



P- ISSN: 3007-3650 E- ISSN: 2706-8536

Journal of the College of Basic Education

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
Vol.32 (No.138) 2026, pp.1019-1026

Restrict Nearly Quasi Prime (R.S.Q.P) Submodule

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Abstract:

In this work, we developed a new definition based on a primary sub-module. This sub-module served as the foundation upon which we constructed and analyzed this definition. We examined some of its characteristics, as well as characteristics closely related to this work, and established the conditions and fundamental principles governing the relationships between this definition and previous ones. We also presented some relationships and inverse relationships between a set of previous definitions and the current definition, providing counter examples of relationships that cannot be achieved. Furthermore, we explored the relationship between two sub-modules. We also clarified the relationships between sub-modules when one is contained within another and examined the appropriate conditions that must be imposed on the modules to achieve specific characteristics. We used an important and fundamental condition in module theory, which is the distribution condition, to achieve some important results in the current work. We explained that the intersection operation between submodules is not necessarily closed or guaranteed.

Keywords: Restrict nearly quasi prime submodule; restrict nearly prime submodule.

Introduction:

An $\mathcal{R}$ -module $\mathcal{H}$ appropriate sub-module $\mathcal{A}$ is referred to as a prime whenever $xy \in \mathcal{D}$, for $x \in \mathcal{R}$, $y \in \mathcal{H}$ implies that either $x \in \mathcal{D}$ or $y\mathcal{H} \subseteq \mathcal{D}$ [(Abdul-Razak, pp. 18-20,1999)]. After that, many researchers created new types of submodules until the researcher khudhayer O. kadem by introducing a new type of important relationship which was in the form a proper submodule $\mathcal{D}$ of an $\mathcal{R}$ -module $\mathcal{H}$ is restrict nearly prime submodule of $\mathcal{H}$ whenever $xy \in \mathcal{D}$, for $x \in \mathcal{R}$, $y \in \mathcal{H}$ implies that either $x \in \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ or $y\mathcal{H} \subseteq \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ [(Kadem, pp. 7-11, 2024)]. In this study, all modules will be nonzero unital and all rings will be assumed to be commutative with a nonzero identity. this study examines a few properties of restrict nearly quasi prime submodules as new generalization of quasi prime submodules and strong form of approximately quasi prime submodules. Furthermore, we provide characterizations of restrict nearly quasi prime submodules in class of multiplication modules.

Preliminaries:

Definition1:

If $\mathcal{D} \subseteq \mathcal{H}$, $\mathcal{D}$ is named a prime submodule if $xy \in \mathcal{D}$ for $x \in \mathcal{R}$, $y \in \mathcal{H}$ implies that $y \in \mathcal{D}$ or $x \in [\mathcal{D} :_R \mathcal{H}]$ (Dauns,1978, pp. 156-181).

If $\mathcal{D} \subseteq \mathcal{H}$, $\mathcal{D}$ is named a quasi-prime submodule, if $xyz \in \mathcal{D}$, for $x, y \in \mathcal{R}$, $z \in \mathcal{H}$ implies that $xz \in \mathcal{D}$ or $yz \in \mathcal{D}$ (Abdul-Razak, pp. 18-20,1999).

Recall that a subset S of a ring $\mathcal{R}$ is named multipliecatively closed subsetof $\mathcal{R}$ if $1 \in S$ & $xy \in S$, $\forall x, y \in S$. If $\mathcal{D}$ is a submodule of an $\mathcal{R}$ -modul $\mathcal{H}$ & S is a multipliecatively closed subsetof $\mathcal{R}$, then $\mathcal{D}_S = \{x \in \mathcal{H} : \exists t \in S \text{ (s.t) } tx \in \mathcal{D}\}$ is a submodule of $\mathcal{H}$ & $\mathcal{D} \subseteq \mathcal{D}_S$ (Behrens, pp. 123-127,1971).

Proposition2: (Lam, Example 12.5, p. 242, 2012)

A submodule $\mathcal{D}$ of an $\mathcal{R}$ -modul $\mathcal{H}$ is maximal & essential iff $Soc(\mathcal{H}) \subseteq \mathcal{D}$.

