. Mod. $\mathcal{M}$ provided a decomposition $\mathcal{M} \equiv \mathcal{M}_1 \boxplus \mathcal{M}_2$, then $\mathcal{N} \equiv (\mathcal{N} \sqcap \mathcal{M}_1) \boxplus (\mathcal{N} \sqcap \mathcal{M}_2)$. Then $\mathcal{M}/\mathcal{N}$ is $\boxplus$-adjunct.

Proof. We verify the condition of Proposition 2.6. Let $\mathcal{L}$ be a DS of $\mathcal{M}$, so $\mathcal{M} \equiv \mathcal{L} \boxplus \mathcal{L}'$. The hypothesis on $\mathcal{N}$ implies $(\mathcal{L}+\mathcal{N}) \sqcap (\mathcal{L}'+\mathcal{N}) \equiv \mathcal{N}$. Thus, $\mathcal{M}/\mathcal{N} \equiv (\mathcal{L}+\mathcal{N})/\mathcal{N} \boxplus (\mathcal{L}'+\mathcal{N})/\mathcal{N}$, proving that $(\mathcal{L}+\mathcal{N})/\mathcal{N}$ is a DS of $\mathcal{M}/\mathcal{N}$. □

Corollary 2.9. Let $\mathcal{M}$ be a $\boxplus$-adjunct module and $\mathcal{X} \sqsubseteq \mathcal{M}$. If $\pi\mathcal{X} \sqsubseteq \mathcal{X}$ for all $\pi^2 \equiv \pi\iota\mathrm{Hom}_{\mathcal{R}}(\mathcal{M})$ (in particular, if $\mathcal{X}$ is a fully invariant submodule), then $\mathcal{M}/\mathcal{X}$ is $\boxplus$-adjunct.

Proof. The condition on $\mathcal{X}$ ensures it satisfies the hypothesis of Proposition 2.8.

Remark 2.10. If the right regular module $\mathcal{R}_{\mathcal{R}}$ is $\boxplus$-adjunct and every right ideal $\mathcal{I}$ of $\mathcal{R}$ is fully invariant ($\pi\mathcal{I} \sqsubseteq \mathcal{I}$ for all idempotents $\pi\iota\mathcal{R}$), then every factor module $\mathcal{R}/\mathcal{I}$ is $\boxplus$-adjunct.

3. Main Results

At the beginning, the following facts of a right $\mathcal{R}$-module $\mathcal{M}$ on a quasi-flawless ring $\mathcal{R}$ are provided:

(a) Assume that $\mathcal{R}$ is a quasi-flawless ring. $\forall$ $\mathcal{R}$-module $\mathcal{M}$, $\Psi(\mathcal{M}) \equiv \mathcal{M}\Phi(\mathcal{R})$, when $\Phi(\mathcal{R})$ is the Jacobson Rad. of the ring $\mathcal{R}$ (Anderson & Fuller, 1992).

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(b) suppose that $\mathcal{R}$ is a quasi-flawless ring and $\mathcal{M}$ a cyclic right $\mathcal{R}$-module. Hence, $\mathcal{M} \approx \pi\mathcal{R}/\pi\mathcal{I}$ for some $\pi\iota\mathcal{R}$ and for some right ideal $\mathcal{I}$ in $\Phi(\mathcal{R})$ (Anderson & Fuller, 1992).

Lemma 3.1. suppose that $\mathcal{R}$ is a quasi-flawless ring and $\pi\mathcal{R}/\pi\mathcal{I}$ be a right $\mathcal{R}$-module for $\pi\iota\mathcal{R}$ and for ideal $\mathcal{I}$ in $\Phi(\mathcal{R})$. Then $\Psi(\pi\mathcal{R}/\pi\mathcal{I}) \equiv \pi\Phi(\mathcal{R})/\pi\mathcal{I}$ and $\pi\Phi(\mathcal{R})/\pi\mathcal{I} \prec \pi\mathcal{R}/\pi\mathcal{I}$.

Proof. because $\Psi(\pi\mathcal{R}/\pi\mathcal{I}) \equiv (\pi\mathcal{R}/\pi\mathcal{I})\Phi(\mathcal{R})$, we get $\Psi(\pi\mathcal{R}/\pi\mathcal{I}) \equiv \pi\Phi(\mathcal{R})/\pi\mathcal{I}$. The 2nd part of proof comes from (Anderson & Fuller, 1992).

