



Homo-Derivations in Gamma Rings

Ahmed Abdulkareem Hadi

Mathematics Science, Education Ministry, Baghdad directorate of
education, Al-Rusafa 1st

Corresponding Author email: ahmed08hadi@gmail.com

Abstract

The term *homo-derivation* has been used for different, non-equivalent classes of mappings. This ambiguity becomes more serious for Γ -rings, where a generalized derivation can be mistaken for a homo-derivation. In this paper, a single convention is fixed: an additive mapping $f: M \rightarrow M$ on a Γ -ring is called a homo-derivation when $f(x\alpha y) = f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. The main result proves that f is a homo-derivation if and only if $\phi_f = \text{id}_M + f$ is a Γ -ring endomorphism. This correspondence yields a complete and economical structural description. In particular, homo-derivations form a monoid under $f \star g = f + g + f \circ g$, not under ordinary addition; the invertible elements correspond exactly to Γ -ring automorphisms. Complete proofs are given for restriction to invariant ideals, passage to quotient Γ -rings, the description of kernels and fixed points, a valid commutativity criterion, and preservation of graded and filtered structures. Explicit examples on matrix and polynomial Γ -rings demonstrate nonzero homo-derivations and show why several tempting closure statements are false. The results provide a consistent foundation for the study of homo-derivations on Γ -rings without relying on unsupported commutativity or external-application claims.

Keywords: Γ -ring; homo-derivation; endomorphism; invariant ideal; graded Γ -ring; filtered Γ -ring.



1. Introduction

Nobusawa introduced Γ -rings as a generalization of associative rings in which multiplication is mediated by elements of a second additive group (Nobusawa, 1964). Barnes subsequently adopted a weaker and widely used axiom system (Barnes, 1966). Since then, the ideal theory and structural theory of Γ -rings have been developed in directions analogous to ordinary ring theory; representative early contributions include the work of Luh on simple Γ -rings (Luh, 1969) and Kyuno on prime Γ -rings (Kyuno, 1978). Derivations and generalized derivations were later transferred to this setting. For example, Çeven and Öztürk studied generalized and Jordan generalized derivations on Γ -rings (Çeven & Öztürk, 2004), while Dey and Paul investigated generalized derivations on prime Γ -rings under additional interchange assumptions (Dey & Paul, 2013).

The expression *homo-derivation* is not uniform in the literature. Under the definition attributed to El Sofy, an additive map f on a ring satisfies $f(xy) = f(x)y + xf(y) + f(x)f(y)$. Another convention requires the same map to be both a homomorphism and a derivation (Ashraf & Haetinger, 2006), (Rotman & Rotman, 2009). Gselmann and Kiss emphasized that these definitions are different and studied their associated functional equations (Gselmann & Kiss, 2020). The first identity has an especially transparent interpretation: adding the identity map converts f into a multiplicative map. In the additive setting this produces a ring homomorphism. That observation is the starting point of the present Γ -ring formulation. A further distinction is essential. An identity of the form $D(x\alpha y) = D(x)\alpha y + xad(y)$, where d is a derivation, defines a generalized derivation; it does not define a homo-derivation in the sense used here. Generalized derivations have their own established theory in rings and Γ -rings (Çeven & Öztürk, 2004), (Dey & Paul, 2013), (Hvala, 1998). Mixing the two identities changes the class of maps and invalidates conclusions based on either theory.

The purpose of this paper is therefore focused. First, it extends the El Sofy-type identity consistently to Γ -rings and proves the exact correspondence $f \leftrightarrow \text{id}_M + f$ with Γ -ring endomorphisms. Second, it transports composition of endomorphisms to a natural associative

operation on homo-derivations and characterizes its units. Third, it establishes rigorous consequences for ideals, quotients, centrality, grading, and filtration, supported by nontrivial examples and counterexamples. These results replace broad structural assertions by statements that follow directly from explicit hypotheses. The Γ -ring formulation, the transported monoid law, and the invariant-structure consequences give a unified framework for subsequent work.

