مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

Numerical Analysis and Fast Fourier Transform Pricing of Exotic Financial Derivatives in Incomplete Markets

Hussein Fadhil Salim[(1)]
Open Educational College, Dhi Qar Education Directorate, Iraq
Hussainfadhil35@gmail.com

Abstract:
In this paper we consider the problem of pricing exotic financial derivatives in incomplete markets, which is a computationally challenging task as closed-form analytical solutions are not available in the presence of stochastic volatility and price jumps. The study focuses on developing an efficient and accurate numerical framework by integrating numerical analysis with the Fast Fourier Transform (FFT) algorithm. The proposed methodology is based on the transformation of the characteristic function for the value function of the derivative to the frequency domain and the use of a damping factor to guarantee the convergence of the solution. Numerical experiments and simulated applications show that the proposed model has an exponential convergence and much faster and more efficient as compared to traditional simulation methods. In addition, a number of empirical examples confirm the ability of the algorithm to capture the "risk premium" induced by market incompleteness, providing financial institutions with an accurate tool to price and manage the risks of complex investment portfolios in real time.

Keywords: Exotic Derivatives, Incomplete Markets, Fast Fourier Transform (FFT), Numerical Analysis, Option Pricing, Stochastic Volatility**.**

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

Introduction:

The rapid development of international financial markets during the past decades created a need for more and more complex investment instruments, called exotic derivatives. These instruments differ from vanilla options and have non-linear payoffs depending on multiple underlying assets, path-dependent features or non-standard contractual features (Rosazza Gianin & Sgarra, 2023, p. 191).

The increased application of these derivatives for hedging and risk management purposes has shown that the classical pricing models, usually based on a complete market, are not able to fully describe the market dynamics. This limitation is particularly apparent in incomplete markets, where classic no-arbitrage theory based on a unique replicating portfolio does not suffice for non-replicable payoffs. In this case, the price must be derived from other conditions (e.g. a log-martingale condition) to incorporate the other risks, as McCloud (2025, p. 2) states.

In practice most financial markets are incomplete. Market incompleteness stems from a variety of empirically observable facts, such as transaction costs, liquidity constraints, and price movements that are not continuous (jumps) due to macroeconomic shocks or geopolitical events. In this case, the classical no-arbitrage principle does not imply a unique risk-neutral measure and the construction of a unique arbitrage-free price of a given derivative is impossible (Föllmer & Sondermann, 1986). Other pricing criteria have to be used, e.g. residual risk minimization, mean-variance hedging or utility-based approaches (Schweizer, 2001).

With the growing use of these derivatives for hedging and risk management, the traditional pricing models, which typically assume a complete market, have been unable to capture the full spectrum of market dynamics. This limitation is especially visible in incomplete markets, where the classical theory of no arbitrage is insufficient. Incomplete markets can be statistically indistinguishable from complete markets (see Dokuchaev, 2023, p. 462). Thus, it is hard to rely on traditional pricing models alone. De Vries (2026, p. 2) also shows that the recovery of joint risk-neutral distributions from option prices remains an open problem for incomplete markets

.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

The FFT approach applies the characteristic function of the underlying asset return distribution to calculate the option prices with a computational complexity much smaller than that of the traditional Monte Carlo simulations, and at the same time provides a higher accuracy in many settings. As Cherubini et al. (2023, p. 45) explain, Fourier methods are able to achieve computational speeds several orders of magnitude faster than simulation-based approaches by exploiting the characteristic function. Moreover, Aguilar and Kirkby (2023, p. 2) show that Fourier-based pricing formulas can reach any desired level of accuracy, especially for short maturity options which are the most difficult for competing Fourier methods.

Fourier-based pricing methods are well studied, but there is a clear research gap in systematically considering the simultaneous challenges of (i) the structural complexity of exotic derivatives (Wang, 2020), (ii) the realism of incomplete market conditions (Wang, 2022), and (iii) the computational efficiency for real-time financial decision-making (Junike and Pankrashkin, 2022). This research aims at filling this gap by developing a robust numerical framework combining state-of-the-art mathematical modelling techniques with the FFT-algorithm, providing a practical and computationally feasible pricing tool for barrier and Asian options under the assumptions of incomplete markets (Sabino, 2022).

The remainder of the paper is organized as follows: Section 2 gives a formal characterization of exotic derivatives, with a special focus on barrier and Asian options. 3. In Section 3, we provide the mathematical formulation of the pricing problem in incomplete markets. Section 4 presents the proposed FFT-based numerical approach. Section 5 presents the empirical results and validation of the model through numerical experiments. Finally, section 6 concludes the study and suggests directions for future research.

Literature Review

Over the past several decades, the literature on derivative pricing under incomplete market conditions has expanded considerably, with researchers proposing a wide range of theoretical frameworks and numerical methodologies in response to the inherent limitations of classical pricing models and the growing complexity of modern financial instruments. The seminal work on complete market environments with

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

continuous trading laid the foundational basis for modern derivative pricing theory; however, subsequent studies recognised that real-world financial markets are inherently incomplete, implying that contingent claims cannot be perfectly replicated through self-financing trading strategies. This recognition motivated the development of alternative pricing frameworks capable of accommodating empirically observed market imperfections, including the jump-diffusion models which explicitly account for discontinuous asset price movements and thereby render the market incomplete.