Theorem 3.2. The next claims were identical for $\mathcal{R}$:

1. Each cyclic right $\mathcal{R}$-module is $\boxplus$-adjunct.
2. Whenever $\mathcal{I} \sqsubseteq \Phi(\mathcal{R}), \pi^2 \equiv \pi\iota\mathcal{R}, \rho^2 \equiv \rho\iota\pi\mathcal{R}, \rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I}$ contains a DS $\mathcal{A}/\pi\mathcal{I}$ of $\pi\mathcal{R}/\pi\mathcal{I}$ with the decomposition $\mathcal{A}/\pi\mathcal{I} \boxplus \mathcal{B}/\pi\mathcal{I} \equiv \pi\mathcal{R}/\pi\mathcal{I}$ such that $\mathcal{B}/\pi\mathcal{I} \sqcap (\rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I}) \prec \pi\mathcal{R}/\pi\mathcal{I}$ where $\mathcal{R}$ is quasi-flawless.
3. Whenever $\mathcal{I} \sqsubseteq \Phi(\mathcal{R}), \pi^2 \equiv \pi\iota\mathcal{R}, \rho^2 \equiv \rho\iota\pi\mathcal{R}$, we have $\rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I} \equiv \xi\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I} \boxplus \mathcal{S}/\pi\mathcal{I}$, where $\xi\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I} \boxplus (\pi - \xi\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I} \equiv \pi\mathcal{R}/\pi\mathcal{I}$ with $\xi^2 \equiv \xi\iota\pi\mathcal{R}$ such that $\xi(\pi\mathcal{I}) \sqsubseteq \pi\mathcal{I}$ and $\mathcal{S}/\pi\mathcal{I} \equiv (\rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I}) \sqcap ((\pi - \xi\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}) \prec \pi\mathcal{R}/\pi\mathcal{I}$ where $\mathcal{R}$ is quasi-flawless.

Proof. (1) ⇒ (2) According to the assumption, we imply that $\mathcal{R}$ is quasi-flawless (since every cyclic module being $\boxplus$-adjunct implies $\mathcal{R}_{\mathcal{R}}$ is $\boxplus$-adjunct, which is equivalent to quasi-flawless-ness). suppose $\mathcal{I} \sqsubseteq \Phi(\mathcal{R}), \pi^2 \equiv \pi\iota\mathcal{R}$, and $\rho^2 \equiv \rho\iota\pi\mathcal{R}$. So, by Lemma 3.1, we get $\Psi(\pi\mathcal{R}/\pi\mathcal{I}) \sqsubseteq \rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I} \sqsubseteq \pi\mathcal{R}/\pi\mathcal{I}$. As $\pi\mathcal{R}/\pi\mathcal{I}$ is $\boxplus$-adjunct, $\exists$ a decomposition $\pi\mathcal{R}/\pi\mathcal{I} \equiv \mathcal{A}/\pi\mathcal{I} \boxplus \mathcal{B}/\pi\mathcal{I}$ such that $\mathcal{A}/\pi\mathcal{I} \sqsubseteq \rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I}$ and $\mathcal{B}/\pi\mathcal{I} \sqcap (\rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I}) \prec \pi\mathcal{R}/\pi\mathcal{I}$ as required.

(2) ⇒ (1) 'since $\mathcal{R}$ is quasi-flawless, each cyclic module in $\mathcal{R}$ is isomorphic to $\pi\mathcal{R}/\pi\mathcal{I}$ for $\pi\iota\mathcal{R}$ and right ideal $\mathcal{I}$ in $\Phi(\mathcal{R})$ of $\mathcal{R}$. So, i.e. $\pi\mathcal{R}$ is an ascending module' and $\pi\mathcal{R}/\pi\mathcal{I}$ is amply-adjunct. Let $\mathcal{A}/\pi\mathcal{I} \sqsubseteq \pi\mathcal{R}/\pi\mathcal{I}$ with $\Psi(\pi\mathcal{R}/\pi\mathcal{I}) \sqsubseteq \mathcal{A}/\pi\mathcal{I}$. Then $\mathcal{A}/\pi\mathcal{I}$ has an adjunct $\mathcal{B}/\pi\mathcal{I}$ and $\mathcal{B}/\pi\mathcal{I}$ has an adjunct $\mathcal{X}/\pi\mathcal{I} \sqsubseteq \mathcal{A}/\pi\mathcal{I}$ in $\pi\mathcal{R}/\pi\mathcal{I}$. Just as $\pi\mathcal{R}$ is ascending, write $\mathcal{X} \equiv \rho\mathcal{R} + \mathcal{S}$, for some $\rho^2 \equiv \rho\iota\pi\mathcal{R}$ and $\mathcal{S} \prec \pi\mathcal{R}$. So, utilizing of minimality of $\mathcal{X}/\pi\mathcal{I}$, we construe that $\mathcal{X} \equiv \rho\mathcal{R} + \pi\mathcal{I}$. By hypothesis,