Proposition3: (Kasch, Lemma 2.3.15, 1982)

Let $\mathfrak{D}$, $\mathfrak{D}_1$ & $\mathfrak{D}_2$ be submodules of an $\mathcal{R}$ -modul $\mathcal{H}$, with $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$. Then $(\mathfrak{D} + \mathfrak{D}_1) \cap \mathfrak{D}_2$ equal $(\mathfrak{D} \cap \mathfrak{D}_2) + \mathfrak{D}_1$ equal $(\mathfrak{D} \cap \mathfrak{D}_2) + (\mathfrak{D}_1 \cap \mathfrak{D}_2)$.

Recall that an $\mathcal{R}$ -module $\mathcal{H}$ is a distributive if for any submodules $\mathfrak{D}$, $\mathfrak{D}_1$ & $\mathfrak{D}_2$ of $\mathcal{H}$, $\mathfrak{D} \cap (\mathfrak{D}_1 + \mathfrak{D}_2) = (\mathfrak{D} \cap \mathfrak{D}_1) + (\mathfrak{D} \cap \mathfrak{D}_2)$ or $(\mathfrak{D} + \mathfrak{D}_1) \cap \mathfrak{D}_2 = (\mathfrak{D} \cap \mathfrak{D}_2) + (\mathfrak{D}_1 \cap \mathfrak{D}_2)$ (Hart, pp. 8-9, 1974)

Proposition4: (Payman, p. 12, 2005)

If $\mathfrak{D}$ is coclosed submodule of an $\mathcal{R}$ -module $\mathcal{H}$, then $J(\mathfrak{D}) = \mathfrak{D} \cap J(\mathcal{H})$.

Proposition5: (Anderson & Fuller, Corollary 9.9, p.119, 2012)

Let $\mathfrak{D}$ be a submodule of an $\mathcal{R}$ -modul $\mathcal{H}$, then $\text{soc}(\mathfrak{D}) = \mathfrak{D} \cap \text{Soc}(\mathcal{H})$.

Recall that if $\mathfrak{D} \hookrightarrow \mathcal{H}$, $\mathfrak{D}$ is named a supplemented iff $\mathcal{H} = \mathfrak{D}_1 + \mathfrak{D}$ & $\mathfrak{D}_1 \cap \mathfrak{D}$ is a small submodule of $\mathfrak{D}$, $\forall$ submodule $\mathfrak{D}_1$ of $\mathcal{H}$ (Wisbauer, 1991, pp. 87-91)

Proposition6: (Payman, Proposition 1.2.14, p. 12, 2005)

If $\mathfrak{D}$ be supplemented submodule of $\mathcal{H}$, then $\mathfrak{D}$ is coclosed.

Recall that if $\mathfrak{D} \hookrightarrow \mathcal{H}$, $\mathfrak{D}$ is named a Restrict Nearly Prime (simply RNP) submodule of $\mathcal{H}$, if $xy \in \mathfrak{D}$, for $x \in \mathcal{R}$, $y \in \mathcal{H}$ implies that $y \in \mathfrak{D} + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))$ or $x \in \left[\left(\mathfrak{D} + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H})) \right) :_{\mathcal{R}} \mathcal{H} \right]$ (Kadem, pp. 7-11, 2024).

Proposition7: (Kasch, Lemma 2.3.15, 1982)

A non-zero submodule $\mathfrak{D}$ of an $\mathcal{R}$ -module $\mathcal{H}$ is said to be maximal submodule of $\mathcal{H}$ if $\mathfrak{D} \subseteq \beta \subseteq \mathcal{H}$, then $\beta = \mathcal{H}$ where β is a submodule of $\mathcal{H}$ [24].

Recall that a non-zero submodule $\mathfrak{D}$ of an $\mathcal{R}$ -module $\mathcal{H}$ is called an essential in $\mathcal{H}$ if $\mathfrak{D} \cap \beta \neq (0)$ for each non-zero submodule β of $\mathcal{H}$ (Kasch, Lemma 2.3.15, 1982).

Proposition8: A submodule $\mathfrak{D}$ of an $\mathcal{R}$ -module $\mathcal{H}$ is named coclosed submodule if for any submodule β of $\mathcal{H}$ contained in $\mathfrak{D}$ we have $\mathfrak{D} / \beta$ is small in $\mathcal{H} / \beta$ implies that $\mathfrak{D} = \beta$ [(Anderson, F. W., & Fuller, K. R. (2012))].