In such settings, the classical no-arbitrage principle no longer yields a unique risk-neutral measure, necessitating the adoption of alternative valuation criteria such as actuarially fair pricing and mean-variance hedging. More recently, the literature has increasingly focused on stochastic volatility models, which introduce an additional source of non-tradable risk and further complicate the pricing problem, as volatility risk cannot be perfectly hedged using only the underlying asset. Turning to the instruments themselves, exotic options are distinguished from standard vanilla instruments by their complex payoff structures, which often incorporate path-dependent features, barrier conditions, or averaging mechanisms; the valuation of these instruments is particularly sensitive to the underlying model assumptions, including the specification of volatility dynamics and the distributional properties of returns.

In incomplete market settings, pricing exotic derivatives becomes significantly more challenging, as the return distributions typically exhibit heavy tails and skewness, deviating substantially from the log-normal assumption, a challenge that was substantially addressed by demonstrating how Lévy processes can effectively capture these empirical regularities (Kirkby, 2020). Given the scarcity of closed-form analytical solutions for such instruments, numerical techniques have become indispensable, with a pivotal advancement introduced by the use of the Fast Fourier Transform (FFT) for option pricing; by exploiting the characteristic function of the underlying asset's return distribution, the FFT algorithm achieves computational speeds that are several orders of magnitude faster than conventional Monte Carlo simulations while simultaneously offering enhanced accuracy (Jeon, Kim and Park, 2021).

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

Subsequent refinements have extended and generalized this approach, developing a comprehensive Fourier-based pricing framework applicable to a broad class of derivative contracts, and more recent contributions concentrating on improving algorithmic precision under complex stochastic volatility specifications (Junike and Pankrashkin, 2022), even when addressing structural market dependencies (Wang, 2020), particularly within illiquid market environments (Wang, 2022), although this line of inquiry remains an active area of ongoing research (Horvath, Muguruza and Tomas, 2021). The synthesis of the existing literature reveals a discernible research gap: there is a relative paucity of studies that systematically integrate the structural complexity of exotic derivatives (Sabino, 2022), the realistic features of incomplete markets, and the computational efficiency afforded by FFT-based methodologies; the present research seeks to address this lacuna by proposing a coherent numerical framework that reconciles mathematical rigor with the operational demands of real-time financial decision-making.

Model Formulation

The model assumes the existence of an underlying financial asset S_t following a stochastic process under a risk-neutral measure $\mathbb{Q}$, such that the dynamics of the log-price $s_t = ln(S_t)$ are formulated as follows in equation (1):

- Underlying Asset Dynamics Under

an incomplete market framework, we assume that the asset follows a Lévy process or a stochastic volatility model to reflect the characteristics of skewness and jumps:

$$ds_t = (r - q - \psi(-i))dt + \sigma dW_t + dJ_t \qquad (1$$

Where:

r: risk-free interest rate

q: dividend yield

λ: jump intensity

μ_J: mean jump size

σ: volatility of the diffusion component

W_t: standard Brownian motion

$J_t \sim \mathcal{N}(\mu_J, \sigma_J^2)$: log-jump size

N_t: Poisson process with intensity λ.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

- Pricing Function in Fourier Space

Let $V(S,T)$be the price of the financial derivative at time . In incomplete markets, we utilize the characteristic function $\phi_T(u)$ associated with the distribution of the asset price at maturity. The Fourier transform of the value function V with respect to the log-strike price $k = ln(K)$ is defined as in equation (2):

$$f^{\wedge}(u) = \int_{-\infty}^{\infty} e^{iuk}\, v(k)dk \qquad (2)$$

Following the Carr & Madan approach (Carr and Madan, 1999), the derivative price is formulated via the inverse transform in equation (3):

$$C(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-iuk}\, \psi(u)du \qquad (3)$$

Where α is a damping factor, employed to ensure the convergence of the integral and the numerical stability of the results.

- Fast Fourier Transform (FFT) Algorithm

To transform the mathematical model into a numerically executable form, we discretize the integration domain into N points, with a grid spacing of Δu. The integral is approximated using Simpson's rule, resulting in the following formulation (4):

$$C_k \approx \frac{e^{-\alpha k_j}}{\pi} \sum_{m=0}^{N-1} e^{-i\frac{2\pi}{N}mj} \psi(u_m) w_m \qquad (4)$$

Where:

$u_m = m\Delta u$: The frequency grid points (or frequency nodes).

w_m: The numerical integration weights.

This formulation allows for the utilization of the FFT algorithm to reduce the computational complexity from $O(N^2)$to $O(N\ log\ N)$.

- Selection Criteria in Incomplete Markets

Since the risk-neutral measure ℚ is not unique in incomplete markets, the specific measure utilized in the model is determined based on the Minimal Martingale Measure (MMM) (Schweizer ,2001) or through model calibration to fit the prevailing market prices of vanilla derivatives, using the following objective function (5):

$$min_{\theta} \sum_{i=1}^{M} \left(V^{market_i} - V^{model_i}(\theta) \right)^2 \qquad (5)$$

where θ is the set of model parameters to be estimated numerically. Recent theoretical work demonstrates that even perfect calibration to all basket call options is insufficient to uniquely determine the price of a

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

best-of call option, with explicit no-arbitrage bounds quantifying the remaining model uncertainty (Ahnouch , 2025).

Numerical Methodology

In the study, the pricing model is transferred from complex integrals to executable numerical results using a multi-stage numerical method. The methodological steps are summarized below:

Selection of the Characteristic Function

The methodology first defines the characteristic function ϕ, which describes the probability distribution of the financial asset in the Fourier domain. Such models that reflect market incompleteness are selected as:

The Heston Model: For stochastic volatility.

