Solution Of Riccati Equation Of Fractional Order Using See Transform

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Abstract:
In this paper, a certain type of Riccati equation of arbitrary order have been considered and solved using SEE integral transform in the aid of the definition of Caputo derivative. Some related theorems and properties of SEE transform with selected cases for specific values of SEE parameter n have been presented.

Keywords: SEE transform, Riccati equation, Caputo derivative, Laplace transform.

1. Introduction:

The method of integral transformation considered to be good tool for giving an analytic solution for different types of differential equations of ordinary or fractional types. In this paper a fractional order of Riccati equation for certain type have been considered and solved by the transformation of SEE.[1,2,3]

The generalized Riccati equation takes its name from the Italian mathematician Jacopo Riccati, [1,2]. The Riccati differential equation has a wide range of applications in engineering, including control systems, optimal control and stability analysis. In physics: Riccati equations serve as a link between linear quantum mechanics and different fields of physics, such as cosmology and thermodynamics. They are used in the study of nonlinear physics and have applications in areas like conformal mapping and algebraic geometry, while in mathematical finance: Riccati equations are

used to solve problems related to option pricing, portfolio optimization, and risk management [9,10]

Fractional calculus is still a fruitful field of research in mathematics which deals with integro-differential operators with fractional orders. It has found lot of applications in different fields of real life phenomena, namely, in physics: fractional calculus has recently found many applications[4]. It has been used to model viscoelastic damping in certain types of materials like polymers. It has also been used to study the behavior and long-term memory of physical systems.

In engineering: Fractional derivative has been used in engineering to model and analyze complex systems. It has been used to study the dynamics of electrical transmission lines, control systems, and signal processing.[11]

Overall, fractional calculus provides a powerful tool for modeling and analyzing complex systems and phenomena.

In the subsequent section of this paper, the necessary definitions and theorems were presented. Section 3 of the paper introduced the main result, while the final section included some conclusions.

2. Preliminaries

In this paper, the SEE integral transform introduced by Sadiq, Emad, and Eman in [3] is applied to solve a certain type of Riccati equation of arbitrary order. The Riccati equation is a first-order ordinary differential equation that is quadratic in the unknown function. It has the general form as follows:

$$\epsilon D^{\alpha} f(s) = a(s) f^2(s) + b(s) f(s) + c(s) \qquad (1)$$

with initial condition $f(0) = \lambda$

where D^{α} is the Caputo derivative and ϵ is some positive number.

As a special case if $a(s), b(s)$ and $c(s)$ are constant then (1) reads:

$$\epsilon D^{\alpha} f(s) = a f^2(s) + b f(s) + c \qquad s \in [0,1] \qquad (2)$$

where ϵ, a, b, c are constants and $\epsilon > 0$

Definition (1)[8]

Let $\alpha \in (n-1, n), n \in \mathbb{N}$ Then Caputo derivative of α order is given by :

$$^{C}D_0^{\alpha} f(s) = \frac{1}{\Gamma(n-\alpha)} \int_0^s (s-v)^{n-\alpha-1} f^{(n)}(v) dv \qquad (3)$$

where $f(s)$ is continuous absolutely up to order $n-1$.

In particular, if $\alpha \in (0,1)$ then:

$$^{C}D_0^{\alpha}f(s) = \frac{1}{\Gamma(1-\alpha)}\int_0^s (s-v)^{-\alpha} f'(v)dv \qquad (4)$$

Definition (2) SEE integral transform

Let $T(s)$ be the SEE transform of the function $f(\xi)$ which is a member of the set K defined by:

$$K = \{f(\xi): \exists M, k_1, k_2 > 0. |f(\xi)| < Me^{k_1|\xi|}, \text{if } \xi \in (-1)^i \times [0,\infty)\}$$

then we have [13]:

$$T(s) = \frac{1}{s^n}\int_0^{\infty} f(\xi)e^{-s\xi}d\xi, \qquad n \in \mathbb{Z} \qquad l_1 \le s \le l_2 \qquad (5)$$

Theorem (1) (Linearity)

SEE transform is linear operator, i.e.,

$$T\big(\alpha g(s) + \beta h(s)\big) = \alpha T\big(g(s)\big) + \beta T\big(h(s)\big) \qquad (6)$$