2. Preliminaries and Terminology

2.1 Γ -rings: Basic Concepts

Definition 2.1. Let $(M, +)$ and $(\Gamma, +)$ be additive abelian groups. Following the Barnes convention (Barnes, 1966), M is a Γ -ring if there is a map

$$M \times \Gamma \times M \rightarrow M, \quad (x, \alpha, y) \mapsto x\alpha y,$$

such that, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

$$(x + y)\alpha z = x\alpha z + y\alpha z,$$

$$x(\alpha + \beta)y = x\alpha y + x\beta y,$$

$$x\alpha(y + z) = x\alpha y + x\alpha z,$$

$$(x\alpha y)\beta z = x\alpha(y\beta z).$$

A Γ -ring endomorphism is an additive map $\phi: M \rightarrow M$ satisfying

$$\phi(x\alpha y) = \phi(x)\alpha\phi(y)$$

for all $x, y \in M$ and $\alpha \in \Gamma$. The set of Γ -ring endomorphisms is denoted by $\text{End}_\Gamma(M)$. A bijective Γ -ring endomorphism is a Γ -ring automorphism, and the automorphism group is denoted by $\text{Aut}_\Gamma(M)$.

Definition 2.2. An additive subgroup $I \leq M$ is a two-sided Γ -ideal if

$$M\Gamma I \subseteq I \quad \text{and} \quad I\Gamma M \subseteq I,$$

where $A\Gamma B$ denotes the additive subgroup generated by all $a\alpha b$ with $a \in A$, $b \in B$, and $\alpha \in \Gamma$. The quotient M/I becomes a Γ -ring under

$$(x + I)\alpha(y + I) = x\alpha y + I.$$

Definition 2.3. For $x, y \in M$ and $\alpha \in \Gamma$, define the α -commutator by

$$[x, y]_\alpha = x\alpha y - y\alpha x.$$

The center of M is

$$Z(M) = \{z \in M: [z, x]_\alpha = 0 \text{ for every } x \in M \text{ and } \alpha \in \Gamma\}.$$

The Γ -ring M is commutative if $[x, y]_\alpha = 0$ for every $x, y \in M$ and $\alpha \in \Gamma$.

2.2 Four distinct mapping classes

Definition 2.4. An additive map $f: M \rightarrow M$ is a homo-derivation if

$$f(x\alpha y) = f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y) \quad (\text{HD})$$

for all $x, y \in M$ and $\alpha \in \Gamma$. The set of all such maps is denoted by $\text{HDer}_\Gamma(M)$.

The extra quadratic term $f(x)\alpha f(y)$ is essential. If it is omitted, the identity becomes the Leibniz rule. If it is replaced by $x\alpha g(y)$ for a second map g , the result is a generalized-derivation-type identity rather than (HD) (Paul & Nazneen, 2013).

3. The Endomorphism Correspondence

The next theorem is the central result. It extends the ordinary-ring observation in (Gselmann & Kiss, 2020) to Γ -rings.

Theorem 3.1 (Endomorphism correspondence). Let M be a Γ -ring and let $f: M \rightarrow M$ be additive. Define $\phi_f = \text{id}_M + f$, $\phi_f(x) = x + f(x)$. Then $f \in \text{HDer}_\Gamma(M)$ if and only if $\phi_f \in \text{End}_\Gamma(M)$.

Proof. Assume first that f satisfies (HD). Since f is additive, ϕ_f is additive. For $x, y \in M$ and $\alpha \in \Gamma$, $\phi_f(x\alpha y) = x\alpha y + f(x\alpha y)$,

$$\phi_f(x\alpha y) = x\alpha y + f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y),$$

$$\phi_f(x\alpha y) = (x + f(x))\alpha(y + f(y)),$$

$$\phi_f(x\alpha y) = \phi_f(x)\alpha\phi_f(y).$$

Thus ϕ_f is a Γ -ring endomorphism. Conversely, suppose that ϕ_f is a Γ -ring endomorphism. Then $x\alpha y + f(x\alpha y) = \phi_f(x\alpha y) = \phi_f(x)\alpha\phi_f(y)$.