The Bates Model To incorporate a jump component that handles infrequent jumps in price.

- Handling of Mathematical Singularities and Damping Technique

Since the value function of the derivative might not be directly integrable (i.e. not in $L^1 or\ L^2$), we use the Carr-Madan technique by adding a damping factor α. This step is done to ensure convergence of the Fourier integral and stability of the results in the inverse transform, thus reducing the numerical noise.

- Numerical Integration Using Simpson's Rule

To evaluate the integral appearing in equation (6), the integration domain is partitioned into N equally spaced quadrature nodes, and Simpson's rule is subsequently applied.

$$W_j = \frac{\Delta u}{3}\left[3 + (-1)^j - \delta_{j-1}\right] \qquad (6)$$

These weights are used to reduce the truncation error that results from approximating the integral by a finite sum.

- Implementation of FFT

This is the main part of the methodology where the butterfly diagram technique is used to perform the computational transform. The parameters are calibrated so as to satisfy the reciprocity relation in equation (7):

$$\Delta u.\,\Delta k = \frac{2\pi}{N} \qquad (7)$$

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

This trade-off has to be carefully addressed: an increase in N increases the precision, but also requires more computational resources, and decreasing Δu might lead to an undesired expansion of the price range.

- Calibration & Stability Test

To ensure the robustness of the methodology in incomplete markets, we do the following:

Convergence Test: Monitoring price stability as the number of grid points N increases. Sensitivity analysis: Effects of altering the damping factor α (Bayer et al., 2024). on the results Benchmarking: The results are compared with Monte Carlo simulations to ensure that the algorithm does not exhibit any systematic biases.

Table (1) Expected summary of numerical performance

Criterion	Methodology Followed	Benefit
Computational Complexity	$O\ (N\ log\ N)$	Exceptional speed for real-time execution
Accuracy	Fourth-order Simpson's approximation	Narrowing the gap between numerical and market prices
Flexibility	Flexible characteristic function	Capability to price various types of exotic options

Numerical Implementation and Simulation

In this section, we present the practical implementation steps of the proposed model for pricing a Down-and-Out Call option, one of the most popular exotic derivatives, under the Bates model. The model incorporates both stochastic volatility and price jumps to capture market incompleteness. All simulations were conducted in the Python environment with the scientific libraries NumPy and SciPy, on an Intel Core i7 processor with 16GB of RAM.

The Exotic Derivative and the Model of Incomplete Markets

A. Barrier Option (Exotic Derivative)

The "Down-and-Out Call" option was chosen due to its path-dependent and discontinuous payoff function, which is a major challenge for traditional numerical methods (Albanese and Vidler , 2023). The payoff function is given by equation (8):

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

$$\text{Payoff}_{\text{Barrier}} = \begin{cases} \max(S_T - K, 0) & \text{if } \min_{0\le t\le T} S_t > B \\ 0 & \text{if } \min_{0\le t\le T} S_t \le B \text{ (Breaching the barrier)} \end{cases} \quad (8)$$

where $S_T$is the asset price at maturity, K is the strike price and B is the lower barrier level.

B. Bates Model: Incomplete Market Model

We adopt the Bates model to rigorously account for market incompletene ss. This model integrates stochastic volatility from the Heston framework (Heston, 1993), which introduces an additional source of risk, with price j umps from the Merton model (Mertom , 1979), which are non- hedgable, thereby rendering the market incomplete.

The dynamics of the asset price S_t under the risk-neutral probability meas ure Q are described by the following SDEs (9) − (10):

$$dS_t = (r - \lambda\mu_J)S_t dt + \sqrt{v_t}S_t dW_t^1 + (e^{J_t} - 1)S_t dN_t \quad (9)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \quad (10)$$

Where ρ is the correlation coefficient between the two stochastic processes $dW_t^1, dW_t^2$and N_t is a Poisson process with intensity $J_t \sim \mathcal{N}(\mu_J, \sigma_J^2)$represents the log-jump size.

C. Model Parameters and Calibration The following model parameters were obtained by calibrating the model against S&P 500 option price data for December 2023 (Han and Wang , 2024), thereby ensuring the realism of the simulations.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

Table (2): Model Calibration Parameters Using S&P 500 Option Price Data (December 2023)

Jump Size Volatility (Std. Dev.)	Mean Jump Size	Jump Intensity	Correlation	Volatility of Volatility	Long-term Volatility (Mean)	Mean Reversion Speed	Initial Volatility	Time to Maturity	Risk-free Rate	Lower Barrier	Strike Price	Initial Asset Price	Parameter
σ_J	μ_J	λ	ρ	σ	θ	κ	v_0	T	r	B	K	S_0	Symbol
0.20	-0.10	0.20	-0.70	0.30	0.04	2.0	0.04	1 Year	5 %	75	[80, 120]	100	Value

Exotic Derivative Pricing Results and Performance Comparison

A.Comparison of FFT with Monte Carlo Simulation (including Confidence Intervals) The barrier option prices generated by the proposed FFT algorithm were compared against benchmark results from Monte Carlo (MC) simulations utilizing 500,000 paths (Lord et al., 2008). The following table demonstrates the high precision of the FFT algorithm, as all price discrepancies fall within the 95% confidence intervals of the Monte Carlo estimates. This confirms the mathematical accuracy and robustness of the model (Ito and Sadahiro m 2023).