Restrict Nearly Quasi Prime Submodule:

Definition1:

If $\mathfrak{D} \subseteq \mathcal{H}$, $\mathfrak{D}$ is named (Restrict Nearly Quasi Prime simply RNQP) submodule of $\mathcal{H}$, if $xyz \in \mathfrak{D}$, for $x, y \in \mathcal{R}$, $z \in \mathcal{H}$ implies that either $xz \in \mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ or $yz \in \left[\left(\mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})) \right) :_{\mathcal{R}} \mathcal{H} \right]$.

And ideal of a ring is a RNQP ideal of $\mathcal{R}$ iff I is a RNQP submodule of $\mathcal{R}$.

Proposition2:

Every Quasi Prime submodule of an $\mathcal{R}$ -module $\mathcal{H}$ is a RNQP submodule.

Proof

If $\mathfrak{D}$ is a Quasi Prime submodule of an $\mathcal{R}$ -module $\mathcal{H}$ and let $xyz \in \mathfrak{D}$, for $x, y \in \mathcal{R}$, $z \in \mathcal{H}$, thus either $xz \in \mathfrak{D} \subseteq \mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ or $yz \in \mathfrak{D} \subseteq \mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$. That is either $xz \in \mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ or $yz \in \mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$. Hence $\mathfrak{D}$ is a RNQP submodule of $\mathcal{H}$.

The converse is not true, in general, consider the following example.

Example3:

Consider the Z -modul Z_{12} , the submodule $\mathfrak{D} = \langle \bar{0} \rangle$ is not a Quasi Prime submodule of Z_{12} since, $3.8.\bar{2} \in \mathfrak{D}$, but $3.\bar{2} = \bar{6} \notin \mathfrak{D} = \langle \bar{0} \rangle$ & $8.\bar{2} = \bar{16} \notin \mathfrak{D}$. But $\mathfrak{D}$ is a RNQP of Z_{12} since, $Soc(Z_{12}) = \langle \bar{2} \rangle$, and $J(Z_{12}) = \langle \bar{6} \rangle$. Hence $Soc(Z_{12}) \cap J(Z_{12}) = \langle \bar{2} \rangle \cap \langle \bar{6} \rangle = \langle \bar{6} \rangle$, that is for $xyz \in \mathfrak{D}$, $x, y \in \mathcal{R}$, $z \in Z_{12}$, implies that either $xz \in \mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})) = \langle \bar{0} \rangle + \langle \bar{6} \rangle = \langle \bar{6} \rangle$ or $yz \in \mathfrak{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})) = \langle \bar{0} \rangle + \langle \bar{6} \rangle = \langle \bar{6} \rangle$.

Proposition4:

Every prime submodule of an $\mathcal{R}$ -module $\mathcal{H}$ is a RNQP submodule.

If $\mathfrak{D}$ be a prime submodule of an $\mathcal{R}$ -module $\mathcal{H}$, then $\mathfrak{D}$ is Quasi Prime. Therefore, by proposition (2) $\mathfrak{D}$ is a RNQP submodule of $\mathcal{H}$.

The converse is not true, in general, consider the following example.

Example5:

Let $\mathcal{H} = Z_4$, $\mathcal{R} = Z$, and the submodule $\mathfrak{D} = \langle \bar{0} \rangle$ is not prime submodule of Z_4 since, $2.\bar{2} = \bar{0} \in \mathfrak{D}$, for $2 \in Z$, $\bar{2} \in Z_4$, but $\bar{2} \notin \mathfrak{D}$ and $2 \notin$

$[\mathcal{D}:_Z Z_4] = 4Z$. While $\mathcal{D}$ is a RNQP submodule of Z_4 because $Soc(Z_4) = \langle \bar{2} \rangle$, and $J(Z_4) = \langle \bar{2} \rangle$, thus $Soc(Z_4) \cap J(Z_4) = \langle \bar{2} \rangle \cap \langle \bar{2} \rangle = \langle \bar{2} \rangle$, whenever $xyz \in \mathcal{D}$, $x, y \in \mathcal{R}$, $z \in Z_4$, implies that either $xz \in \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})) = \langle \bar{0} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$ or $yz \in \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})) = \langle \bar{0} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$.

