مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

A Hybrid Shifted Vieta- Pell Wavelet and Variational Iteration Method for Delay Differential Equation Models

Raya M. Almuslih [(1)]
Mustansiriyah University Department of Mathematics, Baghdad , Iraq

Abdul Khaleq O. Al-Jubory [(2)]
Mustansiriyah University Department of Mathematics, Baghdad , Iraq

rayaan2024@uomustansiriyah.edu.iq khaleqmazeel@uomustansiriyah.edu.iq

Abstract:
The primary objective of this paper is to propose a novel approach. In particular, it will show that the delay differential equations using the variational iteration method (VIM), which combines shifted vieta-pell wavelets. The properties of shifted vieta-pell wavelets studied and the convergence analysis are discussed. The delay differential equations can be simplified to a system of algebraic equations by using a truncated shifted vite-pell wavelets series. To illustrate the novel methods, a few numerical examples were solved to show how accurate and effective. The computations of the residual errors are employed these residuals are shown to trend to zero.

Keywords: Variational Iteration Method, Linear Differential Equations, Shifted Vieta-Pell wavelet polynomials , Lagrange Multiplier.
Note: The research is based on an M.A thesis: yes.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

Introduction:

Differential equations are the formulation of scientific theory for many real-world physical problems, Modeling changes in fundamental physical parameters, such as pressure, temperature, acceleration, etc., in general. Occur frequently in the fields of engineering and science. Differential equation theory is the foundation for several fields, including physical variation or structural dynamics, heat transfer, and the theory of electric circuits (Abdul, Ayad, & Ahmed, 2024) (Ayad & Al Jubory, 2023) (Ayad, Abdul, & Abdullah, 2025) . The derivative of an unknown function at a given period is given in terms of the function's values at earlier periods in delay differential equations (DDEs), a type of functional differential equation form with a wide range of applications that obtain numerical and analytical answers in the fields of physics, as well as engineering. It has long been utilized in control theory, and in recent times, it has been widely employed in numerous biological simulations (Salah, 2022) (Abdul k & Ihab, 2025) (Alrebdi & H, 2024) (Jafari & Noori, 2021).

The Variational Iteration Method (VIM) was developed by the Chinese mathematician He (2000). and it has shown great success in tackling a variety of problems. As we all know the variation iteration method (VIM) is a strong mathematical tool for solving linear and nonlinear problems that may be simply used in reality. There are various publications regarding the use of VIM for a delay differential equation solving . Sara Barati and Karim Ivaz in (Barati, 2012) , employ (VIM) to obtain the approximate analytical solutions. Syed T. and Ahmet Y. in (Dina & Yildirimb, 2010), apply (VIM) using He's polynomials to solve delay differential equations. M.M. Khader (Khader, 2013), devoted the (VIM) to find the exact solution. Asim Rauf, etc. (Asim, Guo Hui1, & Younis, 2020)consider the (VIM) for solving Legendre differential equation.

Wavelet Methods: Wavelets are a family of functions generated by translating and scaling a single function known as the mother wavelet. It allows the exact modeling of various operators and functions. Additionally, wavelet functions establish a link with various kinds of numerical techniques. Recently, certain numerical approaches have been applied to the numerical solutions of delay differential equations via

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

wavelet polynomials, both linear and nonlinear. As an example, in the Haar wavelet collocation (Aziz, 2016). Euler wavelete (Mutaz & Alexander, 2022). Laguerre wavelete approach (Kumbinarasaiah & Ravikiran, 2023). Chebyshev wavelet (Sevin, 2019). Legendre Wavelet Method (Sadeghi, Karimi, & J, 2011). Fibonacci wavelet-based Galerki method (Angadim, 2024). Boubaker Wavelets Functions (Eman & Najeeb, 2021).