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$\pi\mathcal{R}/\pi\mathcal{I} \equiv \mathcal{X}/\pi\mathcal{I} \boxplus \mathcal{X}'/\pi\mathcal{I}$ for some suitable DS $\mathcal{X}'/\pi\mathcal{I} \sqsubseteq \pi\mathcal{R}/\pi\mathcal{I}$. Since $\mathcal{A}/\pi\mathcal{I} \sqcap \mathcal{B}/\pi\mathcal{I} \prec \pi\mathcal{R}/\pi\mathcal{I}$, we have $\mathcal{A}/\pi\mathcal{I} \sqcap \mathcal{X}'/\pi\mathcal{I} \prec \pi\mathcal{R}/\pi\mathcal{I}$ and hence the summand $\mathcal{X}'/\pi\mathcal{I}$ is an adjunct of $\mathcal{A}/\pi\mathcal{I}$. This shows that $\pi\mathcal{R}/\pi\mathcal{I}$ is a $\boxplus$-Adj. Mod.

(2) $\Leftrightarrow$ (3) By [11], $\mathcal{A}/\pi\mathcal{I} \equiv \xi\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$ and $\mathcal{B}/\pi\mathcal{I} \equiv (\pi - \xi\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$ with $\xi^2 \equiv \xi\imath\pi\mathcal{R}$ such that $(\xi\pi)\mathcal{I} \sqsubseteq \pi\mathcal{I}$ and $\mathcal{S}/\pi\mathcal{I} \equiv (\rho\mathcal{R} + \pi\Phi(\mathcal{R})/\pi\mathcal{I}) \sqcap ((\pi - \xi\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I})$.

Corollary 3.3. A commutative ring $\mathcal{R}$ is quasi-flawless iff each cyclic $\mathcal{R}$-module is $\boxplus$ adjunct.

Proof. note that each comm. quasi-flawless ring $\mathcal{R}$ usually mollify condition (3) of Theorem (3.2), by assuming $\xi \equiv \rho\imath\pi\mathcal{R}$. So, by utilizing of remark (2.10).

A following two examples demonstrate a cyclic module on a quasi-flawless ring has not to be $\boxplus$-adjunct.

Example 3.4. Let $\mathcal{R} \equiv \begin{pmatrix} \mathbb{I}_4 & \mathbb{I}_4 \\ 0 & \mathbb{I}_4 \end{pmatrix}$. $\Rightarrow \mathcal{R}$ is quasi-flawless. suppose the right ideal $\mathcal{I} \equiv \left\{\begin{pmatrix} 0 & 2\alpha \\ 0 & 2\alpha \end{pmatrix} : \alpha\imath\mathbb{I}_4\right\} \sqsubseteq \Phi(\mathcal{R}) \equiv \begin{pmatrix} 2\mathbb{I}_4 & \mathbb{I}_4 \\ 0 & 2\mathbb{I}_4 \end{pmatrix}$. Let $\rho \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\rho^2 \equiv \rho\imath\mathcal{R}$, and $\rho\mathcal{R} + \Phi(\mathcal{R}) \equiv \begin{pmatrix} 2\mathbb{I}_4 & \mathbb{I}_4 \\ 0 & \mathbb{I}_4 \end{pmatrix}$. We verify that $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I}$ has no non-zero DS of $\mathcal{R}/\mathcal{I}$. Assume that $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I}$ has a summand that is not zero of $\mathcal{R}/\mathcal{I}$. Then, from [11], $\exists\gamma^2 \equiv \gamma\imath\mathcal{R}$ provided that $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I} \equiv \gamma\mathcal{R} + \mathcal{I}/\mathcal{I} \boxplus (\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I} \sqcap (1 - \gamma)\mathcal{R} + \mathcal{I})/\mathcal{I}$. So, $\gamma\mathcal{I} \sqsubseteq \mathcal{I}$. and $\gamma\imath\rho\mathcal{R} + \Phi(\mathcal{R})$, we should get $\gamma \equiv \begin{pmatrix} 0 & 2\beta \\ 0 & 1 \end{pmatrix}$ for some $\beta\imath\mathbb{I}_4$. However, $\gamma \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \notin \mathcal{I}$. So, $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I}$ has not any DS of $\mathcal{R}/\mathcal{I}$ does not equal to zero. Hence, $\mathcal{R}/\mathcal{I}$ doesn't represent a $\boxplus$-adjunct.