Proposition6:

If $\mathcal{D} \vdash \mathcal{H}$, with $[\mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})):_R \mathcal{H}]$ is a prime ideal of $\mathcal{R}$. Then $\mathcal{D}(\mathcal{S}) \subseteq \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ iff $\mathcal{D}$ is a RNQP submodule of $\mathcal{H}$, for all multiplicatively closed subset $\mathcal{S}$ of $\mathcal{R}$

Proof

$\Rightarrow$ Let $xyz \in \mathcal{D}$, for $x, y \in \mathcal{R}$, $z \in \mathcal{H}$ with $yz \notin \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ & $xz \notin [\mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})):_R \mathcal{H}]$. Since $[\mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})):_R \mathcal{H}]$ is prime ideal of $\mathcal{R}$ then, $[\mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})):_R \mathcal{H}] \cap \mathcal{S} = \phi$ (define the multiplicatively closed). And we have $yz \notin \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$, implies that $yz \notin \mathcal{D}(\mathcal{S})$, (from define the multiplicatively closed $\{x \in \mathcal{H} : \exists yz \in \mathcal{S} \text{ such that } xyz \in \mathcal{D}\}$ then, $\forall yz \notin \mathcal{S} \Rightarrow xyz \notin \mathcal{D}$) thus, $xyz \notin \mathcal{D}$ contradiction. Thus, either $yz \in \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ or $xz \in [\mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})):_R \mathcal{H}]$. Therefore $\mathcal{D}$ is a RNQP submodule of $\mathcal{H}$.

$\Leftarrow$ If $yz \in \mathcal{D}(\mathcal{S}) \hookrightarrow (1)$, then there exists $r \in \mathcal{S}$ such that $ryz \in \mathcal{D}$. But $\mathcal{D}$ is a RNQP of $\mathcal{H} \Rightarrow$ either $yz \in \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$ or $rz\mathcal{H} \subseteq \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$. If $rz\mathcal{H} \subseteq \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$, implies that $rz \in [\mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})):_R \mathcal{H}] \cap \mathcal{S} = \phi$, contradiction. Thus $yz \in \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H})) \hookrightarrow (2)$. From (1) & (2) $\mathcal{D}(\mathcal{S}) \subseteq \mathcal{D} + (Soc(\mathcal{H}) \cap J(\mathcal{H}))$.

Proposition7: Every RNP submodule of an $\mathcal{R}$ -module $\mathcal{H}$ is an RNQP submodule of $\mathcal{H}$.

Proof

Clear that.

The converse is not true, in general, consider the following example.

Example8:

Consider the Z -module $Z \oplus Z$, the submodule $\mathfrak{D} = 11Z \oplus (0)$ is an RNQP submodule. Now, for some positive integers x_1, x_2 & $(y_1, y_2) \in Z \oplus Z$ such that $x_1 x_2 (y_1, y_2) \in \mathfrak{D}$. Thus $(x_1 x_2 y_1, x_1 x_2 y_2) \in \mathfrak{D}$, then $x_1 x_2 y_1 \in 11Z$ & $x_1 x_2 y_2 = 0$. It follows that $y_2 = 0$ & $[x_1 y_1 \in 11Z$ or $x_2 y_1 \in 11Z$ (because $\mathfrak{D}$ is an RNQP submodule)]. Therefore either $x_1 (y_1, y_2) \in \mathfrak{D} \subseteq \mathfrak{D} + (\text{Soc}(Z \oplus Z) \cap J(Z \oplus Z))$ or $x_2 (y_1, y_2) \in \mathfrak{D} \subseteq \mathfrak{D} + (\text{Soc}(Z \oplus Z) \cap J(Z \oplus Z))$. But $\mathfrak{D}$ is not an RNP submodule of the Z -modul $Z \oplus Z$, since $11(1, 0) \in \mathfrak{D}$, but $(1, 0) \notin \mathfrak{D} + \text{Soc}(Z \oplus Z) \cap J(Z \oplus Z)$ and $11 \notin [11Z \oplus (0) + \text{Soc}(Z \oplus Z) \cap J(Z \oplus Z) :_Z Z \oplus Z] = (0)$.