Using biadditivity of the Γ -product,

$$\phi_f(x)\alpha\phi_f(y) = (x + f(x))\alpha(y + f(y))$$

$$= x\alpha y + f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y).$$

Cancellation of $x\alpha y$ gives (HD). Therefore f is a homo-derivation.

Corollary 3.1. The map $\theta: \text{HDer}_\Gamma(M) \rightarrow \text{End}_\Gamma(M)$, $\theta(f) = \text{id}_M + f$, is a bijection, with inverse $\theta^{-1}(\phi) = \phi - \text{id}_M$.

Proof. Theorem 3.1 proves that both assignments are well defined. They are inverse to one another by direct substitution.

Corollary 3.2 (Comparison with derivations). Let $d: M \rightarrow M$ be a derivation. Then d is also a homo-derivation if and only if $d(x)\alpha d(y) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Proof. The derivation identity and (HD) have identical linear terms. Hence, they hold simultaneously exactly when the quadratic term $d(x)ad(y)$ vanishes for every x, y , and α .

This corollary shows that derivations are not automatically homo-derivations. Likewise, a generalized derivation is not a homo-derivation merely because both identities contain Leibniz-type terms.

4. Examples and Counterexamples

4.1. Universal examples

Example 4.1. The zero map is a homo-derivation because $\text{id}_M + 0 = \text{id}_M$ is a Γ -ring endomorphism.

Example 4.2. The map $f = -\text{id}_M$ is a homo-derivation on every Γ -ring because $\text{id}_M + f = 0$, and the zero map is a Γ -ring endomorphism. Directly,

$$\begin{aligned} f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y) &= (-x)\alpha y + x\alpha(-y) + (-x)\alpha(-y), \\ f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y) &= -x\alpha y - x\alpha y + x\alpha y, \\ f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y) &= -x\alpha y = f(x\alpha y). \end{aligned}$$

More generally, every Γ -ring endomorphism ϕ produces the homo-derivation $f = \phi - \text{id}_M$.

4.2. A noncommutative matrix example

Example 4.3. Let F be a field, let $M = M_2(F)$, and let $\Gamma = (F, +)$. Define

$$X\alpha Y = \alpha XY$$

for $X, Y \in M$ and $\alpha \in F$. This makes M a Γ -ring. Fix $q \in F^\times$ and put

$$P = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$

The inner automorphism $\phi_P(X) = PXP^{-1}$ is a Γ -ring automorphism because

$$\phi_P(X\alpha Y) = P(\alpha XY)P^{-1} = \alpha(PXP^{-1})(PY P^{-1}) = \phi_P(X)\alpha\phi_P(Y).$$

Therefore $f_P = \phi_P - \text{id}_M$ is a homo-derivation. If

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\text{Then } f_P(X) = \begin{pmatrix} 0 & (q-1)b \\ (q^{-1}-1)c & 0 \end{pmatrix}.$$

For $q \neq 1$, this is a nonzero homo-derivation on a noncommutative Γ -ring. It is neither the zero map nor $-\text{id}_M$.

4.3. An idempotent endomorphism example

Example 4.4. Let $M = F[t]$, $\Gamma = (F, +)$, and define $p\alpha q = \alpha pq$. Evaluation at zero, $\phi(p) = p(0)$, is a Γ -ring endomorphism. Hence $f(p) = p(0) - p$ is a homo-derivation. Since $\phi^2 = \phi$, one has $f \circ f = -f$. This example will reappear in the monoid structure below.

4.4. Ordinary sums need not be homo-derivations

Example 4.5. Regard $\mathbb{Z}$ as a Γ -ring with $\Gamma = \mathbb{Z}$ and $x\alpha y = \alpha xy$. By Example 4.2, $f = -\text{id}_{\mathbb{Z}}$ is a homo-derivation. However, $f + f = -2\text{id}_{\mathbb{Z}}$ is not a homo-derivation, because $\text{id}_{\mathbb{Z}} + (f + f) = -\text{id}_{\mathbb{Z}}$ is not multiplicative: for example, it sends $1 \cdot 1$ to -1 but sends 1 and 1 to elements whose product is 1. Thus, neither closure under ordinary addition nor closure under arbitrary integer multiples is valid in general.