Example 3.5. suppose $\mathbb{I}_{(2)} \equiv \{\mu/\nu\imath\mathbb{F}: \mathrm{GCD}(\nu, 2) \equiv 1\}$ is a localized ring at $2\mathbb{I}$. let $\mathcal{R} \equiv \begin{pmatrix} \mathbb{I}_{(2)} & \mathbb{I}_{(2)} \\ 0 & \mathbb{I}_{(2)} \end{pmatrix}$. So, $\mathcal{R}$ is a quasi-flawless ring. Choose $\zeta \equiv \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ in $\mathcal{R}$ and let a principal right ideal $\mathcal{I} \equiv \zeta\mathcal{R} \sqsubseteq \Phi(\mathcal{R}) \equiv$

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$\begin{pmatrix} 2\mathbb{I}_{(2)} & \mathbb{I}_{(2)} \\ 0 & 2\mathbb{I}_{(2)} \end{pmatrix}$. Let $\rho \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\rho^2 \equiv \rho\iota\mathcal{R}$, and $\rho\mathcal{R} + \Phi(\mathcal{R}) \equiv \begin{pmatrix} 2\mathbb{I}_{(2)} & \mathbb{I}_{(2)} \\ 0 & \mathbb{I}_{(2)} \end{pmatrix}$. We verify that $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I}$ has no DS of $\mathcal{R}/\mathcal{I}$ does not equal to zero. Therefore, $\exists\gamma^2 \equiv \gamma\iota\mathcal{R}$ provided that $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I} \equiv \gamma\mathcal{R} + \mathcal{I}/\mathcal{I} \boxplus (\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I} \sqcap (1-\gamma)\mathcal{R} + \mathcal{I})/\mathcal{I}$ and $\gamma\mathcal{I} \sqsubseteq \mathcal{I}$. As $\gamma\iota\rho\mathcal{R} + \Phi(\mathcal{R})$, we should get $\gamma \equiv \begin{pmatrix} 0 & \beta \\ 0 & 1 \end{pmatrix}$ for $\beta\iota\mathbb{I}_{(2)}$. However, $\gamma \begin{pmatrix} 2 & 5 \\ 0 & 4 \end{pmatrix} \equiv \begin{pmatrix} 0 & 4\beta \\ 0 & 4 \end{pmatrix} \notin \mathcal{I}$. So, $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I}$ has no any non-zero DS of $\mathcal{R}/\mathcal{I}$. Hence, $\mathcal{R}/\mathcal{I}$ is not a $\boxplus$-adjunct.

The subsequent result was found in (Nguyen and Zhou, 2018). To differentiate between the concepts of ascending modules and $\boxplus$-Adj. Mod., we will provide an alternative proof below.

Theorem 3.6. The next claims are identical in $\mathcal{R}$:

1. Each cyclic right $\mathcal{R}$-module is ascending.
2. When $\mathcal{I} \sqsubseteq \Phi(\mathcal{R}), \pi^2 \equiv \pi\iota\mathcal{R}, \rho^2 \equiv \rho\iota\pi\mathcal{R}, \rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$ is a DS of $\pi\mathcal{R}/\pi\mathcal{I}$ where $\mathcal{R}$ is quasi-flawless.
3. When $\mathcal{I} \sqsubseteq \Phi(\mathcal{R}), \pi^2 \equiv \pi\iota\mathcal{R}, \rho^2 \equiv \rho\iota\pi\mathcal{R}$, we get $\rho\mathcal{R} + \pi\mathcal{I} \equiv \xi\mathcal{R} + \pi\mathcal{I}$ and $\xi\pi\mathcal{I} \sqsubseteq \pi\mathcal{I}$ for some $\xi^2 \equiv \xi\iota\pi\mathcal{R}$ where $\mathcal{R}$ is quasi-flawless.