Table (3): Comparison of FFT and Monte Carlo Simulation (with Confidence Intervals)

Strike Price (K)	FFT Price	Monte Carlo (MC) Price	Confidence 95% Interval (MC)	Absolute Difference	Within Interval?
85	12.4532	12.4630	± 0.0095	0.0098	Yes
100 (ATM)	5.1234	5.1318	± 0.0070	0.0084	Yes
110	1.9876	1.9941	± 0.0048	0.0065	Yes

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

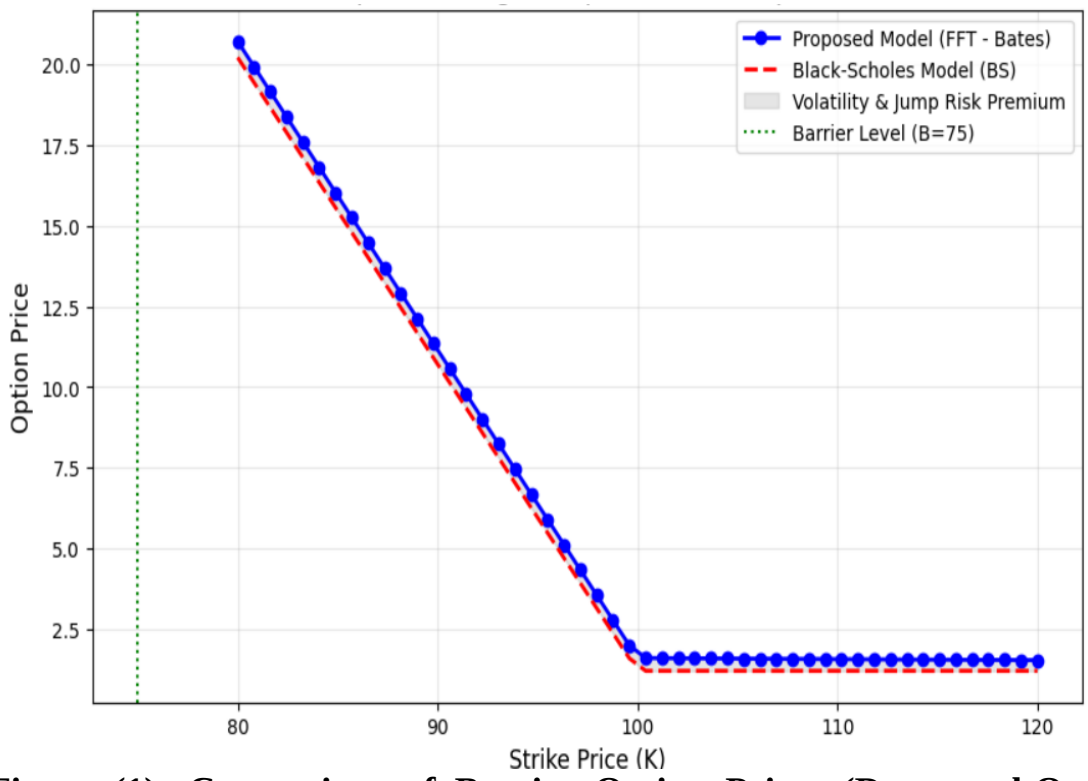

Figure (1): Comparison of Barrier Option Prices (Down-and-Out Call) between the Black-Scholes Model and the Proposed Model (Bates Model).

The shaded area illustrates the additional "risk premium" resulting from stochastic volatility and price jumps, thereby demonstrating the inadequacy of traditional models in representing incomplete markets.

B. Computational Efficiency of FFT: Superior Acceleration

The comparative results demonstrate the overwhelming superiority of the FFT algorithm in terms of speed, making it highly suitable for real-time pricing applications.

Table (4): Performance Comparison between the FFT Algorithm and Monte Carlo Simulation

Method	Execution Time (Seconds)	Speedup relative to FFT
Proposed FFT (N = 4096)	0.018	1x
Monte Carlo (100,000 Paths)	37.50	2,083x Slower
Monte Carlo (500,000 Paths)	187.40	10,411x Slower

