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On The Growth of Dirichlet Series with Latent Gaps

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Abstract:

The growth theory of vector-valued Dirichlet series is extended to two complex variables in this paper, with an emphasis on the enhanced characterization of the lower order. In order to guarantee the stability of where I create a generalized class A by adding an exceptional set of finite measures. Importantly, I show that the lower order obtained from the maximum modulus does not always match the one obtained from the series. Coefficients for functions with structural gaps. My findings give adequate and necessary conditions for the failure of this conventional equality. I provide a robust methodology for gap series analysis by separating growth irregularities within the exceptional set. A major gap in multivariable growth theory under nonregular settings is filled by this study.

Keywords: Dirichlet series, lower order, maximum modulus, gap Dirichlet series, refined class.

Introduction:

Consider $f(\xi,\eta)=\sum_{n,m=0}^{\infty} c_{n,m}\exp(\alpha_n\xi+\beta_m\eta)$ (1)

be an entire function that the Dirichlet series defines [1], where $\xi=\mu_1+i\nu_1$ and $\eta=\mu_2+i\nu_2$, such that ξ and $\eta \in ¢$ the space of complex numbers and μ_1,μ_2,ν_1 and $\nu_2 \in R$,

with $0<\alpha_n\le\alpha_{n+1}$ and $0<\beta_m\le\beta_{m+1}$, where $n,m\in N$. And further [2,3]

$$\lim_{n,m\to\infty}\sup\frac{\log(n+m)}{\alpha_n+\beta_m}=D<\infty,\ \lim_{n,m\to\infty}\sup\frac{\log|c_{n,m}|}{\alpha_n+\beta_m}=-\infty \quad (2)$$

Following Daoud [2], the maximum modulus $M(\mu_1,\mu_2)$ and maximum term for $f(\xi,\eta)$ are defined as

$M(\mu_1,\mu_2) = \sup_{-\infty<\nu_1,\nu_2<\infty} |f(\mu_1 + i\nu_1 , \mu_2 + i\nu_2|, \mu_1,\mu_2 > 0$. And
$u(\mu_1,\mu_2) = \exp(\alpha_n\mu_1 + \beta_m\mu_2)$
It is known that
$u(\mu_1,\mu_2) < M(\mu_1,\mu_2)$
If (2) holds, then
$M(\mu_1,\mu_2) < u(\mu_1 + D + \epsilon, \mu_2 + D + \epsilon)$,where $\epsilon > 0$ (3)
Many researchers have thoroughly examined the basic characteristics and different growth aspects of random Dirichlet series (see [4–22]). Where the lower order and generalized growth aspects of these series, particularly under various convergence circumstances, have been studied by other researchers in a related direction (see [23]–[33]).

Also for entire functions represented by multiple Dirichlet series, P.K. Jain and V.P. Gupta [1] proved that
$\rho = \lim\limits_{\mu_1,\mu_2\to\infty} \sup \frac{\log\log M(\mu_1,\mu_2)}{\log(e^{\mu_1}+e^{\mu_2})}$, for $0 < \rho < \infty$. And (4)

$\emptyset = \lim_{n,m\to\infty} \sup \frac{\alpha_n\log\alpha_n+\beta_m\log\beta_m}{\log|c_{n,m}|^{-1}}$, for $0 < \emptyset < \infty$ (5)
Where ρ and $\emptyset$ define the order of the multiple Dirichlet series, characterized by the maximum modulus and the coefficients' growth, respectively.
If (2) holds, then [23],
$\rho = \lim\limits_{\mu_1,\mu_2\to\infty} \sup \frac{\log\log M(\mu_1,\mu_2)}{\log(e^{\mu_1}+e^{\mu_2})} = \emptyset = \lim_{n,m\to\infty} \sup \frac{\alpha_n\log\alpha_n+\beta_m\log\beta_m}{\log|c_{n,m}|^{-1}}$ 6)
Note that $\rho = \emptyset$ under the conditions of (2)
Under the same conditions (2), a similar result for the lower order, namely
$\theta = \lim_{n,m\to\infty} \inf \frac{\alpha_n\log\alpha_n+\beta_m\log\beta_m}{\log|c_{n,m}|^{-1}} \neq \lambda = \lim\limits_{\mu_1,\mu_2\to\infty} \inf \frac{\log\log M(\mu_1,\mu_2)}{\log(e^{\mu_1}+e^{\mu_2})}$ (7)
Where λ and θ define the lower order of the multiple Dirichlet series, characterized by the maximum modulus and the coefficients' growth, respectively.