In this paper, we propose an approximation method. The proposed method variational iteration method(VIM) in order to solve various nonlinear delay differential equations (DDEs) analytically via shifting Vieta-Pell wavelets basis of equation (1). The following nonlinear delay differential equation are our main focus:

$$u''(x) = f\left(x, u(t), u\left(\frac{t}{\tau}\right), u'\left(\frac{t}{\tau}\right)\right), \in [0,1]$$

(1) with initial conditions: $u(0) = \alpha,\ u'(0) = \beta\ where\ 0 < \tau < 1,$

2. Vieta-Pell polynomials on the interval [-1,1] [2]:

The Vieta-pell polynomials, denoted via $V_m^{0,1}(\mathrm{t})$ as a degree of m are interested when the range of the independent variable is $[-1,1]$. They are orthogonal polynomials sequence and can be recursively obtained as below

$$V_0^{0,1}(\mathrm{x}) = 2, \quad V_1^{0,1}(\mathrm{x}) = x,$$
$$V_m^{0,1}(x) = xV_{m-1}^{0,1}(\mathrm{x}) - V_{m-2}^{0,1}(x), \qquad m \geq 2$$

(2) The first four are given as below:

$$V_2^{0,1}(\mathrm{x}) = x^2 - 2,$$
$$V_3^{0,1}(\mathrm{t}) = x^3 - 3x,$$
$$V_4^{0,1}(\mathrm{t}) = x^4 - 4x^2 + 2.$$

3. Shifted Vieta-Pell polynomials on the interval [0,1] [1]:

The Vieta-pell polynomials, denoted via $V_m^{0,1}(\mathrm{t})$ as a degree of m are

interested when the range of the independent variable is $[0,1]$..We use the Shifted:

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

$x = 4t - 2, \quad 0 \le t \le 1.$ The shifted polynomials thus become:

$V_m^{0,1}(t) = V_m(4t-2)$.

They are orthogonal polynomials sequence and can be recursively obtained as below:

$$V_m^{0,1}(t) = (4t-2)V_{m-1}^{0,1}(t) - V_{m-2}^{0,1}(t) \quad (3)$$

The first three are given as below:

$V_0^{0,1}(t) = 2,$

$V_1^{0,1}(t) = 4t - 2,$

$V_2^{0,1}(t) = 16t^2 - 16t + 2,$

$V_3^{0,1}(t) = 64t^3 - 96t^2 + 36t - 2.$

4.Shifted Vieta-Pell Wavelets polynomials on the interval[0,1]:

The family of continuous wavelet can be call shifted vieta-pell wavelets $\Psi_{n,m}(t) = \Psi_{k,n,m}(t)$, it's define as follows:

$$\Psi_{n,m}(t) = \begin{cases} 2^{k/2} Vm\,(2kt-n) & \frac{n}{2k} \le t \le n + \frac{1}{2k} \\ 0 & other\ wise \end{cases} \quad (4)$$

Considering

$n = 1,2,\dots,2^k - 1,\ m = 0,1,2,\dots,m-1\ and\ k = 1,2\dots,for\ instance,\ k = 1\ and\ m = 4$

, we get shifted vieta – pell wavelets bases as follows: $\Psi_{1,0}(t) = 2\sqrt{2}$ (2t-1) , $\Psi_{1,1}(t) = 2\sqrt{2}\,(2t-1)^2$, $\Psi_{1,2}(t) = 2\sqrt{2}\,(16t^3 - 24t^2 + 10t - 1)$ and $\Psi_{1,3}(t) = 2\sqrt{2}\,(63t^4 - 128\,t^3 + 84t^2 - 20t + 1)$ ...

5. Delay Operational Matrices of wavelet vite-pell basis:

An unknown function $u(t)$ is expanded into this basis via truncated matrix format : $u(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\Psi_{n,m}(t) = u(t) \approx C^{\mathrm{T}}\Psi(t)$ (5) We collect these elements into a single functional basis vector

$\Psi(t) = \left[\Psi_{00}(t), \dots, \Psi_{0,M-1}(t), \dots, \Psi_{2-1,M-1}^{k}(t)\right]^{\mathrm{T}}$

$C(t) =$

$\left[C_{00}(t), \dots, C_{0,M-1}(t), \dots, C_{2-1,M-1}^{k}(t)\right]^{\mathrm{T}}$