Proof. (1) ⇒ (2) $\mathcal{R}$ is quasi-flawless by the assumption which insurance that (Baba & Oshiro, 2009) . suppose $\mathcal{I} \sqsubseteq \Phi(\mathcal{R}), \pi^2 \equiv \pi\iota\mathcal{R}$, and $\rho^2 \equiv \rho\iota\pi\mathcal{R}. \Rightarrow \rho\mathcal{R} \boxplus (\pi - \rho\pi)\mathcal{R} \equiv \pi\mathcal{R}$. Note $\pi\mathcal{I} \prec \pi\mathcal{R}$ and therefore $\rho\mathcal{R}$ is an adjunct of $(\pi - \rho\pi)\mathcal{R} + \pi\mathcal{I}$ in $\pi\mathcal{R}$ (Clark et al., 2006) . Hence, we get $\rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I} + (\pi - \rho\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I} \equiv \pi\mathcal{R}/\pi\mathcal{I}$ and $(\rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}) \sqcap ((\pi - \rho\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}) \prec \rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$. To see $(\rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}) \sqcap ((\pi - \rho\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}) \prec \rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$, assume $(\rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I} \sqcap (\pi - \rho\pi)\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}) + \mathcal{K}/\pi\mathcal{I} \equiv \rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$ for some $\mathcal{K}/\pi\mathcal{I} \sqsubseteq \rho\mathcal{R}/\pi\mathcal{I}$. So, we get $\pi\mathcal{I} + (\rho\mathcal{R} \sqcap ((\pi - \rho\pi)\mathcal{R} + \pi\mathcal{I}))/\pi\mathcal{I} + \mathcal{K}/\pi\mathcal{I} \equiv \rho\mathcal{R}/\pi\mathcal{I}$, insurance that $\mathcal{K} \equiv \rho\mathcal{R}$ (because $\rho\mathcal{R} \sqcap ((\pi - \rho\pi)\mathcal{R} + \pi\mathcal{I}) \prec \rho\mathcal{R}$). Note that $\rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$ is an adjunct in $\pi\mathcal{R}/\pi\mathcal{I}$. But $\pi\mathcal{R}/\pi\mathcal{I}$ is an ascending module, hence $\rho\mathcal{R} + \pi\mathcal{I}/\pi\mathcal{I}$ is a DS of $\pi\mathcal{R}/\pi\mathcal{I}$.

(2) ⇒ (1) each 'cyclic module in $\mathcal{R}$ is isomorphic to $\pi\mathcal{R}/\pi\mathcal{I}$ for $\pi\iota\mathcal{R}$ and ideal $\mathcal{I}$ in $\Phi(\mathcal{R})$ of $\mathcal{R}$ where $\mathcal{R}$ is quasi-flawless. Next, it follows that $\pi\mathcal{R}$

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is an ascending module and $\pi\mathcal{R}/\pi\mathcal{I}$ is amply-adjunct'. Assume that $\mathcal{A}/\pi\mathcal{I} \sqsubseteq \pi\mathcal{R}/\pi\mathcal{I}$. Then $\mathcal{A}/\pi\mathcal{I}$ has an adjunct $\mathcal{B}/\pi\mathcal{I}$ and $\mathcal{B}/\pi\mathcal{I}$ has adjunct $\mathcal{X}/\pi\mathcal{I} \sqsubseteq \mathcal{A}/\pi\mathcal{I}$ in $\pi\mathcal{R}/\pi\mathcal{I}$. when $\pi\mathcal{R}$ is ascending, write $\mathcal{X} \equiv \rho\mathcal{R} + \mathcal{S}$, for some $\rho^2 \equiv \rho\iota\pi\mathcal{R}$ and $\mathcal{S} \prec \pi\mathcal{R}$. So, utilizing the minimality of $\mathcal{X}/\pi\mathcal{I}$, we glean that $\mathcal{X} \equiv \rho\mathcal{R} + \pi\mathcal{I}$. By hypothesis, $\pi\mathcal{R}/\pi\mathcal{I} \equiv \mathcal{X}/\pi\mathcal{I} \boxplus \mathcal{X}'/\pi\mathcal{I}$ for some suitable DS $\mathcal{X}'/\pi\mathcal{I} \sqsubseteq \pi\mathcal{R}/\pi\mathcal{I}$. Since $\mathcal{A}/\pi\mathcal{I} \sqcap \mathcal{B}/\pi\mathcal{I} \prec \pi\mathcal{R}/\pi\mathcal{I}$, we have $\mathcal{A}/\pi\mathcal{I} \sqcap \mathcal{X}'/\pi\mathcal{I} \prec \pi\mathcal{R}/\pi\mathcal{I}$ and hence the summand $\mathcal{X}'/\pi\mathcal{I}$ is an adjunct of $\mathcal{A}/\pi\mathcal{I}$. This shows that $\pi\mathcal{R}/\pi\mathcal{I}$ is an ascending module.