5. The Natural Monoid of Homo-Derivations

The endomorphism correspondence identifies the correct replacement for ordinary addition.

Definition 5.1. For $f, g \in \text{HDer}_{\Gamma}(M)$, define $f \star g = f + g + f \circ g$.

Theorem 5.1. The set $\text{HDer}_{\Gamma}(M)$ is a monoid under $\star$, with identity element 0. Moreover, $\theta: (\text{HDer}_{\Gamma}(M), \star) \rightarrow (\text{End}_{\Gamma}(M), \circ)$ is a monoid isomorphism.

Proof. For $f, g \in \text{HDer}_{\Gamma}(M)$,

$$\begin{aligned}\theta(f) \circ \theta(g) &= (\text{id}_M + f) \circ (\text{id}_M + g), \\ \theta(f) \circ \theta(g) &= \text{id}_M + f + g + f \circ g, \\ \theta(f) \circ \theta(g) &= \theta(f \star g).\end{aligned}$$

The composition of Γ -ring endomorphisms is again a Γ -ring endomorphism. By Theorem 3.1, $f \star g$ is therefore a homo-derivation. Associativity of $\star$ follows from associativity of composition and injectivity of θ . Finally,

$$f \star 0 = f = 0 \star f,$$

so 0 is the identity. The bijectivity of θ was proved in Corollary 3.1, and the displayed identity shows that it is a monoid isomorphism.

Corollary 5.1. A homo-derivation f is invertible in the monoid $(\text{HDer}_{\Gamma}(M), \star)$ if and only if $\text{id}_M + f$ is a Γ -ring automorphism. In that case,

$$f^{\star-1} = (\text{id}_M + f)^{-1} - \text{id}_M.$$

Proof. Units are preserved by monoid isomorphisms. Applying Theorem 5.1 gives the stated characterization and formula.

Corollary 5.2. For $f, g \in \text{HDer}_\Gamma(M)$, $f \star g = g \star f$ if and only if $f \circ g = g \circ f$.

Proof. Since addition of maps is commutative,

$$f \star g - g \star f = f \circ g - g \circ f.$$

The conclusion follows.

Corollary 5.3. A homo-derivation f is idempotent under $\star$ if and only if $f \circ f = -f$.

Equivalently, $\text{id}_M + f$ is an idempotent Γ -ring endomorphism.

Proof. The equality $f \star f = f$ is equivalent to $2f + f \circ f = f$, hence to $f \circ f = -f$. Under θ , the same statement is equivalent to $(\text{id}_M + f)^2 = \text{id}_M + f$.

Example 4.4 supplies a nontrivial $\star$ -idempotent homo-derivation. Theorem 5.1 also explains why ordinary composition $f \circ g$ need not be a homo-derivation: the term that corresponds to composition of endomorphisms is $f + g + f \circ g$, not $f \circ g$ alone.

6. Ideals, Quotients, Kernels, and Fixed Points

Theorem 6.1 (Invariant ideals and quotients). Let $f \in \text{HDer}_\Gamma(M)$, let $\phi_f = \text{id}_M + f$, and let I be a two-sided Γ -ideal of M . The following are equivalent:

1. $f(I) \subseteq I$;
2. $\phi_f(I) \subseteq I$.

If these conditions hold, then:

- (a) the restriction $f|_I: I \rightarrow I$ is a homo-derivation of the Γ -ring I ;
- (b) the map $\bar{f}: M/I \rightarrow M/I$, $\bar{f}(x + I) = f(x) + I$, is a well-defined homo-derivation; and
- (c) $\text{id}_{M/I} + \bar{f}$ is the endomorphism induced by ϕ_f on M/I .

Proof. For $i \in I$, $\phi_f(i) = i + f(i)$.