Proof

From properties of integration we have :

$$T\big(\alpha g(s) + \beta h(s)\big) = \frac{1}{s^n}\int_0^{\infty} (\alpha g(\xi) + \beta h(\xi))e^{-s\xi}d\xi$$

$$= \frac{1}{s^n}\left[\int_0^{\infty}\big(\alpha g(\xi)\big)e^{-s\xi}d\xi + \int_0^{\infty}\big(\beta h(\xi)\big)e^{-s\xi}d\xi\right]$$

$$= \frac{1}{s^n}\left[\alpha\int_0^{\infty}\big(g(\xi)\big)e^{-s\xi}d\xi + \beta\int_0^{\infty}\big(h(\xi)\big)e^{-s\xi}d\xi\right]$$

$$= \frac{\alpha}{s^n}\int_0^{\infty}\big(g(\xi)\big)e^{-s\xi}d\xi + \frac{\beta}{s^n}\int_0^{\infty}\big(h(\xi)\big)e^{-s\xi}d\xi$$

$$= \alpha T\big(g(s)\big) + \beta T\big(h(s)\big)$$

Theorem (2)

If $G(v)$ and $H(v)$ are the SEE of respective functions $g(t)$ and $h(t)$ both defined in set K defined by:

$$K = \{f(\xi): \exists M, k_1, k_2 > 0. |f(\xi)| < Me^{k_1|\xi|}, if\ \xi \in (-1)^i \times [0,\infty)\}$$

then the convolution of g and h defined by $g * h$ is the function on $u \ge 0$ given by :

$$(g * h)(x) = \int_0^x g(u)h(x-u)du \qquad (7)$$

Then

$$S\big((g * h)(x)\big) = v^n S\big(g(x)\big)S\big(h(x)\big) \qquad (8)$$

$$= v^n G(v) \cdot H(v)$$

Theorem (3) [3]

For $\mu-1<\alpha\leq\mu,\ \mu\in\mathbb{N}$ and if $T(v)$ is SEE transform of $f(x)$, then SEE transform of Caputo α fractional derivative of function $f(x)$ is

$$S({}^{C}D_{0+}^{\alpha}f(x))=\frac{1}{v^{-\alpha+\mu}}[v^{\mu}T(v)-\frac{1}{v^{n-\mu+1}}f^{\mu-1}(0)-\cdots-\frac{1}{v^{n-1}}f^{\mu-1}(0)-\frac{1}{v^{n}}f^{(\mu-1)}(0)]$$

(9)

3. Solving Riccati equation by using SEE transform

Using SEE transform to solve the equation of Riccati mentioned in equation (2) could be summarized following:

Apply SEE transform to the equation of Riccati:

$$T\left(\epsilon\ {}^{C}D_{0+}^{\alpha}f(x)\right)=T(af^{2}(t)+bf(t)+c)$$

by linearity of SEE transform, we get:

$$T\left(\epsilon\ {}^{C}D_{0+}^{\alpha}f(x)\right)=aT(f^{2}(t))+bTf(t)+T(c))$$

$$T\left(\ {}^{C}D_{0+}^{\alpha}f(x)\right)=\frac{a}{\epsilon}T(f^{2}(t))+\frac{b}{\epsilon}Tf(t)+\frac{c}{\epsilon(n^{2}+1)}\qquad(10)$$

Solving for $T(v)$ to get the transformation of the desired solution and then take T^{-1} to find the exact analytic solution for the Riccati equation (2).

Notes:

1- If $n=-2$ then eq. (9) reads:

$$S({}^{C}D_{0+}^{\alpha}f(x))=\frac{1}{v^{-\alpha+\mu}}[v^{\mu}T(v)-\frac{1}{v^{-1-\mu}}f^{\mu-1}(0)-\cdots-\frac{1}{v^{-3}}f^{\mu-1}(0)-\frac{1}{v^{-2}}f^{(\mu-1)}(0)]$$

Especially in SEE integral transform with $n=-2$ we get from (10):

$$\frac{1}{v^{-\alpha+\mu}}\left[v^{\mu}T(v)-\frac{1}{v^{-1-\mu}}f^{\mu-1}(0)-\cdots-\frac{1}{v^{-3}}f^{\mu-1}(0)-\frac{1}{v^{-2}}f^{(\mu-1)}(0)\right]=\frac{a}{\epsilon}v^{2}T(v)+\frac{b}{\epsilon}T(v)+\frac{c}{5\epsilon}$$

This is equivalent to "Mohanad Transform", [14].