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

Using the standard wavelet operational matrix of derivative

$$u'(t) \approx C^{\mathrm{T}}\mathrm{M}\Psi(t) \quad (6)$$

$$u''(t) \approx C^{\mathrm{T}}M^{2}\Psi(t) \quad (7)$$

$$g(t) \approx G^{\mathrm{T}}\Psi(t) \quad (8)$$

we introduce the wavelet operational matrix of delay $D_\tau\left(\frac{t}{\tau}\right)$:

$$\Psi\left(\frac{t}{\tau}\right) \approx D_\tau\Psi(\mathrm{t}) \quad (9)$$

We Substituting the scaling identity into our function definitions yields:

$$u\left(\frac{t}{\tau}\right) \approx C^{\mathrm{T}}\, D_\tau\, \Psi(t) \quad (10)$$

$$u'\left(\frac{t}{\tau}\right) \approx C^{\mathrm{T}}\mathrm{M}\, D_\tau\Psi(t) \quad (11)$$

Remove the constant coefficient matrix C^{T} to focus integration on time-dependent functions and the basis vector $\Psi(t)$.

6. Variational Iteration Method

Determining the correct solution is the main purpose of the variational iteration method. Consider the general problem that follows

$$\mathfrak{u}''(t) = a(t)\mathfrak{u}\left(\frac{t}{\tau}\right) + b(t)\acute{u}\left(\frac{t}{\tau}\right) + d(t)\mathfrak{u}(t) + g(t) \quad (12)$$

With initial : $\mathfrak{u}(0) = \alpha \;\; \mathfrak{u}'(0) = \beta$ (13)

Consider overall differential equation of (VIM) :

$$\mathfrak{l}\mathfrak{u} + \mathcal{N}\mathfrak{u} = \mathfrak{g}(x) \quad (14)$$

where the driving term is $\mathfrak{g}$ (x), the operators $\mathfrak{l}$ and $\mathcal{N}$ are linear and nonlinear, respectively.

$$\mathfrak{u}_{n+1}(x) = \mathfrak{u}_n(x) + \int_0^x \lambda(t)\left[\mathfrak{l}\mathfrak{u}_n\left(\frac{t}{\tau}\right) + \mathcal{N}\hat{\mathfrak{u}}_n\left(\frac{t}{\tau}\right) - \mathfrak{g}(t)\right]dt \quad (15)$$

from VIM , so-called corrective functional of eq. (2) can be created.

$$\mathfrak{u}_{n+1}(x) = \mathfrak{u}_n(x) + \int_0^x \lambda(t)\left[\mathfrak{u}''(t) - \left(a(t)\hat{\mathfrak{u}}\left(\frac{t}{\tau}\right) + b(t)\hat{\acute{\mathfrak{u}}}\left(\frac{t}{\tau}\right) + d(t)\hat{\mathfrak{u}}(t)\right) - \mathfrak{g}(t)\right]dt \quad (16)$$

If $\tilde{\boldsymbol{u}}$ is a restricted value and $\lambda(t)$ is a Lagrange multiplier, and $\boldsymbol{\delta\tilde{u}_n = 0}$. With reference to the distinct variable $\mathfrak{u}_n$, we can find the variation on both sides of eq. (16):

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

$$\frac{\delta u_{n+1}(x)}{\delta u_n(x)} = 1 + \frac{\delta}{\delta u_n(x)}\left(\int_0^x \lambda(t)\left[\, u''(t) - \left(a(t)\hat{u}\left(\frac{t}{\tau}\right) + b(t)\hat{u}'\left(\frac{t}{\tau}\right) + d(t)\hat{u}(t)\right) - g(t)\right] dt\right) \quad (17)$$

Or equivalently

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\left(\int_0^x \lambda(t)\left[\, u''(t) - \left(a(t)\hat{u}\left(\frac{t}{\tau}\right) + b(t)\hat{u}'\left(\frac{t}{\tau}\right) + d(t)\hat{u}(t)\right) - g(t)\right] dt\right) \quad (18)$$

That gives: $\delta\, u_{n+1}(x) = \delta u_n(x) + \left(\int_0^x \lambda(t)[\delta u''_n(t)dt]\right)$ (19)