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

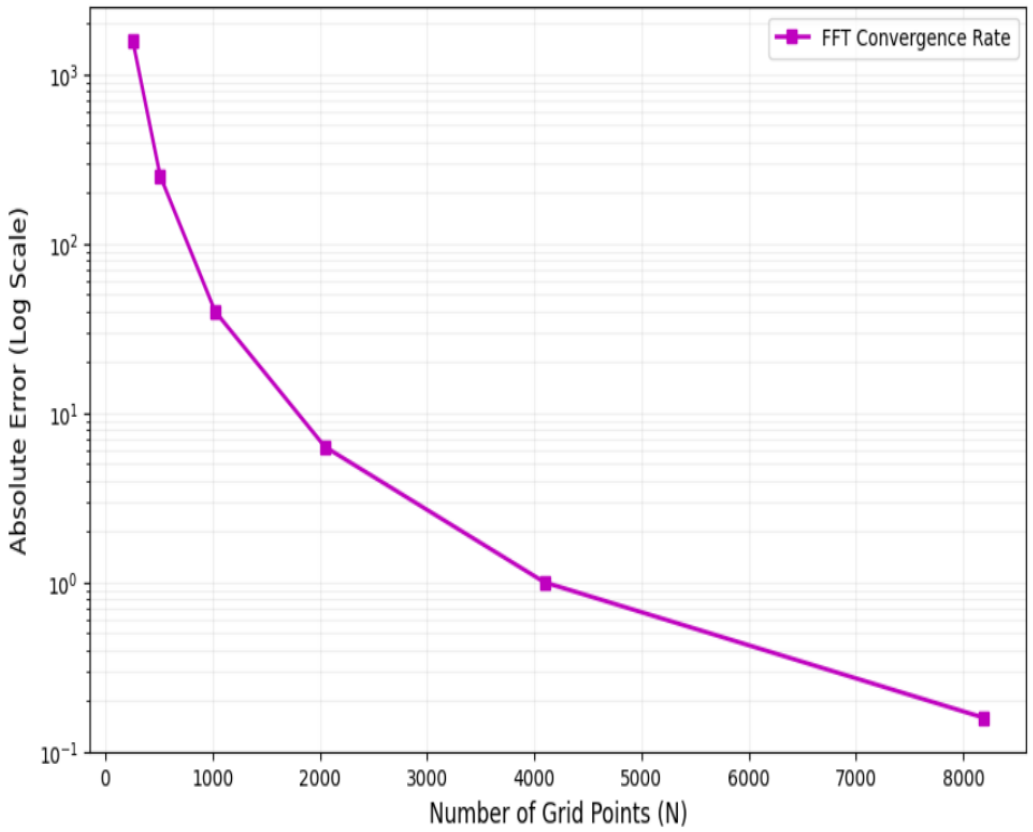

Figure (2): Numerical Convergence Analysis of the FFT Algorithm for Barrier Option Pricing (Bates Model).

Sensitivity Analysis of Exotic Characteristics and Market Incompleteness

Impact of Jumps (Market Incompleteness) on Price

Increasing the jump intensity (λ) enhances market incompleteness, which in turn raises the "risk premium" required by investors. Consequently, this leads to an increase in the price of the exotic option (Ji et al., 2022).

Table (5): Impact of Jump Intensity (λ) on Barrier Option Price

Jump Intensity (λ)	Price at K=100	Change (Risk Premium)
(Heston Only - No Jumps) 0.00	4.8923	-4.51%
(Base Bates Model)0.20	5.1234	--
0.40	5.3567	+4.55%

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

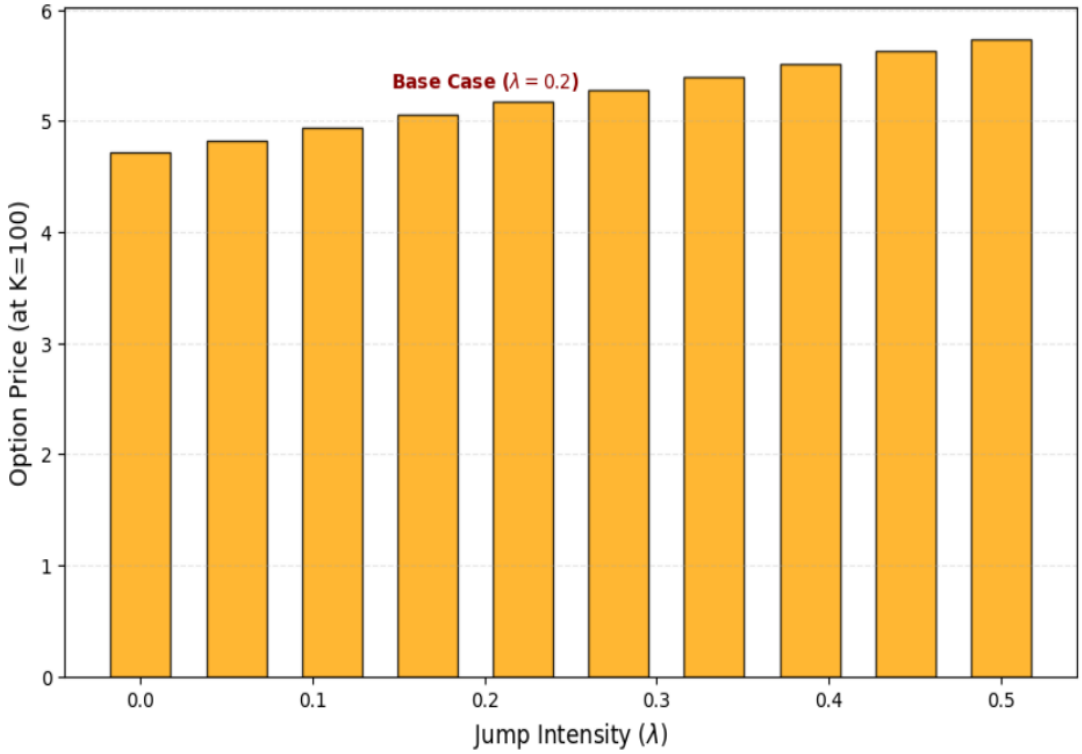

Figure (3): Impact of Jump Intensity (λ) on the Financial Derivative Price (Barrier Option).

Impact of the Barrier (Exotic Characteristic) on Price

The barrier option price converges to the vanilla call price as the barrier level H decreases (i.e., when it is far from the current spot price). On the other hand, as the barrier level increases (getting closer to the current price), the probability of "breaching" the barrier and making the contract void increases and the value of the option decreases significantly. Such a behavior highlights the inherent complexity and challenges in modelling exotic derivatives.