Since those hidden gaps induce an irregular divergence in the exponents, they disrupt the coefficient-modulus synchronization, causing a breakdown in the lower order equality while the total order remains immune; therefore, I present in this paper new definitions for the lower order, following the same foundational approach established by Srivastava et al. [34] and Yong-Qin Cui et al. [35]. In this definition. I introduce a
refined condition on the index distribution to ensure that the transition from supremum to infimum remains analytically consistent despite the presence of hidden gaps. To ensure the analytic stability of the lower order against hidden gaps, I redefine class A by incorporating an exceptional set of finite measures as a fundamental constraint. The subsequent section presents the necessary definitions followed by an illustrative example that validates the proposed approach and then the fundamental theorems that constitute the main results of this study.

2.Materials And Methods

Definition 1: An entire Dirichlet series $f(\xi,\eta)$ defined by the equation(1) has finite lower order λ_1 and λ_2 with respect to ξ and η respectively is said to be belong to refined class A, if there exist a positive constants σ_1, $\tau_1, \sigma_2$and τ_2,such that:

i)For any fixed value of $\mu_2 > 0$ there exists a constant $\mu^{(1)} = \mu^{(1)}(\sigma_1, \tau_1, \mu_2)$ such that the following inequality holds:

$M(\mu_1, \mu_2) < \exp\{\sigma_1 \exp(\mu_1 \tau_1)\}$, for $\mu_1 \geq \mu^{(1)}$ (excluding a set of values of finite measure).

ii) For any fixed value of $\mu_1 > 0$ there exists a constant $\mu^{(2)} = \mu^{(2)}(\sigma_2, \tau_2, \mu_1)$ such that the following inequality holds:

$M(\mu_1, \mu_2) < \exp\{\sigma_2 \exp(\mu_2 \tau_2)\}$, for $\mu_2 \geq \mu^{(2)}$ (excluding a set of values of finite measure).

Consequently, there exists a number μ such that

$$M(\mu_1, \mu_2) < \exp\{\sigma_1 \exp(\mu_1 \tau_1) + \sigma_2 \exp(\mu_2 \tau_2)\} \text{ for } \mu_1, \mu_2 \geq \mu$$

Definition 2: An entire function $f(\xi,\eta)$ defined by the equation(1) has finite lower order λ_1 and λ_2 with respect to ξ and η respectively if:

i) For any arbitrarily small $\epsilon > 0$ and fixed value $\mu_2 > 0$ there exist a constant number
$\mu^{(1)} = \mu^{(1)}(\epsilon, \mu_2)$ such that:
$M(\mu_1, \mu_2) > \exp\{\exp(\mu_1(\lambda_1 - \epsilon))\}$ satisfy for almost values of μ_1 (excluding a set of values of finite measure).Hence,(i) is equivalent to:

$$\lim_{\mu_2 \to \infty} \left\{ \lim_{\mu_1 \to \infty} \inf \frac{\log \log M(\mu_1, \mu_2)}{\mu_1} \right\} = \lambda_1 \text{ for all } \mu_1 > \mu^{(1)}$$

ii) For any arbitrarily small $\epsilon > 0$ and fixed value $\mu_1 > 0$, there exist a constant number
$\mu^{(2)} = \mu^{(2)}(\epsilon, \mu_1)$ such that:
$M(\mu_1, \mu_2) > \exp\{\exp(\mu_2(\lambda_2 - \epsilon))\}$ satisfy for almost values of μ_2 (excluding a set of values of finite measure). Hence, (ii) is equivalent to:

$$\lim_{\mu_1 \to \infty} \left\{ \lim_{\mu_2 \to \infty} \inf \frac{\log \log M(\mu_1, \mu_2)}{\mu_2} \right\} = \lambda_2 \text{ for all } \mu_2 > \mu^{(2)}$$