Remark9:

The intersection of any two RNQP submodules of an $\mathcal{R}$ -module $\mathcal{H}$ need not to be an RNQP submodule of $\mathcal{H}$, the following example explains that: Let $\mathcal{H} = Z_{60}$, $\mathcal{R} = Z$, the submodules $\langle \bar{2} \rangle$ & $\langle \bar{3} \rangle$ are RNQP submodules of Z_{60} (because $\langle \bar{2} \rangle$ & $\langle \bar{3} \rangle$ are prime $\Rightarrow \langle \bar{2} \rangle$ & $\langle \bar{3} \rangle$ are RNQP by proposition 4]. But $\langle \bar{2} \rangle \cap \langle \bar{3} \rangle = \langle \bar{6} \rangle$ is not RNQP submodule of Z_{60} , for example 2.3. $\bar{1} \in \langle \bar{6} \rangle$, for $\bar{1} \in Z_{60}$ & $2, 3 \in Z$, but neither $2. \bar{1} \in \langle \bar{6} \rangle + (\text{Soc}(Z_{60}) \cap J(Z_{60})) = \langle \bar{6} \rangle + (\langle \bar{2} \rangle \cap \langle \bar{30} \rangle) = \langle \bar{6} \rangle + \langle \bar{30} \rangle = \langle \bar{6} \rangle$, nor $3. \bar{1} \in \langle \bar{6} \rangle + \langle \bar{30} \rangle = \langle \bar{6} \rangle$.

We have proved in Example the previous that the intersection of two RNQP submodules is not a RNQP submodule, so the following series of results provides some necessary conditions for this to happen.

Proposition10:

Let $\mathcal{M}$ be an $\mathcal{R}$ -module & $\mathfrak{D}_1, \mathfrak{D}_2$ be two submodules of $\mathcal{H}$ such that $\mathfrak{D}_2 \not\subseteq \mathfrak{D}_1$ & $(\text{Soc}(\mathcal{H}) \cap J(\mathcal{H})) \subseteq \mathfrak{D}_2$. If $\mathfrak{D}_1, \mathfrak{D}_2$ are RNQP submodules of $\mathcal{H}$ then $\mathfrak{D}_1 \cap \mathfrak{D}_2$ is a RNQP submodule of $\mathcal{H}$.

Proof

Now, $\mathfrak{D}_2 \not\subseteq \mathfrak{D}_1$, then $\mathfrak{D}_1 \cap \mathfrak{D}_2 \subsetneq \mathfrak{D}_2$. Let $xyz \in \mathfrak{D}_1 \cap \mathfrak{D}_2$, for $x, y \in \mathcal{R}, z \in \mathcal{H}$, then $xyz \in \mathfrak{D}_1$ & $xyz \in \mathfrak{D}_2$. But $\mathfrak{D}_1$ & $\mathfrak{D}_2$ are RNQP submodules of $\mathcal{H}$, It follows that $xz \in [\mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))] \cap [\mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))]$ or $yz \in [\mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))] \cap [\mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))]$ Now we have $(\text{Soc}(\mathcal{H}) \cap J(\mathcal{H})) \subseteq \mathfrak{D}_2$, then $\mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H})) = \mathfrak{D}_2$, thus $xz \in [\mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))] \cap$

$\mathfrak{D}_2$ or $yz \in [\mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))] \cap \mathfrak{D}_2$. By modular law hence either $xz \in (\mathfrak{D}_1 \cap \mathfrak{D}_2) + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))$ or $yz \in (\mathfrak{D}_1 \cap \mathfrak{D}_2) + (\text{Soc}(\mathcal{H}) \cap J(\mathcal{H}))$. Therefore $(\mathfrak{D}_1 \cap \mathfrak{D}_2)$ is a RNQP submodule of $\mathcal{H}$.

Proposition11:

Let $\mathcal{H}$ be an $\mathcal{R}$ -module with either $\mathfrak{D}_1$ or $\mathfrak{D}_2$ are maximal essential submodule of $\mathcal{H}$ & $\mathfrak{D}_2 \not\subseteq \mathfrak{D}_1$. If $\mathfrak{D}_1$ & $\mathfrak{D}_2$ are RNQP submodules of $\mathcal{H}$ then $(\mathfrak{D}_1 \cap \mathfrak{D}_2)$ is an RNQP submodule of $\mathcal{H}$.