2- If $n=-1$ then eq. (9) reads:

$$S({}^{C}D_{0+}^{\alpha}f(x)) = \frac{1}{v^{-\alpha+\mu}}\left[v^{\mu}T(v) - \frac{1}{v^{-\mu}}f^{\mu-1}(0) - \cdots - \frac{1}{v^{-2}}f^{\mu-1}(0) - vf^{(\mu-1)}(0)\right]$$

Especially in SEE integral transform with $n=-1$ we get from (10):

$$\frac{1}{v^{-\alpha+\mu}}\left[v^{\mu}T(v) - \frac{1}{v^{-\mu}}f^{\mu-1}(0) - \cdots - \frac{1}{v^{-2}}f^{\mu-1}(0) - vf^{(\mu-1)}(0)\right] = \frac{a}{\epsilon}v^{2}T(v) + \frac{b}{\epsilon}T(v) + \frac{c}{2}$$

This is equivalent to "Mahgoub Transform", [5].

3- If $n=0$ then eq. (9) reads:

$$S({}^{C}D_{0}^{\alpha}f(x)) = \frac{1}{v^{-\alpha+\mu}}\left[v^{\mu}T(v) - \frac{1}{v^{1-\mu}}f^{\mu-1}(0) - \cdots - vf^{\mu-1}(0) - f^{(\mu-1)}(0)\right]$$

Especially in SEE integral transform with $n=0$ we get from (10):

$$\frac{1}{v^{-\alpha+\mu}}\left[v^{\mu}T(v) - \frac{1}{v^{1-\mu}}f^{\mu-1}(0) - \cdots - vf^{\mu-1}(0) - f^{(\mu-1)}(0)\right] = \frac{a}{\epsilon}v^{2}T(v) + \frac{b}{\epsilon}T(v) + \frac{c}{\epsilon}$$

This is equivalent to "Laplace Transform", [6].

4- If $n=1$ then eq. (9) reads

$$S({}^{C}D_{0+}^{\alpha}f(x)) = \frac{1}{v^{-\alpha+m}}\left[v^{m}T(v) - \frac{1}{v^{2-m}}f^{m-1}(0) - \cdots - f^{m-1}(0) - \frac{1}{v}f^{(m-1)}(0)\right]$$

Especially in SEE integral transform with $n=1$ we get from (10):

$$\frac{1}{v^{-\alpha+\mu}}\left[v^{\mu}T(v)-\frac{1}{v^{2-\mu}}f^{\mu-1}(0)-\cdots-f^{\mu-1}(0)-\frac{1}{v}f^{(\mu-1)}(0)\right]=\frac{a}{\epsilon}v^{2}T(v)+\frac{b}{\epsilon}T(v)+\frac{c}{2\epsilon}$$

This is equivalent to "Aboodh Transform", [7].

4. Results

In order to declare the applicability of our approach for solving a certain type of nonlinear Riccati equation of fractional order, we present following examples

Example 1.

This example consider a nonlinear equation of Riccati type, which reads:[12]

$$D^{\alpha}f(t)=1-f^{2}(t),\quad t\geq 0,\ \alpha\in(0,1] \tag{11}$$

With initial condition: $f(0)=0$

To solve (11) by SEE transform, we first apply the SEE transform to the problem to get:

$$T\left(D^{\alpha}f(t)\right)=T\left(1-f^{2}(t)\right)$$
$$=T(1)-T(f^{2}(t))$$
$$=\frac{1}{\epsilon(n^{2}+1)}-T(f^{2}(t))$$
$$=\frac{1}{\epsilon(n^{2}+1)}-v^{n}T^{2}(f(t))$$

Using theorem 3 to determine the fractional derivative , we get:

$$\frac{1}{v^{-\alpha+\mu}}v^{\mu}T(v)\ =\frac{1}{\epsilon(n^{2}+1)}-v^{n}T^{2}(v)$$