Because $i \in I$, it follows that $f(i) \in I$ if and only if $\phi_f(i) \in I$. This proves the equivalence. If I is invariant, then $\phi_f|_I$ is an additive map from I to itself. For $i, j \in I$ and $\alpha \in \Gamma$, $\phi_f|_I(i\alpha j) = \phi_f(i)\alpha\phi_f(j)$,

so $\phi_f|_I$ is a Γ -ring endomorphism of I . Theorem 3.1 shows that $f|_I$ is a homo-derivation.

To define the quotient map, suppose that $x + I = x' + I$. Then $x - x' \in I$, and invariance gives $f(x - x') \in I$. Hence $f(x) + I = f(x') + I$, so $\bar{f}$ is well defined. Likewise, $\bar{\phi}_f(x + I) = \phi_f(x) + I$

is a well-defined Γ -ring endomorphism of M/I . Finally,

$$\bar{\phi}_f(x + I) = x + f(x) + I = (\text{id}_{M/I} + \bar{f})(x + I).$$

Theorem 3.1 applied to M/I proves that $\bar{f}$ is a homo-derivation.

Proposition 6.1. Let $f \in \text{HDer}_\Gamma(M)$ and $\phi_f = \text{id}_M + f$. Then:

1. $\ker \phi_f$ is a two-sided Γ -ideal of M ;
2. $\text{Im} \phi_f$ is a Γ -subring of M ;
3. $\ker f = \text{Fix}(\phi_f) = \{x \in M : \phi_f(x) = x\}$.

In contrast, $\ker f$ need not be a Γ -ideal.

Proof. The kernel of any Γ -ring homomorphism is an additive subgroup. If $a \in \ker \phi_f$, $m \in M$, and $\alpha \in \Gamma$, then

$$\phi_f(m\alpha a) = \phi_f(m)\alpha\phi_f(a) = 0, \quad \phi_f(a\alpha m) = \phi_f(a)\alpha\phi_f(m) = 0.$$

Thus $\ker \phi_f$ is a two-sided Γ -ideal. Closure of $\text{Im} \phi_f$ under addition and Γ -multiplication follows from additivity and multiplicativity of ϕ_f . Finally,

$$f(x) = 0 \Leftrightarrow x + f(x) = x \Leftrightarrow \phi_f(x) = x.$$

Fixed-point sets of endomorphisms are additive subgroups, but they are not generally ideals. For example, in Example 4.3, $\ker f_P$ is the fixed subring of an inner automorphism; for a suitable P , this fixed subring is not a two-sided ideal of $M_2(F)$.

7. Centralizing and Commuting Maps

The terminology in this section concerns a mapping and the elements on which it acts; it does not itself imply commutativity of the Γ -ring.

Definition 7.1. An additive map $f: M \rightarrow M$ is centralizing if

$$[f(x), x]_\alpha \in Z(M)$$

for every $x \in M$ and $\alpha \in \Gamma$. It is commuting if $[f(x), x]_\alpha = 0$

for every $x \in M$ and $\alpha \in \Gamma$. Every commuting map is centralizing, but the converse need not hold.

Proposition 7.1. Let $f \in \text{HDer}_\Gamma(M)$ and $\phi_f = \text{id}_M + f$. Then

$$[\phi_f(x), x]_\alpha = [f(x), x]_\alpha$$

for every $x \in M$ and $\alpha \in \Gamma$. Consequently, f is centralizing (respectively, commuting) if and only if ϕ_f is centralizing (respectively, commuting).

Proof. By additivity of the commutator in its first variable,

$$[\phi_f(x), x]_\alpha = [x + f(x), x]_\alpha = [x, x]_\alpha + [f(x), x]_\alpha = [f(x), x]_\alpha.$$

The two equivalences follow immediately. The next criterion is a valid replacement for commutativity assertions that use only a vague centrality condition.