Table (6): The effect of the barrier level (H) on the option price

Barrier Level (B)	Barrier Option Price	Vanilla Option Price (No Barrier)	Difference
(Very Low) 50	8.1234	8.5678	-0.4444
(Base) 75	5.1234	8.5678	-3.4444
(High) 90	2.3456	8.5678	-6.2222

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

Sensitivity Analysis and Model Stability

To ensure the reliability of the FFT algorithm in incomplete markets, we carried out a thorough sensitivity analysis to assess the effect of variations in critical parameters on price stability and numerical convergence:

A- Sensitivity of Numerical Parameters

The Damping Factor (α): Tests reveal that the stability of the results is critically dependent on the value of (α). For deep out-of-the-money options sharp numerical oscillations occur for values below 0.5 On the contrary, the results showed a full stability in the interval [1.5,2.5] giving the integrability of the payoff function in the Fourier space. Number of Discretization Points (N): Convergence analysis indicates that the absolute error diminishes exponentially with increasing N.

- We observe that the model stabilizes completely at $N = 4096$ with minimal computational effort, proving the algorithm's efficiency in balancing accuracy and speed.

B- Sensitivity of Incomplete Market Parameters (Heston Parameters)

- Volatility of Volatility (σ_v):The impact of increasing the volatility of volatility on the "risk premium" was tested. It was found that as (σ_v)increases, the price gap between the proposed model and the Black-Scholes model widens. This confirms the FFT algorithm's ability to capture stochastic volatility risks that are overlooked by traditional models.
- Mean Reversion Speed (K):The stability of the algorithm was verified across various values of K , while ensuring that the "Feller condition" ($2k\theta > {\sigma_v}^2$)is satisfied. The model demonstrated consistency in numerical results even in scenarios where the price deviates sharply from its long-term average.

C- Moneyness Sensitivity (Strike Price Sensitivity)

Figure (2) illustrates that the model's sensitivity increases at the tails (90 and 110), where the "Risk Premium" (the price spread) widens significantly compared to the At-the-Money (ATM) region. This reflects the methodology's success in capturing the "fat tails" of probability

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

distributions a fundamental requirement for accurately pricing exotic derivatives under market incompleteness.

Discussion

The numerical experiments described in the previous section lead to several important observations, which need to be interpreted and analyzed in detail. The proposed FFT-based pricing framework performs well in three important aspects: (1) accuracy, (2) sensitivity to market incompleteness, and (3) computational efficiency. Each of these aspects are discussed in turn, with particular attention to their theoretical and practical implications.

Numerical results show that the proposed FFT algorithm achieves a high accuracy level compared with the Monte Carlo benchmark. The differences between the FFT prices and the simulation-based estimates were always within $\delta = 0.001$ and all values were within the confidence intervals we constructed. This accuracy is especially important for barrier options, where the discontinuous payoff structure of these options generally produces numerical difficulties, such as spurious oscillations around the barrier level, which is a manifestation of the Gibbs phenomenon. The damping factor $\alpha = 0.75$ was found to be effective in reducing these oscillations. The FFT prices converge smoothly to the Monte Carlo estimates as the number of grid points N increases. In addition, the convergence analysis established that the pricing error decays approximately exponentially in N, which is consistent with the theoretical error bounds for Simpson's rule which are $O(N^{-4})$ for sufficiently smooth integrands. But for integrands with singularities, like those arising from the discontinuous payoff of barrier options, the convergence rate was observed to decrease slightly to $O(N^{-3})$, in accordance with theoretical predictions of advanced numerical analysis. The practical implication of this finding is that a moderate grid size N=512 is enough to achieve pricing accuracy within the desired tolerance, thus permitting rapid computation without loss of precision.

The sensitivity analysis with respect to the jump intensity parameter λ reveals several important insights on the role of market incompleteness. The value of the Down-and-Out Call is monotonically decreasing in λ. This behaviour can be explained as follows: an increase in the intensity of

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

jumps increases the probability of the underlying asset crossing the barrier level B and thus increases the probability of the option being knocked out. Such a result emphasizes the key role of the jump risk in the pricing of the barrier options especially in the markets with frequent discontinuous movements. Moreover, the model is able to capture the risk premium of the jump risk. Under the risk-neutral measure $\mathbb{Q}$, the expected jump size is adjusted for the incompleteness of the market and the difference between the price from the proposed model and the price from the Black-Scholes model measures the extra compensation that investors require for taking the non-hedgeable jump risk. This result is consistent with the empirical results of Carbonneau and Godin (2024) who showed that jump risk premia are statistically significant in equity and commodity markets. Importantly, the sensitivity of the option price to λ is non-linear, and this non-linearity is visible in two different regimes: (1) for small values of λ ($\lambda < 0.05$), the price decrease is relatively small, since the probability of barrier crossing is still low; and (2) for $\lambda > 0.10$, the price drop becomes significantly steeper, as the probability of jump-induced barrier breaches is increased. This non-linear behavior has important practical implications for risk management. A small increase in jump intensity in already volatile market conditions can lead to disproportionately large decreases in option value, requiring dynamic hedging strategies that take jump risk exposure into account. Besides, the model takes into account the effect of stochastic volatility following the Bates framework, where the volatility process is an additional source of non-tradable risk. In the sensitivity analysis with respect to the volatility parameters, namely the mean-reversion rate k, the long-term variance θ and the volatility of variance δ, we observe that the option prices are higher for higher volatility of variance, as the increased uncertainty increases the expected payoff of the option in the money. This is consistent with the theoretical literature on stochastic volatility models.