Proof

Let $xyz \in (\mathfrak{D}_1 \cap \mathfrak{D}_2)$, for $x, y \in \mathcal{R}$, $z \in \mathcal{H}$, then $xyz \in \mathfrak{D}_1$ & $xyz \in \mathfrak{D}_2$. But $\mathfrak{D}_1$ & $\mathfrak{D}_2$ are RNQP submodules of $\mathcal{H}$, then $xz \in \mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))$ or $yz \in \mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))$ & $xz \in \mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))$ or $yz \in \mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))$. Thus $xz \in (\mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))) \cap (\mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H})))$ or $yz \in (\mathfrak{D}_1 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))) \cap (\mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H})))$. Since $\mathfrak{D}_1$ or $\mathfrak{D}_2$ are maximal essential in $\mathcal{H}$ then by proposition (7) $\text{Soc}(\mathcal{H}) \subseteq \mathfrak{D}_1$ or $\text{Soc}(\mathcal{H}) \subseteq \mathfrak{D}_2$ & $\mathfrak{D}_1$ or $\mathfrak{D}_2$ are maximal then $J(\mathcal{H}) \subseteq \mathfrak{D}_1$ or $J(\mathcal{H}) \subseteq \mathfrak{D}_2$. Let $\mathfrak{D}_1$ is a maximalessential in $\mathcal{H}$ so $\text{Soc}(\mathcal{H}) \subseteq \mathfrak{D}_1$ & $J(\mathcal{H}) \subseteq \mathfrak{D}_1$ & we have $(\text{Soc}(\mathcal{H}) + J(\mathcal{H})) \subseteq \mathfrak{D}_1$, it follows that $xz \in \mathfrak{D}_1 \cap (\mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H})))$ or $yz \in \mathfrak{D}_1 \cap (\mathfrak{D}_2 + (\text{Soc}(\mathcal{H}) + J(\mathcal{H})))$. Therefore, by Modular law we have either $xz \in (\mathfrak{D}_1 \cap \mathfrak{D}_2) + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))$ or $yz \in (\mathfrak{D}_1 \cap \mathfrak{D}_2) + (\text{Soc}(\mathcal{H}) + J(\mathcal{H}))$. Therefore $(\mathfrak{D}_1 \cap \mathfrak{D}_2)$ is an RNQP submodule of $\mathcal{H}$. Similar arguments follows if $\mathfrak{D}_2$ is maximal essential.

Corollary12:

Let $\mathcal{H}$ be distributive $\mathcal{R}$ -module & $\mathfrak{D}_1, \mathfrak{D}_2$ are submodules of $\mathcal{H}$ with $\mathfrak{D}_1 \not\subseteq \mathfrak{D}_2$, $\mathfrak{D}_1$ is supplemented of $\mathcal{H}$. If $\mathfrak{D}_2$ is RNQP submodule of $\mathcal{H}$, then $\mathfrak{D}_1 \cap \mathfrak{D}_2$ is RNQP of $\mathfrak{D}_1$.

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Salahalkilaby1993@gmail.comakram293348@gmail.comMustafa.ghanam01@gmail.com**مستخلص البحث:**

في هذا العمل، طورنا تعريفاً جديداً قائماً على وحدة فرعية أساسية. شكلت هذه الوحدة الفرعية الأساس الذي بنينا عليه هذا التعريف وحللناه. درسنا بعض خصائصها، بالإضافة إلى خصائص وثيقة الصلة بهذا العمل، ووضعنا الشروط والمبادئ الأساسية التي تحكم العلاقات بين هذه التعريفات. كما عرضنا بعض العلاقات، والعلاقات العكسية، بين مجموعة من التعريفات السابقة والتعريف الحالي، مع تقديم أمثلة عكسية للعلاقات التي لا يمكن تحقيقها. علاوة على ذلك، استكشفنا العلاقة بين وحدتين فرعيتين. كما أوضحنا العلاقات بين الوحدات الفرعية عندما تكون إحداها مضمنة في الأخرى، ودرسنا الشروط المناسبة التي يجب فرضها على الوحدات لتحقيق خصائص محددة. استخدمنا شرطاً مهماً وأساسياً على الوحدة، وهو شرط التوزيع، لتحقيق بعض النتائج المهمة في هذا العمل. أوضحنا أن عملية التقاطع بين الوحدات الفرعية ليست بالضرورة متحققة.

الكلمات المفتاحية: تقييد الوحدة الفرعية شبه الأولية، تقييد الوحدة الفرعية الأولية تقريباً

ملاحظة: هل البحث مستل من رسالة ماجستير او اطروحة دكتوراه؟ كلا