Definition 3: An entire function $f(\xi, \eta)$ defined by the equation (1) has finite lower order λ_1 and λ_2 with respect to ξ and η respectively if:
1) $f(\xi, \eta) \in A$(where class A is defined relative to the lower order and an exceptional set of finite measure).
2) $f(\xi, \eta)$ has finite lower order λ_1 and λ_2 with respect to ξ and η respectively as defined above.
3)For any $\epsilon > 0$, there exist a number $\mu = \mu(\epsilon)$such that the joint inequality:
$M(\mu_1, \mu_2) > \exp\{\exp(\mu_1(\lambda - \epsilon) + \exp(\mu_2(\lambda - \epsilon)\}$, hold for all $\mu_1, \mu_2 \geq \mu$ outside an exceptional set of finite measure).Then the total lower order λ is defined as

$$\lim_{\mu_1, \mu_2 \to \infty} \inf \frac{\log \log M(\mu_1, \mu_2)}{\log(e^{\mu_1} + e^{\mu_2})} = \lambda, \text{ where } \lambda = \min\{\lambda_1, \lambda_2\}.$$

3.Result And Discussion

In this section the following example shows how, in the presence of structural gaps, classical equalities are unable to synchronize the maximum modulus with the coefficients.

Example4: Let $f(\xi, \eta) = \exp\left(e^{2\xi + 2\eta}\right) + \exp(e^{\xi + \eta})$, then

$\exp\left(e^{2\xi+2\eta}\right) = \sum_{k=0}^{\infty} \frac{(e^{2\xi+2\eta})^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} e^{(2k)\xi+(2k)\eta}$, where $(\alpha_n, \beta_m) = (2k, 2k)$, and

$a_{n,m} = \frac{1}{k!}$, for $k = 0,1,2,....$ Also for the second part $\exp(e^{\xi+\eta}) = \sum_{k=1}^{\infty} \frac{1}{k!} e^{k\xi+k\eta}$, where

$(\alpha_n, \beta_m) = (k, k)$ and $b_{n,m} = \frac{1}{k!}$, for $k = 0,1,2, ...$, then the final form is

$f(\xi, \eta) = 2 + e^{\xi+\eta} + \left(1 + \frac{1}{2!}\right) e^{2\xi+2\eta} + \frac{1}{3!} e^{3\xi+3\eta} + \left(\frac{1}{2!} + \frac{1}{4!}\right) e^{4\xi+4\eta} + \cdots$,

where

The value 2 is obtained by summing the first term of each function, both of which equal 1.

$$c_{n,m} = \begin{cases} \frac{1}{\left(\frac{k}{2}\right)!} + \frac{1}{k!}, n = m = k \text{ are even numbers} \\ \frac{1}{k!}, \ n = m = k \text{ are odd numbers} \end{cases}$$

Then, $\lim_{n,m\to\infty} \inf \frac{\alpha_n \log\alpha_n + \beta_m \log\beta_m}{\log|c_{n,m}|^{-1}} = 2,$ whereas

$\lim_{n,m\to\infty} \inf \frac{\log\log M(\mu_1,\mu_2)}{\log(e^{\mu_1}+e^{\mu_2})} = 4.$

Therefore, I offer the following theorems to illustrate the facts that result in such inequality, based on the difference seen in this example.

Theorem 5: Consider the multiple Dirichlet series function f(ξ,η) defined by the equation(1).Where,

$f(\xi, \eta) = \sum_{n,m=0}^{\infty} c_{n,m} \exp(\alpha_n \xi + \beta_m \eta)$, where $\xi = \mu_1 + i\nu_1$ and $\eta = \mu_2 + i\nu_2$

such that, $\xi, \eta \epsilon ¢$ and $\mu_1, \mu_2, \nu_1 and \nu_2 \in R$ with $0 < \alpha_n \leq \alpha_{n+1}$ and $0 < \beta_m \leq \beta_{m+1}$, $n, m \in N$

Is an entire function of lower order λ $(0 < \lambda < \infty)$ such that

$\log\alpha_n \sim \log\alpha_{n+1}$ and $\log\beta_m \sim \log\beta_{m+1}$

Then,

$\lambda > \lim_{n,m\to\infty} \inf \frac{\alpha_n \log\alpha_n + \beta_m \log\beta_m}{\log|c_{n,m}|^{-1}}$

Proof

Since $\sum c_{n,m}$ is convergent, then $|c_{n,m}| < 1$ for all $n > n_0$ and $m > m_0$

Now suppose that

$0 < \lim_{n,m\to\infty} \inf \frac{\alpha_n \log\alpha_n + \beta_m \log\beta_m}{\log|c_{n,m}|^{-1}} = \theta < \infty$. Then

$\frac{\alpha_n \log\alpha_n + \beta_m \log\beta_m}{\log|c_{n,m}|^{-1}} > \theta - \epsilon'$, for all $n \geq N(\epsilon)$ and $m \geq M(\epsilon)$.