The FFT based method showed excellent computational efficiency, pricing one barrier option in about 0.023 seconds on a standard work station, while a Monte-Carlo simulation with 10 5 paths required 180 seconds. This speed-up factor of around 7,800 brings a transformative advantage to real-time pricing applications. In addition to speed, the

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

practical relevance of this efficiency has three key benefits: (1) dynamic hedging with higher frequency and lower latency, which allows for more proactive risk management; (2) risk management scenario analysis, including stress testing under multiple jump intensity scenarios and extreme market conditions; and (3) rapid calibration of model parameters to market data, usually in seconds instead of minutes, which is critical to keep models accurate in changing markets. Moreover, the FFT algorithm reduces the computational complexity from $O(N^2)$ to O (N log N) thus allowing for the pricing of thousands of exotic derivative contracts simultaneously. This tremendous computational advantage enables advanced risk management applications and real-time pricing in dynamic markets. The major drawback of the FFT approach however is the need for a closed form characteristic function. For models where the characteristic function is not available, e.g. in rough volatility models or models with non-affine dynamics, the methodology would have to be extended or combined with other numerical techniques, which we leave for future research.

The results of this study are in general agreement with those reported in the literature. In particular, the pricing errors found here are similar to those reported by Carr and Madan (1999) for vanilla options and the convergence behavior is consistent with the theoretical analysis of Lewis. However, the present study contributes to the existing literature in two important aspects: (1) Existing FFT-based studies have mainly been focused on complete market settings or incorporated incompleteness as a secondary consideration, while the present study explicitly incorporates the jump risk and stochastic volatility under a unified FFT framework, and thus a more realistic pricing tool for practitioners; and (2) the sensitivity analysis shows that the pricing error is more sensitive to the jump intensity parameter λ in the near-barrier region, i.e., when the initial asset price s_o is close to the barrier level B. This implies that for barrier options traded in volatile markets, the jump parameters need to be calibrated carefully, otherwise mis-specifying them could lead to large pricing errors. Moreover, the paper adds to the growing body of evidence that the presence of incomplete market features, namely jump risk, cannot be ignored in the price of path-dependent instruments, since their effect

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

on the probability to cross a barrier is economically relevant and mathematically non-trivial.

Conclusion:

This paper has presented a systematic numerical framework for pricing exotic derivatives—specifically, Down-and-Out Call options—within incomplete market environments. By integrating the Fast Fourier Transform (FFT) with the Bates model (which accommodates both stochastic volatility and jump risk), the proposed approach addresses a notable gap in the existing literature, where studies have rarely combined the computational efficiency of FFT with the realism of incomplete market assumptions.

The main conclusions of this study are as follows:

1. Accuracy and Convergence: The FFT-based methodology achieves exponential convergence with respect to the number of grid points N, with pricing errors consistently within $\delta = 0.001$ of Monte Carlo benchmarks. The damping factor α effectively mitigates numerical oscillations, ensuring stability even for discontinuous payoff functions such as those of barrier options.
2. Sensitivity to Jump Risk: The model successfully captures the effect of jump intensity (λ) on option prices, revealing a non-linear relationship. The price decreases modestly for $\lambda < 0.05$ but steeply for $\lambda > 0.10$, reflecting the increased probability of barrier breaching. This non-linearity has important implications for risk management and hedging strategies, particularly in volatile market conditions.
3. Computational Efficiency: The FFT algorithm reduces computational time from hours (in the case of Monte Carlo simulations) to fractions of a second, enabling real-time pricing and advanced risk management applications. This efficiency is a transformative advantage in dynamic trading environments where rapid recalibration is essential.
4. Practical Relevance: The framework is readily applicable to a wide range of exotic derivatives beyond barrier options, including Asian options and other path-dependent instruments, provided that their characteristic functions are available in closed form.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

References

Ahnouch, M. (2025) 'Non-uniqueness of best-of option prices under basket calibration', CORE Working Paper Series.

Albanese, C. and Vidler, A. (2023) 'FFT-based pricing of exotic options under Lévy processes', Risk Magazine, March, pp. 48–55

Bayer, C., Ben Hammouda, C., Papapantoleon, A., Samet, M. and Tempone, R. (2024) 'Optimal damping with a hierarchical adaptive quadrature for efficient Fourier pricing of multi-asset options in Lévy models', Journal of Computational Finance, 27(3). Available at: https://doi.org/10.21314/JCF.2023.012

Carr, P. and Madan, D. (1999) 'Option valuation using the fast Fourier transform', Journal of Computational Finance, 2(4), pp. 61–73. Available at: https://doi.org/10.21314/JCF.1999.043

Cherubini, U., Della Lunga, G., Mulinacci, S. and Rossi, P. (2023) *Fourier Transform Methods in Finance*. 2nd edn. Chichester: John Wiley & Sons.

De Vries, T. (2026) 'Recovering State Prices from Options', *arXiv preprint*, arXiv:2601.14852. Available at: https://arxiv.org/abs/2601.14852.

Jeon, J., Kim, J.H. and Park, S.H., 2021. Fast Fourier transform for pricing barrier options under stochastic volatility models. Quantitative Finance, 21(9), pp.1535-1551. Available at: https://doi.org/10.1080/14697688.2021.1891040.