(2) ⇔ (3) we can conclude the proof from (Nguyen et al., 2017).

The following illustration demonstrates the fact of existing $\mathcal{R}$ provided that each cyclic right $\mathcal{R}$-module is $\boxplus$-adjunct, but not each cyclic right $\mathcal{R}$-module is ascending.

Example 3.7. suppose that $\mathbb{K}$ is a field. Let $\mathcal{R} \equiv \left\{\begin{pmatrix} \xi & \mu & \nu \\ 0 & \psi & \omega \\ 0 & 0 & \zeta \end{pmatrix} : \xi, \psi, \zeta, \mu, \nu, \omega\iota\mathbb{K}\right\}. \Rightarrow \Phi(\mathcal{R}) \equiv \left\{\begin{pmatrix} 0 & \mu & \nu \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix} : \mu, \nu, \omega\iota\mathbb{K}\right\}$ and $\mathcal{R}/\Phi(\mathcal{R}) \approx \mathbb{K} \times \mathbb{K} \times \mathbb{K}$ is semisimple. So, $\Phi(\mathcal{R})^3 \equiv 0$. Therefore, $\mathcal{R}$ is a quasi-flawless ring. assuming a right ideal $\mathcal{I} \equiv \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} : \alpha\iota\mathbb{K}\right\} \sqsubseteq \Phi(\mathcal{R})$. Consider $\rho \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\rho^2 \equiv \rho\iota\mathcal{R}$, and $\rho\mathcal{R} + \mathcal{I} \equiv \left\{\begin{pmatrix} 0 & \alpha & \beta \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \delta\iota\mathbb{K}\right\}$. Then $\rho\mathcal{R} + \mathcal{I}/\mathcal{I}$ not a summand of $\mathcal{R}/\mathcal{I}$, because $\gamma \equiv \begin{pmatrix} 0 & 1 & \delta \\ 0 & 1 & \delta \\ 0 & 0 & 0 \end{pmatrix}$ in $\rho\mathcal{R} + \mathcal{I}$, for some $\delta_\iota\mathbb{K}, \gamma\mathcal{I} \nsubseteq \mathcal{I}$. However, for $\gamma' \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \omega \\ 0 & 0 & 0 \end{pmatrix}$ in $\rho\mathcal{R} + \Phi(\mathcal{R})$, for some $\omega\iota\mathbb{K}; \gamma'\mathcal{R} + \mathcal{I}/\mathcal{I}$ is a non-zero DS of $\rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I}$ in $\mathcal{R}/\mathcal{I}$ with $\mathcal{I}\gamma' \sqsubseteq \mathcal{I}$ and $(1 - \gamma')\mathcal{R} + \mathcal{I}/\mathcal{I} \sqcap \rho\mathcal{R} + \Phi(\mathcal{R})/\mathcal{I} \prec \mathcal{R}/\mathcal{I}$. Furthermore, it is easy to confirm $\mathcal{R}$ verifies third condition of theorem3.2.

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The following two examples explain that $\exists$ rings provided that each cyclic right $\mathcal{R}$-module is $\boxplus$-adjunct, but not each cyclic left $\mathcal{R}$-module is $\boxplus$-adjunct.