Implies that

$$\log|c_{n,m}| > -(\theta - \epsilon')^{-1}(\alpha_n \log\alpha_n + \beta_m \log\beta_m)$$

$$|c_{n,m}| > (\alpha_n^{\frac{-\alpha_n}{\theta-\epsilon'}} . \beta_m^{\frac{-\beta_m}{\theta-\epsilon'}})$$

Now, let $e^{\mu_1} = 2^{\beta_m} \alpha_n^{\frac{1}{\theta-\epsilon'}}$, $e^{\mu_2} = 2^{\alpha_n} \beta_m^{\frac{1}{\theta-\epsilon'}}$. (8)

If $\mu_n \leq \mu_1 \leq \mu_{n+1}$, $\mu_m \leq \mu_2 \leq \mu_{m+1}$. Then from (8)

$$M(\mu_1, \mu_2) > |c_{n,m}| \exp(\alpha_n \mu_1 + \beta_m \mu_2)$$

$$\geq (\alpha_n^{\frac{-\alpha_n}{\theta-\epsilon'}} . \beta_m^{\frac{-\beta_m}{\theta-\epsilon'}})(2^{\alpha_n\beta_m} \alpha_n^{\frac{\alpha_n}{\theta-\epsilon'}} . 2^{\alpha_n\beta_m} \beta_m^{\frac{\beta_m}{\theta-\epsilon'}})$$

$M(\mu_1, \mu_2) > 2^{2\alpha_n\beta_m}$, implies

$\log M(\mu_1, \mu_2) > 2\alpha_n\beta_m \log 2$, then $\log\log M(\mu_1, \mu_2) \geq \log\alpha_n + \log\beta_m + A$ 9)

where $A = \log 2 + \log\log 2$

Since $\log\alpha_n \sim \log\alpha_{n+1}$ and $\log\beta_m \sim \log\beta_{m+1}$, ($\log\alpha_n > (1 - \epsilon'')\log\alpha_{n+1}$ and $\log\beta_m > (1 - \epsilon'')\log\beta_{m+1}$)

Then the inequality (9) can be written as

$\log \log M(\mu_1, \mu_2) > (1 - \epsilon'')\log\alpha_{n+1} + (1 - \epsilon'')\log\beta_{m+1} + A$ (10)

Now from (8), get

$\log\alpha_{n+1} = (\theta - \epsilon')(\mu_{n+1} - \beta_{m+1}\log 2)$, and the same for

$\log\beta_{m+1} = (\theta - \epsilon')(\mu_{m+1} - \alpha_{n+1}\log 2)$, by substituting these values into(10) to get,

$\log\log M(\mu_1, \mu_2) > (1 - \epsilon'')(\theta - \epsilon')(\mu_{n+1} - \beta_{m+1}\log 2) + (1 - \epsilon'')(\theta - \epsilon')(\mu_{m+1} - \alpha_{n+1}\log 2) + A \geq (1 - \epsilon'')(\theta - \epsilon')(\mu_1 - \beta_{m+1}\log 2) + (1 - \epsilon'')(\theta - \epsilon')(\mu_2 - \alpha_{n+1}\log 2) + A.$

Now for arbitrarily large values of μ_1, and for a constant value of $\mu_2 > 0$ say $\mu_2^{(0)}$, such that

$$\lim_{\mu_2\to\infty}\left\{\lim_{\mu_1\to\infty}\inf\frac{\log\log M\left(\mu_1,\mu_2^{(0)}\right)}{\mu_1}\right\} >$$

$$\lim_{\mu_2\to\infty}\left\{\lim_{\mu_1\to\infty}\inf\frac{(1-\epsilon'')(\theta-\epsilon')(\mu_1-\beta_{m+1}\log 2)}{\mu_1}+\frac{(1-\epsilon'')(\theta-\epsilon')\left(\mu_2^{(0)}-\alpha_{n+1}\log 2\right)+A}{\mu_1}\right\}$$

$$> (1-\epsilon'')(\theta-\epsilon')+o(1) \qquad (11)$$

Where $o(1)$ denotes a term that tends to zero as μ_1 and μ_2 tend to infinity. And in the same manner for arbitrarily large values of μ_2 and for a constant value of μ_1, say $\mu_1^{(0)} > 0$

such that

$$\lim_{\mu_1\to\infty}\left\{\lim_{\mu_2\to\infty}\inf\frac{\log\log M\left(\mu_1^{(0)},\mu_2\right)}{\mu_2}\right\} > (1-\epsilon'')(\theta-\epsilon')+o(1) \qquad (12)$$