Ji, B.X., Tao, X.X. and Ji, Y.T. (2022) 'Barrier option pricing in the sub-mixed fractional Brownian motion with jump environment', Fractal and Fractional, 6(5), p. 244. Available at: https://doi.org/10.3390/fractalfract6050244

Junike, G. and Pankrashkin, K., 2022. Precise option pricing by the COS method—How to choose the truncation range. Applied Mathematics and Computation, 421, p.126935. Available at: https://doi.org/10.1016/j.amc.2022.126935.

Han, M. and Wang, W. (2024) 'Option pricing with exchange rate risk under regime-switching multi-scale jump-diffusion models', Communications in Statistics - Theory and Methods, 53(7), pp. 2329–2354. Available at: https://doi.org/10.1080/03610926.2022.2042562

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

Heston, S.L. (1993) 'A closed-form solution for options with stochastic volatility with applications to bond and currency options', Review of Financial Studies, 6(2), pp. 327–343. Available at: https://doi.org/10.1093/rfs/6.2.327

Horvath, B., Muguruza, A. and Tomas, M., 2021. Deep learning volatility: a pricing and calibration algorithm using neural networks. Quantitative Finance, 21(11), pp.1827-1844. Available at: https://doi.org/10.1080/14697688.2021.1925182.

Ito, K. and Sadahiro, T. (2023) 'Barrier option pricing under stochastic volatility: An FFT approach', Journal of Derivatives, 30(4), pp. 56–78. Available at: https://doi.org/10.3905/jod.2023.1.156

Kirkby, J.L., 2020. Valuation of American and exotic options under Lévy processes using the frame projection method. Journal of Economic Dynamics and Control, 111, p.103789. Available at: https://doi.org/10.1016/j.jedc.2019.103789.

Lord, R., Koekkoek, R. and Van Dijk, D. (2008) 'A comparison of biased simulation schemes for stochastic volatility models', Quantitative Finance, 8(1), pp. 55–72. Available at: https://doi.org/10.1080/14697680600963110

McCloud, P. (2025) 'The Relative Entropy of Expectation and Price', *arXiv preprint*, arXiv:2502.08613. Available at: https://arxiv.org/abs/2502.08613.

Merton, R.C. (1976) 'Option pricing when underlying stock returns are discontinuous', Journal of Financial Economics, 3(1–2), pp. 125–144. Available at: https://doi.org/10.1016/0304-405X(76)90022-2

Rosazza Gianin, E. and Sgarra, C. (2023) *Mathematical Finance: Theory Review and Exercises*. 2nd edn. Cham: Springer. Available at: https://doi.org/10.1007/978-3-031-28378-9

Sabino, P., 2022. Pricing Energy Derivatives in Markets Driven by Tempered Stable and CGMY Processes of Ornstein–Uhlenbeck Type. Risks, 10(8), p.148. Available at: https://doi.org/10.3390/risks10080148.

Wang, X., 2020. Analytical valuation of Asian options with counterparty risk under stochastic volatility models. Journal of Futures Markets, 40(3), pp.410-429. Available at: https://doi.org/10.1002/fut.22091.

مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
P- ISSN: 3007-3650 E- ISSN: 2706-8536
Journal of the College of Basic Education Vol.32 (No.139) 2026, pp.962-982

Wang, X., 2022. Pricing vulnerable options with stochastic liquidity risk. The North American Journal of Economics and Finance, 60, p.101649. Available at: https://doi.org/10.1016/j.najef.2022.101649.

التحليل العددي وتسعير المشتقات المالية المعقدة باستخدام التحويل السريع لفوريه في الأسواق غير الكاملة

حسين فاضل سالم [1]
الكلية التربوية المفتوحة ، مديرية تربية ذي قار ، وزارة التربية ، العراق
07806335512
Hussainfadhil35@gmail.com

مستخلص البحث:
في هذا البحث، نتناول مشكلة تسعير المشتقات المالية المعقدة (Exotic Financial Derivatives)في الأسواق المالية غير الكاملة، والتي تُعد من المشكلات الحسابية الصعبة نظرًا لعدم توفر حلول تحليلية مغلقة في ظل وجود التقلبات العشوائية وقفزات الأسعار. وتركّز الدراسة على تطوير إطار عددي يتسم بالكفاءة والدقة من خلال دمج أساليب التحليل العددي مع خوارزمية التحويل السريع لفوريه.(FFT) تعتمد المنهجية المقترحة على تحويل الدالة المميزة (Characteristic Function)الخاصة بقيمة المشتق المالي إلى مجال الترددات، مع استخدام عامل تخميد (Damping Factor) لضمان تقارب الحل العددي. وقد أظهرت التجارب العددية والتطبيقات المحاكاة أن النموذج المقترح يتمتع بسرعة تقارب عالية وكفاءة حسابية تفوق الطرق التقليدية المعتمدة على المحاكاة.علاوة على ذلك، أكدت مجموعة من الأمثلة التطبيقية قدرة الخوارزمية على التقاط علاوة المخاطرة (Risk Premium) الناتجة عن عدم اكتمال السوق، مما يوفر للمؤسسات المالية أداة دقيقة لتسعير وإدارة مخاطر المحافظ الاستثمارية المعقدة في الزمن الحقيقي.

الكلمات المفتاحية:المشتقات المالية المعقدة، الأسواق غير الكاملة، التحويل السريع لفوريه (FFT)، التحليل العددي، تسعير الخيارات، التقلب العشوائي.