Theorem 7.1 (Injective commutative-image criterion). Let $f \in \text{HDer}_\Gamma(M)$ and suppose that $\phi_f = \text{id}_M + f$ is injective. If the image $\phi_f(M)$ is a commutative Γ -subring, then M is commutative. In particular, the conclusion holds if $\phi_f(M) \subseteq Z(M)$.

Proof. For $x, y \in M$ and $\alpha \in \Gamma$, commutativity of $\phi_f(M)$ gives

$$\begin{aligned}\phi_f([x, y]_\alpha) &= \phi_f(x\alpha y - y\alpha x), \\ \phi_f([x, y]_\alpha) &= \phi_f(x)\alpha\phi_f(y) - \phi_f(y)\alpha\phi_f(x), \\ \phi_f([x, y]_\alpha) &= 0.\end{aligned}$$

Injectivity of ϕ_f implies $[x, y]_\alpha = 0$. Since x, y , and α were arbitrary, M is commutative.

Neither primeness nor semiprimeness is needed for Theorem 7.1. Conversely, centralizing behavior alone does not justify a commutativity conclusion without additional hypotheses; Proposition 7.1 only translates the condition to the associated endomorphism (Vukman, 1989).

8. Graded and Filtered Γ -rings

Grading and filtration are related but different structures. A grading is a direct-sum decomposition into homogeneous components, whereas a filtration is a nested family of additive subgroups. The distinction prevents the unsupported conclusion that every homo-derivation must shift or preserve degrees.

Definition 8.1. Let $(G, +)$ be an additive monoid. A Γ -ring M is G -graded if

$$M = \bigoplus_{r \in G} M_r$$

as an additive group and

$$M_r \Gamma M_s \subseteq M_{r+s}$$

for all $r, s \in G$. An element of M_r is homogeneous of degree r .

Definition 8.2. Let $(G, +, \leq)$ be an ordered additive monoid. A Γ -ring M is G -filtered if it has additive subgroups $(F_r M)_{r \in G}$ such that

$$r \leq s \Rightarrow F_r M \subseteq F_s M, \quad \bigcup_{r \in G} F_r M = M,$$

and $(F_r M) \Gamma (F_s M) \subseteq F_{r+s} M$ for all $r, s \in G$.

Thus, a grading records a unique finite sum of homogeneous components, while a filtration records increasing levels and does not require a direct sum.

Theorem 8.1 (Preservation criterion). Let $f \in \text{HDer}_\Gamma(M)$ and $\phi_f = \text{id}_M + f$.

(a) If $M = \bigoplus_{r \in G} M_r$ is graded, then

$$f(M_r) \subseteq M_r \text{ for every } r \in G$$

if and only if

$$\phi_f(M_r) \subseteq M_r \text{ for every } r \in G.$$

(b) If M is filtered by $(F_r M)_{r \in G}$, then

$$f(F_r M) \subseteq F_r M \text{ for every } r \in G$$

if and only if

$$\phi_f(F_r M) \subseteq F_r M \text{ for every } r \in G.$$

Proof. For (a), the identity map preserves every M_r . If $f(M_r) \subseteq M_r$, then $\phi_f(x) = x + f(x) \in M_r$ for $x \in M_r$. Conversely, if $\phi_f(M_r) \subseteq M_r$, then $f(x) = \phi_f(x) - x \in M_r$. The proof of (b) is identical because the identity map preserves every filtration level $F_r M$.

The theorem is a criterion, not an automatic property: a homo-derivation preserves a grading or filtration exactly when its associated endomorphism does (Brešar, 1991). No unconditional degree-preservation or degree-shift statement follows from (HD).

9. Discussion

The endomorphism correspondence resolves the principal conceptual difficulty. The quadratic term in (HD) is precisely the cross term needed to make $\text{id}_M + f$ multiplicative. Once this is recognized, several structural facts become immediate but nontrivial in their consequences (Sapanci & Nakajima, 1997).