Solving for $T(v)$ to get:

$$T(v)=\frac{1}{2v^{-n}}\left(-v^{\alpha}\mp\sqrt{(v^{2\alpha}+\frac{4v^{n}}{\epsilon n^{2}+\epsilon}}\right)$$
$$=-v^{\alpha-n}\pm\frac{1}{2}\sqrt{v^{2(\alpha-n)}+\frac{4}{\epsilon(n^{2}+1)}}$$

or

$$T_1(v) = -v^{\alpha-n} + \frac{1}{2}\sqrt{v^{2(\alpha-n)} + \frac{4}{\epsilon(n^2+1)}}$$

$$T_2(v) = -v^{\alpha-n} - \frac{1}{2}\sqrt{v^{2(\alpha-n)} + \frac{4}{\epsilon(n^2+1)}}$$

to find solution for (11), we take the inverse See transform for $T_{1,2}(v)$ as following

$$S^{-1}(T_1) = S^{-1}(-v^{\alpha-n} + \frac{1}{2}\sqrt{v^{2(\alpha-n)} + \frac{4}{\epsilon(n^2+1)}})$$

$$S^{-1}(T_2) = S^{-1}(-v^{\alpha-n} - \frac{1}{2}\sqrt{v^{2(\alpha-n)} + \frac{4}{\epsilon(n^2+1)}})$$

After simple calculations we can determine the specific solution for $\alpha = 1$ by following:

$$f(t) = \frac{e^{2t}-1}{e^{2t}+1}$$

Example 2. Consider the following nonlinear fractional Riccati equation :[10]

$$D^{\alpha}f(t) = 1 + 2f(t) - f^2(t), \quad t \geq 0\,, \quad \alpha \in (0,1] \qquad (12)$$

with initial condition: $f(0) = 0$

Again to solve (12) by SEE transform, we apply the SEE transform to the problem to get:

$$T\left(D^{\alpha}f(t)\right) = T\left(1 + 2f(t) - f^2(t)\right)$$

$$\frac{1}{v^{-\alpha+\mu}}v^{\mu}T(v) = \mathrm{T}(1) + 2\mathrm{T}(\mathrm{v}) - \mathrm{T}^2(\mathrm{v})$$

$$\frac{1}{v^{-\alpha+\mu}}v^{\mu}T(v) = \frac{1}{\mathrm{n}^2+1} + 2\,T(v) - v^{n}T^2(v)$$

Solving for $T(v)$ to get following

$$T(v) = \frac{1}{2v^n}((2 - v^\alpha) \pm \sqrt{(v^\alpha - 2)^2 - \frac{4v^n}{n^2+1}})$$

to find solution for (12), we take the inverse SEE transform for $T_{1,2}(v)$ as following

$$S^{-1}(T_1) = S^{-1}(\frac{1}{2v^n}((2 - v^\alpha) + \sqrt{(v^\alpha - 2)^2 - \frac{4v^n}{n^2+1}})$$

$$S^{-1}(T_2) = S^{-1}(\frac{1}{2v^n}((2 - v^\alpha) - \sqrt{(v^\alpha - 2)^2 - \frac{4v^n}{n^2+1}})$$

After simple calculations we can determine the specific solution for $\alpha = 1$ by following:

$$f(t) = 1 + \sqrt{2}\tanh(\sqrt{2}\,t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$$

Conclusion:

Through this paper, a specific type of fractional order Riccati differential equations of was studied, due to their importance in many applied problems, as mentioned in the introduction with a reference, and they were solved using the SEE integral transformation, the most general of many integral transformations, depending on n, which belongs to integers such as Transformation of Laplace, Aboudh, Mahjoub, Muhanad, Rohit, Dinesh, Gupta and Emad-Sara. We would like to mention that each of the aforementioned integral transformations has no preference over the other transformation, whether only in the nature of using this transformation and according to the problem taken, some of which we find to be of high efficiency with the fewest algebraic steps, for example, in cryptography, the problem of pharmacological concentration, nuclear physics, radiation, and others, as mentioned in previous papers.

The analytic solution for any fractional order differential equation is quite important in the area of fractional calculus because analytic solutions

provide a deeper theoretical understanding of the behavior of fractional order differential equations. They allow us to study the properties of the solutions and the underlying physical phenomena that they represent.

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حل معادلة ريكاتي من الرتبة الكسرية باستخدام تحويل SEE

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مستخلص البحث:
في هذا البحث، تمّت دراسة نوع مُحدد من معادلات ريكاتي ذات الرتبة الكسورية وحلّها باستخدام التحويل التكاملي SEE بالاستعانة بتعريف مشتقة كابوتو. كما تمّ عرض بعض النظريات والخصائص ذات الصلة بتحويل SEE مع حالات مختارة لقيم مُحددة لمعامل n.

الكلمات المفتاحية : SEE transform, Riccati equation, Caputo derivative, Laplace transform.

ملاحظة : هل البحث مستل من رسالة ماجستير او اطروحة دكتوراه ؟ نعم : --- كلا : كلا