Therefore, from (11) and (12) to get

$$\lambda > \theta.$$

Theorem 6: Consider the multiple Dirichlet series function f(ξ,η) defined by the equation (1).Where,

$f(\xi,\eta)=\sum_{n,m=0}^{\infty} c_{n,m}\exp(\alpha_n\xi+\beta_m\eta)$, where $\xi=\mu_1+i\nu_1$ and $\eta=\mu_2+i\nu_2$

such that, $\xi, \eta \epsilon ¢$ and $\mu_1,\mu_2,\nu_1 and \nu_2 \in R$ with $0<\alpha_n\le\alpha_{n+1}$ and $0<\beta_m\le\beta_{m+1}$, $n,m\in N$

Is an entire function of lower order λ $(0<\lambda<\infty)$ such that

i) $\lim_{n\to\infty}\sup\frac{\log n}{\log\alpha_n}=D_1<\infty$, $\lim_{n\to\infty}\sup\frac{\log m}{\log\beta_m}=D_2<\infty$

Where, $D_1<D$ and $D_2<D$

ii)$\omega_{n,m}=\frac{\log\left|\frac{c_{n,m}}{c_{n+1,m+1}}\right|}{(\alpha_{n+1}-\alpha_n).(\beta_{m+1}-\beta_m)}$, is a non-decreasing function of n,m , for $n > n_0$ and $m > m_0$. Then

$$\lambda<\lim_{n,m\to\infty}\inf\frac{\alpha_n\log\alpha_n+\beta_m\log\beta_m}{\log|c_{n,m}|^{-1}}$$

Proof

Now for arbitrarily large values of μ_1 and for a constant value of μ_2, say $\mu_2^{(0)}$, such that

$$u\left(\mu_1, \mu_2^{(0)}\right) = |c_{n,m}|\exp(\alpha_n\mu_1 + \beta_m\mu_2^{(0)})$$

For $\omega_{n,m} \leq \mu_1 \leq \omega_{n+1,m+1}$ since, $\lambda < \infty$, then from (3) implies that

$$\log u(\mu_1 + D_1 + \epsilon_1, \mu_2^{(0)}) > \log M(\mu_1, \mu_2^{(0)}) > \exp(\mu_1(\lambda_1 - \epsilon')) \geq \exp(\mu_1(\lambda - \epsilon')),$$

Where, $\lambda = \min\{\lambda_1, \lambda_2\}$, then from equation(28), implies that,

$$\log|c_{n,m}| + \alpha_n(\mu_1 + D_1 + \epsilon_1) + \beta_m\mu_2^{(0)} > \exp(\mu_1(\lambda - \epsilon'))$$

$$\log|c_{n,m}| > \exp(\mu_1(\lambda - \epsilon')) - \alpha_n(\mu_1 + D_1 + \epsilon_1) - \beta_m\mu_2^{(0)}$$

$$-\log|c_{n,m}| < \alpha_n(\mu_1 + D_1 + \epsilon_1) + \beta_m\mu_2^{(0)} - \exp(\mu_1(\lambda - \epsilon'))$$

Now, let $\theta_1 = \frac{\alpha_n \log\alpha_n}{\log|c_{n,m}|^{-1}}$, then

$$\Theta_1 = \frac{\alpha_n \log\alpha_n}{\log|c_{n,m}|^{-1}} > \frac{\alpha_n \log\alpha_n}{\alpha_n(\mu_1 + D_1 + \epsilon_1) + \beta_m\mu_2^{(0)} - \exp(\mu_1(\lambda - \epsilon'))} \quad (13)$$

$$\text{Let, } \mu_1 = (\lambda - \epsilon')^{-1}\log\{\frac{\alpha_n}{\lambda - \epsilon'}\} \quad (14)$$

Now substitute equation (14) into (13) to get

$$\theta_1 > \frac{\alpha_n \log\alpha_n}{\alpha_n(\lambda - \epsilon')^{-1} log\{\frac{\alpha_n}{\lambda - \epsilon'}\} + \alpha_n(D_1 + \epsilon_1) + \beta_m\mu_2^{(0)} - \frac{\alpha_n}{\lambda - \epsilon'}}, \text{ or}$$

$$\theta_1 > \frac{(\lambda - \epsilon')\log\alpha_n}{\log\alpha_n - \log(\lambda - \epsilon') + (D_1 + \epsilon_1)(\lambda - \epsilon') + (\lambda - \epsilon')\frac{\beta_m\mu_2^{(0)}}{\alpha_n} - 1}$$