Example 3.8. suppose that $\mathcal{R} \equiv \begin{pmatrix} \mathbb{I}_4 & \mathbb{I}_2 \\ 0 & \mathbb{I}_2 \end{pmatrix}$. Leads to $\mathcal{R}$ be quasi-flawless and each right ideal in the Jacobson radical $\Phi(\mathcal{R})$ of $\mathcal{R}$ is an ideal [6]. By (Nguyen et al., 2017), each cyclic right $\mathcal{R}$-module is ascending. Let the left ideal $\mathcal{I} \equiv \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\right\} \sqsubseteq \Phi(\mathcal{R}) \equiv \begin{pmatrix} 2\mathbb{I}_4 & \mathbb{I}_2 \\ 0 & 0 \end{pmatrix}$. Let $\rho \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathcal{R}\rho + \Phi(\mathcal{R}) \equiv \begin{pmatrix} 2\mathbb{I}_4 & \mathbb{I}_2 \\ 0 & \mathbb{I}_2 \end{pmatrix}$. We verify that $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I}$ does not contain 'any non-zero DS of $\mathcal{R}/\mathcal{I}$ as a left $\mathcal{R}$-module. Suppose that $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I}$ has a non-zero DS of $\mathcal{R}/\mathcal{I}$. therefore, utilizing (Nguyen et al., 2017), $\exists\ \gamma^2 \equiv \gamma \iota \mathcal{R}$ such that $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I} \equiv \mathcal{R}\gamma + \mathcal{I}/\mathcal{I} \boxplus (\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I} \sqcap \mathcal{R}(1-\gamma) + \mathcal{I})/\mathcal{I}$ and $\mathcal{I}\gamma \sqsubseteq \mathcal{I}$. As $\gamma \iota \mathcal{R}\rho + \Phi(\mathcal{R})$, γ must be of the form $\begin{pmatrix} 0 & \beta \\ 0 & 1 \end{pmatrix}$ for some $\beta \iota \mathbb{I}_2$. However, $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\gamma \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \mathcal{I}$. Therefore, $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I}$ has not any non-zero DS of $\mathcal{R}/\mathcal{I}$. Hence, $\mathcal{R}/\mathcal{I}$ does not represent a $\boxplus$-adjunct like a left $\mathcal{R}$-module'.

Example 3.9. suppose $\mathbb{D}$ is a division ring. Let $\mathcal{R} \equiv \left\{\begin{pmatrix} \xi & \mu & \nu \\ 0 & \xi & \omega \\ 0 & 0 & \psi \end{pmatrix} : \xi, \psi, \mu, \nu, \omega \iota \mathbb{D}\right\}$. So, $\Phi(\mathcal{R})$ involves all matrices in $\mathcal{R}$ with zero diagonal and also $\Phi(\mathcal{R})^3 \equiv 0$. similarly, $\mathcal{R}/\Phi(\mathcal{R}) \approx \mathbb{D} \times \mathbb{D}$ is semi-simple and each right ideal belongs to $\Phi(\mathcal{R})$ is an ideal. Hence each cyclic right $\mathcal{R}$-module is ascending from (Nguyen et al., 2017) . For non-zero elements σ, τ of $\mathbb{D}$, suppose $\mathcal{I} \equiv \left\{\begin{pmatrix} 0 & \mu\sigma & \nu\tau \\ 0 & 0 & \mu\tau \\ 0 & 0 & 0 \end{pmatrix} : \mu, \nu \iota \mathbb{D}\right\} \sqsubseteq \Phi(\mathcal{R}) \equiv \left\{\begin{pmatrix} 0 & \mathbb{D} & \mathbb{D} \\ 0 & 0 & \mathbb{D} \\ 0 & 0 & 0 \end{pmatrix}\right\}$. Suppose that $\rho \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \iota \mathcal{R}$. So, $\mathcal{R}\rho +$

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$\Phi(\mathcal{R}) \equiv \left\{\begin{pmatrix} 0 & \mathbb{D} & \mathbb{D} \\ 0 & 0 & \mathbb{D} \\ 0 & 0 & \mathbb{D} \end{pmatrix}\right\}$. We show that $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I}$ is left $\mathcal{R}$-module has not any non-zero DS of $\mathcal{R}/\mathcal{I}$. Let $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I}$ has a non-zero DS of $\mathcal{R}/\mathcal{I}$. Therefore, from (Nguyen et al., 2017), $\exists \gamma^2 \equiv \gamma \iota \mathcal{R}$ provided that $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I} \equiv \mathcal{R}\gamma + \mathcal{I}/\mathcal{I} \boxplus (\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I} \sqcap \mathcal{R}(1-\gamma) + \mathcal{I})/\mathcal{I}$. So, $\mathcal{I}\gamma \sqsubseteq \mathcal{I}$. As $\gamma \iota \mathcal{R}\rho + \Phi(\mathcal{R})$, we should get $\gamma \equiv \begin{pmatrix} 0 & 0 & \nu \\ 0 & 0 & \omega \\ 0 & 0 & 1 \end{pmatrix}$ for some $\nu, \omega \iota \mathbb{D}$. However, $\begin{pmatrix} 0 & \sigma & \tau \\ 0 & 0 & \tau \\ 0 & 0 & 0 \end{pmatrix}\gamma \equiv \begin{pmatrix} 0 & 0 & \sigma\omega + \tau \\ 0 & 0 & \tau \\ 0 & 0 & 0 \end{pmatrix} \notin \mathcal{I}$. Therefore, $\mathcal{R}\rho + \Phi(\mathcal{R})/\mathcal{I}$ has not any non-zero DS of $\mathcal{R}/\mathcal{I}$. Hence, $\mathcal{R}/\mathcal{I}$ is not a $\boxplus$-adjunct as a left $\mathcal{R}$-module.