Obtained by using $\delta\hat{u}_n = 0$ and $\delta\hat{u}'_n(x) = 0$

By integrating eq. (19) components twice, yielding the following results.:

$$\delta\, u_{n+1}(x) = \delta u_n(x) + \lambda(t)\delta\, u_n'(t) - \lambda'(t)\, u_n(x) + \delta\int_0^x \lambda''(t)(\, u_n(t)dt) \quad (20)$$

In order to satisfy the extreme requirement of u_{n+1}, $\delta\, u_{n+1}\,(x) = 0$. As a result, the right side of eq.(20) must also be zero since the left side is zero. This results in the stationary conditions, that is:

$$1 + \lambda'|_{t=x} = 0 \quad , \quad \lambda'|_{t=x} = 0 \quad , \quad \lambda''|_{t=x} = 0$$

The optimal Lagrange multiplier for a second-order system is derived using stationary conditions: $\lambda = t - x$ (21)

We write the sequence iteration formula as:

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left[u_n''(t) - \left(a(t)u_n\left(\frac{t}{\tau}\right) + b(t)u_n'\left(\frac{t}{\tau}\right) + d(t)u_n(t)\right) - g(t)\right] dt \quad (22)$$

For the first structural iteration ($n = 0$), we substitute the operational matrix approximations the eq.(5), eq.(7), eq.(8),eq.(10) and eq.(11) into eq.(22), with initial condition $u_0(t) = \alpha + \beta x$.

$$u_1(x) = \alpha + \beta x + \int_0^x (t-x)\left[C^{\mathrm{T}}M^2\Psi(t) - \left(a(t)C^{\mathrm{T}}D_\tau\,\Psi(t) + b(t)C^{\mathrm{T}}\mathrm{M}\,D_\tau\Psi(t) + d(t)C^{\mathrm{T}}\Psi(t) - C^{\mathrm{T}}\Psi(t)\right)\right] dt \quad (23)$$

where the exact solution requires $u(x) \approx u_1(x)$ we declare our approximation equality:

$$C^{\mathrm{T}}\Psi(t) - u_1(t) = 0 \quad (24)$$

The residual function $R(t; \boldsymbol{C})$ by moving all terms to one side:

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

$R(t;\boldsymbol{C}) = C^{\mathrm{T}}\Psi(t) - (\alpha + \beta x) - \int_0^x (t-x)[C^{\mathrm{T}}\Psi(t)(\mathrm{M}^2 - a(t)\,D_\tau - b(t)\,\mathrm{M}D_\tau - d(t) - \mathbf{I})\,]\,dt$ (25)

Let 2^kM independent collocation points t_i distributed within [0,1] .Evaluate the residual function at each point to construct a linear system of algebraic equations:

$$\mathrm{R}(\mathrm{t_i};\mathbf{C}) = 0,\ \mathrm{i} = 0,1,2,\dots,2^{\mathrm{k}}\mathrm{M}$$

7. Numerical illustrations

In this section of the current activity, we suggest a few applications for the handled system.