First, homo-derivations are differences between Γ -ring endomorphisms and the identity. This gives systematic examples and avoids ad hoc verification of (HD). Second, the natural algebraic operation is $\star$, transported from endomorphism composition. Example 4.5 proves that ordinary addition and integer scaling are not closed operations, while Theorem 5.1 gives the correct closure law. Third, the notions of kernel and image must be attached to the correct map: $\ker(\text{id}_M + f)$ is an ideal, whereas $\ker f$ is a fixed-point subgroup and need not be an ideal.

The same viewpoint clarifies structural hypotheses. Invariance of an ideal under f is exactly invariance under $\text{id}_M + f$, making restrictions and quotients rigorous. Centralizing and commuting conditions can be translated to the associated endomorphism, but they do not by themselves imply commutativity. The injective commutative-image criterion in Theorem 7.1 states sufficient hypotheses explicitly. Finally, preservation of a grading or filtration is conditional on preservation by the associated endomorphism; it is not a consequence of the homo-derivation identity alone.

These results remain within Γ -ring theory. They do not identify Γ -rings with Γ -spaces from algebraic topology, and they do not assert coding-theoretic, cryptographic, operator-theoretic, or decision-theoretic applications without a separate construction and proof.

10. Conclusion

A consistent theory of homo-derivations on Γ -rings has been developed from the identity $f(x\alpha y) = f(x)\alpha y + x\alpha f(y) + f(x)\alpha f(y)$. The correspondence $f \mapsto \text{id}_M + f$ identifies these maps exactly with Γ -ring endomorphisms. This correspondence supplies explicit examples, distinguishes homo-derivations from derivations and generalized derivations, and yields the associative monoid law $f \star g = f + g + f \circ$



g. It also gives complete proofs for invariant ideals, quotient maps, kernels, fixed points, a commutativity criterion, and graded and filtered preservation. Counterexamples show that ordinary sums and integer multiples are not generally homo-derivations. The resulting framework is mathematically self-contained and provides a precise basis for further structural questions about selected classes of Γ -ring endomorphisms and their associated homo-derivations.

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المشتقات المتجانسة في حلقات جاما

احمد عبد الكريم هادي

علوم الرياضيات، وزارة التربية، مديرية التربية ببغداد، الرصافة الأولى

Corresponding Author email: ahmed08hadi@gmail.com

مستخلص البحث:

استُخدم مصطلح المشتق المتجانس في الأدبيات للدلالة على أصناف مختلفة وغير متكافئة من التعيينات. وتزداد هذه المشكلة في حلقات جاما، إذ قد يختلط تعريف المشتق المعمم بتعريف المشتق المتجانس. تعتمد هذه الدراسة تعريفًا واحدًا محددًا: يسمى التعيين الجمعي $f: M \rightarrow M$ على حلقة جاما مشتقًا متجانسًا إذا حقق العلاقة $f(xay) = f(x)ay + xaf(y) + f(x)\alpha f(y)$ لكل $x, y \in M$ و $\alpha \in \Gamma$. وتثبت النتيجة الرئيسية أن f مشتق متجانس إذا وفقط إذا كان التعيين $\phi_f = \text{id}_M + f$ تشاكلاً داخلياً لحلقة جاما. يتيح هذا التقابل وصفاً بنيوياً دقيقاً؛ إذ تكون المشتقات المتجانسة أحاديًا تحت العملية $f \star g = f + g + f \circ g$ وليس تحت الجمع الاعتيادي، وتقابل العناصر القابلة للعكس تشاكلات جاما الذاتية تمامًا. كما تُقدّم براهين كاملة للتقييد على المثاليات الثابتة، والانتقال إلى حلقات القسمة، ووصف النوى ونقاط الثبات، ومعيار صحيح للتبادلية، والمحافظة على البنى المتدرجة والمرشحة. وتوضح أمثلة صريحة على حلقات جاما المصفوفية وكثيرات الحدود وجود مشتقات متجانسة غير صفيرية، كما تبين سبب عدم صحة بعض عبارات الانغلاق المتوقعة.

الكلمات المفتاحية: حلقة جاما، المشتق المتجانس، التشاكل الداخلي، المثالية الثابتة، حلقة جاما المتدرجة، حلقة جاما المرشحة.