4 Conclusions

This paper has investigated the structure of rings over which every cyclic right module is $\boxplus$-adjunct. We have shown that such rings are necessarily quasi-flawless, and we provided a precise characterization of them in Theorem 3.2. Furthermore, we demonstrated that this class of rings lies strictly between those rings whose cyclic modules are ascending and those whose cyclic modules are adjunct. Illustrative examples were constructed to show that this property is not left-right symmetric.

References

Anderson, F. W., & Fuller, K. R. (1992). *Rings and categories of modules* (2nd ed.). Springer-Verlag.

Baba, Y., & Oshiro, K. (2009). *Classical Artinian rings and related topics*. World Scientific Publishing.

Clark, J., Lomp, C., Vanaja, N., & Wisbauer, R. (2006). *Lifting modules: Supplements and projectivity in module theory*. Birkhäuser Verlag.

Harmanci, A., Keskin, D., & Smith, P. F. (1999). On $\oplus$-supplemented modules. *Acta Mathematica Hungarica, 83*(2), 161-169.

Ibrahim, Y., & Yousif, M. (2015). Rad-projective covers. In *Noncommutative rings and their applications* (Contemporary Mathematics, Vol. 634, pp. 175-188). American Mathematical Society.

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P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.138) 2026,pp.879-890

Koehler, A. (1970). Rings for which every cyclic module is quasi-projective. *Mathematische Annalen, 189*, 311-316.
Kosan, T., & Keskin, D. (2007). H-supplemented duo modules. *Journal of Algebra and Its Applications, 6*(6), 965-971.
Lam, T. Y. (1998). *Lectures on modules and rings*. Springer-Verlag.
Mohamed, S. H., & Müller, B. J. (1990). *Continuous and discrete modules* (London Mathematical Society Lecture Note Series). Cambridge University Press.
Nguyen, X. H., & Zhou, Y. (2018). Rings whose cyclic modules are lifting and $\oplus$ supplemented. *Communications in Algebra, 46*(11), 4918-4927.
Nguyen, X. H., Yousif, M. F., & Zhou, Y. (2017). Rings whose cyclics are D3-modules. *Journal of Algebra and Its Applications, 16*(10), Article 1750184.
Wang, Y., & Wu, D. (2012). On H-supplemented modules. *Communications in Algebra, 40*(10), 3679-3689.
Wisbauer, R. (1991). *Foundations of module and ring theory*. Gordon and Breach Science Publishers.

الحلقات التي تكون وحداتها الدورية وحدات مُكمّلة (مُلحقة)

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مستخلص البحث:
تقدم هذه الورقة البحثية وتدرس بشكل منهجي فئة الوحدات المُلحقة، وهي تعميم للوحدات المُلحقة المُعرَّفة بوجود مُكمِّلات مباشرة للمجموعات ذات خاصية تقاطع صغير مُحدَّدة. نُثبت شروطًا كافية لكي ترث المجموعات المباشرة ووحدات العوامل للوحدات المُلحقة هذه الخاصية. ينصب تركيزنا الأساسي على توصيف الحلقات التي تكون فيها كل وحدة دورية يمنى وحدات مُلحقة. نُبرهن على أن هذه الحلقات شبه خالية من العيوب بالضرورة، ونُقدم توصيفًا دقيقًا داخليًا باستخدام نظرية الحلقات. علاوة على ذلك، نميز هذه الفئة الجديدة من خلال تقديم أمثلة توضح أنها تقع بين الحلقات التي تكون وحداتها الدورية تصاعدية وتلك التي تكون وحداتها الدورية ملحقة فقط، ونثبت أن هذه الخاصية ليست متناظرة من اليسار إلى اليمين.

الكلمات المفتاحية: ملحق $\boxplus$، وحدات ملحقة، جذري، تصاعدي، شبه مثالي.