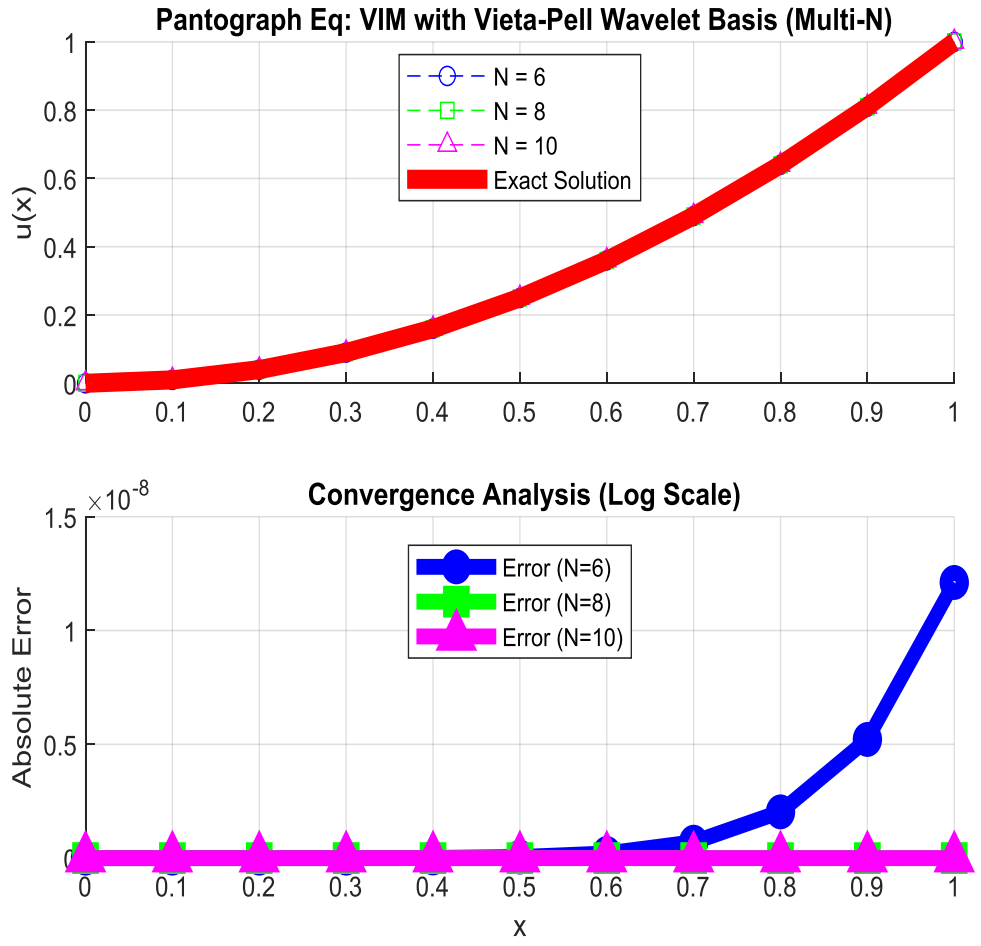
Example1: Consider the delay differential equation:

$\mathrm{u}''(\mathrm{x}) - \mathrm{u}\left(\frac{\mathrm{x}}{2}\right) - 2 + \frac{1}{4}\mathrm{x}^2$, where $u(0) = 1, u'(0) = 1$, accurate solution $\mathrm{u}(x) = x^2$

Table1:

X	Exact solution	Approximate solution	Error absolute
0	0	0	0.0e+00
0.1	0.01	0.01	0.0e+00
0.2	0.04	0.04	0.0e+00
0.3	0.09	0.09	0.0e+00
0.4	0.16	0.16	0.0e+00
0.5	0.25	0.25	0.0e+00
0.6	0.36	0.36	0.0e+00
0.7	0.49	0.49	0.0e+00
0.8	0.64	0.64	0.0e+00
0.9	0.81	0.81	0.0e+00
1	1	1	0.0e+00

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944



8.Conclusion:

In this paper, The variational iteration method was used to solve delay differential equations via shifted Vite-pell wavelet polynomials numerically. After being tested on, several examples. As an analytical approach, the Variational Iteration Method (VIM) constructs a series of iterations that converges directly to the exact solution, thereby achieving zero error in the obtained numerical approximation. This method was shown to yield good results. This approach has broader applicability due to its accuracy and reduced computational cost.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

9.Further Work

1.Employ petrov - Galerkin method (PGM) to solve delay differential equations via shifted Vite-pell wavelet polynomials numerically.

2. Solve integral equation through variational iteration method (VIM) with anther basis (Legendre polynomials, laguerre polynomials,….)

Acknowledgments

I extend my thanks and gratitude to Mustansiriya University, College of Science, and the Department of Mathematics.

References

Abdul k, O., & Ihab, H. J. (2025). Solving pantograph type through shifted Vieta-Pell polynomial. pp. 1227-1235.

Abdul k, O., Ayad, W. A., & Ahmed, H. R. (2024). Approximate approach for solving pantograph type via Genocchi polynomials. *4*, pp. 965–973.

Alrebdi, R., & H, K. (2024). Two different analytical approaches for solving the pantograph delay equation with variable coefficient of exponential order.