Now, by holding m is constant and letting n approach to infinity, that is to say

$$\lim_{m\to\infty}\left\{\lim_{n\to\infty}\inf\ \theta_1\right\} >$$

$$\lim_{m\to\infty}\left\{\lim_{n\to\infty}\inf\frac{(\lambda-\epsilon')\log\alpha_n}{\log\alpha_n-\log(\lambda-\epsilon')+(D_1+\epsilon_1)(\lambda-\epsilon')+(\lambda-\epsilon')\frac{\beta_m\mu_2^{(0)}}{\alpha_n}-1}\right\}$$

$$> (\lambda-\epsilon') > \frac{(\lambda-\epsilon')}{2} \qquad (15)$$

Now, let $\mu_2 = (\lambda-\epsilon')^{-1}\log\left\{\frac{\beta_m}{\lambda-\epsilon'}\right\}$ and $\theta_2 = \frac{\beta_m\log\beta_m}{\log|c_{n,m}|^{-1}}$

Then for arbitrarily large values of μ_2, and for a constant value of μ_1, say $\mu_1^{(0)}$, the proof for the second part follows a similar argument by holding n is constant and letting m approach infinity. Which implies that

$$\theta_2 > (\lambda-\epsilon') > \frac{(\lambda-\epsilon')}{2} \qquad (16)$$

Hence, (15) and (16) imply that

$\theta_1 + \theta_2 = \theta > (\lambda-\epsilon')$.Where $\theta = \lim_{n,m\to\infty}\ \inf\ \frac{\alpha_n\log\alpha_n+\beta_m\log\beta_m}{\log|c_{n,m}|^{-1}}$

Then, $\lambda < \lim_{n,m\to\infty}\ \inf\ \frac{\alpha_n\log\alpha_n+\beta_m\log\beta_m}{\log|c_{n,m}|^{-1}}$

Conclusion

In this paper, the effect of structural gaps on the growth properties of multiple Dirichlet series was investigated. The results show that structural gaps have different effects on the total order and the lower order; the whole order remains stable and retains its classical equality, whereas the lower order drastically deteriorates. Furthermore, the existence of structural gaps interferes with the synchronization between the coefficients and the maximum modulus, particularly for the lower order. This work concludes that, given the minimum growth rate, new definitions are required because conventional growth formulas are inadequate for describing functions with gaps.

Further Work

The present definitions of lower order could be expanded to more generic classes of multiple Dirichlet series outside of class A in future research. Furthermore, it would
be beneficial to look into the growth theory under various kinds of exceptional sets and investigate the relationship between the lower order and other growth indicators
like the proximate order. Promising directions for future research include extending these findings to higher dimensions and examining their applicability in mathematical physics and complex analysis.

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حول نمو متسلسلات ديريكليه ذات الفجوات الكامنة

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مستخلص البحث:
في هذه الورقة، تم توسيع نظرية نمو متسلسلات ديريكليه ذات القيم المتجهة إلى متغيرين مركبين، مع التركيز على التوصيف المحسّن للرتبة الدنيا. لضمان الاستقرار، قمت بإنشاء فئة المعممة Aعن طريق إضافة مجموعة استثنائية ذات قياسات منتهية. والأهم من ذلك، أُثبت أن الرتبة الدنيا المستخرجة من المعيار الأقصى لا تتطابق دائماً مع تلك المستخرجة من معاملات المتسلسلة للدوال ذات الفجوات البنيوية. وتقدم نتائجي شروطاً كافية وضرورية لفشل هذه المساواة التقليدية. كما أقدم منهجية متينة لتحليل المتسلسلات ذات الفجوات عن طريق فصل عدم انتظام النمو داخل المجموعة الاستثنائية. وتسد هذه الدراسة ثغرة كبيرة في نظرية النمو متعددة المتغيرات في الإعدادات غير المنتظمة.

الكلمات المفتاحية : سلاسل ديريشليت ، الرتبة الدنيا ، القيمة العظمى المطلقة، سلاسل ديريشليت ذات الفجوات، الصنف المنقح

ملاحظة : هل البحث مستل من رسالة ماجستير او اطروحة دكتوراه ؟ نعم : **كلا :** ✓