Angadim, L. M. (2024). Numerical solution of differential equations using the wavelet-based Galerkin method with Fibonacci wavelets. *2*, pp. 162-16.

Asim, R., Guo Hui1, Z., & Younis, A. M. (2020). Variational Iteration Method for Solving Legendre Differential Equations. *1*, pp. 43-49.

Ayad, A. W., Abdul, k. O., & Abdullah, A. N. (2025). study the complex linear differential equations whose coefficients are exponential polynomials. *AIP Conference.*

Ayad, W. A., & Al Jubory, A. O. (2023). Infinite order of the solutions of higher order homogeneous linear differential equations with entire coefficients having completely regular growth property and bounded components fatuous set. *1*, pp. 1317-1324.

Aziz, R. A. (2016). The delay differential and delay partial differential equations via Haar wavelet.

Barati, S. I. (2012). Variational iteration method for solving systems of linear delay differential equations. *8*.

Dina, S. T., & Yildirimb, A. (2010). Variational iteration method for delay differential equationsUsing He's Polynomials. pp. 1045 – 1048.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

Eman, H., & Najeeb, S. F. (2021). Boubaker Wavelets Functions: Properties and Applications. *4*, pp. 1226-1233.

He, J. (2000). Variational iteration method for autonomous ordinary differential systems. Applied Mathematics and Computation.

Jafari, H. M., & Noori, S. M. (2021). new numerical method to solve pantograph delay differential equations with convergence analysis.

Khader, M. M. (2013). Numerical and theoretical treatment for solving linear and nonlinear delay differential equations using variational iteration method. *2*, pp. 243–256.

Kumbinarasaiah, S., & Ravikiran, A. M. (2023). Wavelets approach for the solution of nonlinear variable delay differential equations. *2*, pp. 139-148.

Mutaz, M., & Alexander, T. (2022). A New Technique for Solving Neutral Delay Differential Equations Based on Euler Wavelets.

Sadeghi, M., Karimi, S. V., & J, S. (2011). Numerical solution of delay differential equations using Legendre wavelet method. *1*, pp. 27-33.

Salah, S. N. (2022). 3D-Chaotic discrete system of vector borne diseases using environment factor with deep analysis. *1*, pp. 3972-3987.

Sevin, G. (2019). Chebyshev wavelet solution of nonlinear ordinary differential equations with delays. *3*, pp. 67–80.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.934-944

طريقة هجينة تجمع بين مويجات "فييتا-بيل" (Vieta–Pell) المزاحة وطريقة التكرار التغايرية لنماذج المعادلات التفاضلية ذات التأخير الزمني

ريا عبد المنعم المصلح (1) عبد الخالق عويد مزعل الجبوري (2)

الجامعة مستنصرية / كلية العلوم / قسم الرياضيات
khaleqmazeel@uomustansiriyah.edu.iq
rayaan2024@uomustansiriyah.edu.iq
07722005359 07816639440

مستخلص البحث:
الهدف الأساسي من هذه البحث هو اقتراح منهج جديد. على وجه الخصوص، سيُظهر أنه يمكن استخدام معادلات الفروق المتأخرة باستخدام طريقة التكرار المتغيّر (VIM)، والتي تجمع بين موجات شيفتد فييتا-بيل. تم دراسة خصائص موجات شيفتد فييتا-بيل ومناقشة تحليل التقارب. يمكن تبسيط معادلات الفروق المتأخرة إلى نظام من المعادلات الجبرية باستخدام سلسلة موجات شيفتد فييتا-بيل المقصورة. لتوضيح الطرق الجديدة، تم حل بعض الأمثلة العددية لإظهار مدى الدقة والفعالية. تُستخدم حسابات الأخطاء المتبقية حيث يتم إظهار أن هذه الأخطاء تميل إلى الصفر.

الكلمات المفتاحية :طريقة التكرار التغايرية، المعادلات التفاضلية الخطية، كثيرات الحدود الموجية لموجة فييتا-بيل ، مضاعف لاجرانج.
ملاحظة : هل البحث مستل من رسالة ماجستير : نعم