The Nonlinear Delay Second Order Eigenvalue Problems Consist Of Delay Ordinary Differential Inequalities

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Abstract

The aim of this paper is to study the nonlinear delay second order eigenvalue problems consist of delay ordinary differential inequalities, one of the expansion methods that called the least square method will be developed to solve this type of problems.

Introduction

The nonlinear delay second order eigenvalue problems consist of delay nonlinear ordinary differential inequalities with the boundary conditions defined on some interval, have many applications in different scientific fields, physical, biological and engineering science, also it is one of the most important application referred to as a delay nonlinear eigenvalue problem, [1].

This delay eigenvalue problem belongs to a wide class of problems whose eigenvalues and eigen-functions have particularly nice properties, [2].

In this paper we study and solve this type of problems using the least square method.

1. Preliminaries

In this section some basic definitions and remarks that needed in this work are recalled. We start with the following definition.

1.1 Definition

A delay differential inequality is an inequality in which the unknown function and some of its derivatives, evaluated at arguments which are different by any of fixed number or function of values.

Consider the n-th order delay differential inequality:

$$\mathrm{K}(x, y(x), y(x-\tau_1),\ldots, y(x-\tau_m), y'(x), y'(x-\tau_1),\ldots, y'(x-\tau_m), y^{(n)}(x),\ldots, y^{(n)}(x-\tau_m)) \geq f(x) \qquad [1.1]$$

where K is a given function and $\tau_1, \tau_2, \ldots, \tau_m$ are given fixed positive numbers called the time delays, [1].

We say that inequality[1.1] is homogenous delay differential inequality in case $f(x) \geq 0$, which we handle in this paper, otherwise this inequality is called non-homogenous delay differential inequality,[2].

1.2 Definition

The delay differential inequality is said to be nonlinear when it is nonlinear with respect to the unknown function that enter with different arguments and their derivatives that appeared in it, [1].

Hence, the new concepts of this work is given by the following definition.

1.3 Definition

The delay eigen-value problem consist of delay ordinary differential inequality is said to be nonlinear when it is nonlinear with respect to the unknown eigen-function enter with different arguments and their derivatives that appeared in it.

Next, consider the following nonlinear delay second order eigen-value problem:

$$-(p(x)y'(x))' + q(x)y(x-\tau) - f(x, \lambda, y(x-\tau)) \geq 0 \qquad [1.2]$$

with the associated boundary conditions:

$$a_1 y(a) + a_2 y'(a) \geq 0 \quad , x \in [\, a-\tau, a] \qquad [1.3]$$
$$b_1 y(b) + b_2 y'(b) \geq 0 \quad , x \in [\, b-\tau, b]$$
$$y(x-\tau) \geq \varphi(x-\tau), \text{if } \; x-\tau < a$$

where p, p' and q are given real-valued continuous functions defined on the interval $[a,b]$, p is positive, not both coefficients in one condition are zero, $\tau > 0$ is the time delay, f is a known nonlinear function with respect to y. φ is the initial function defined on $x \in [x_0 - \tau, x_0]$. The problem here is to determine the eigen-value λ in which a nontrivial solution y for the problem given by inequalities [1.2]-[1.3] occurs. In this case λ is said to be a delay eigenvalue and y is the associated delay eigen-function.

In other words y is an eigen function for the variable x and the nonlinear function $f(x, \lambda, y(x-\tau))$ with respect to the eigen-value λ.

Like the linear second order eigenvalue problems, the problem given by inequalities [1.2]-[1.3] satisfies the following remarks, [2].

1.4 Remarks

1. The linear delay operator: $L \geq -\frac{d^2}{dx^2} p(x) - \frac{d}{dx} p'(x) + A(x)q(x)$, where $A(x)$ is an operator defined by $A(x)y(x) \geq y(x-\tau)$, is self- adjoint, [3] .

2. The delay eigen-functions are orthogonal.

3. There are infinite number of delay eigenvalues forming a monotone increasing sequence with $\lambda_j \to \infty$ as $j \to \infty$**.** Moreover, the delay eigen-functions corresponding to the delay eigenvalues has exactly j roots on the interval (a,b)**.**

4. The delay eigen-functions are complete and normal in $L^2[a,b]$.

5. Each delay eigenvalue corresponds only one delay eigen-function in $L^2[a,b]$.

To check remarks (2-5), see [4]**.**

2. The Least-Square Method

This method is one of the expansion methods that used to solve the linear (nonlinear) differential equations and inequalities with or without delays, [5],[6].

Here we develop this method to solve the problem given by inequalities[1.1][1.2].

The method is based on approximating the unknown function y as a linear combination of n linearly independent functions $\{\phi_i\}^n{}_{i=1}$, that is write

$$y \geq \sum_{i=1}^{n} \phi_i(x) \qquad [2.1]$$

which implies that, $y(x-\tau) \geq \sum_{i=1}^{n} \phi_i(x-\tau)$

this approximated solution must satisfy the boundary conditions given by inequalities [1.2] to get a new approximated solution. By substituting this approximated solution into inequality [1.1] one can get:

$$R(x,\lambda,\vec{c}) \geq -(p(x)\sum_{i=1}^{n} \phi_i'(x))' + q(x)\sum_{i=1}^{n} \phi_i(x-\tau) - f(x,\lambda,\sum_{i=1}^{n} \phi_i(x-\tau)) \quad [2.2]$$

where R is the error in the approximation of inequality[2.2]and $\vec{c}$ is the vector of $n-2$
elements of c_i, $i=1,2,\ldots,n$, [7], [8]

Thus, to minimize the functional:

$$J(\lambda,\vec{c}) \geq \int_a^b (R(x,\lambda,\vec{c}))^2\, dx \qquad [2.3]$$

put $\frac{\partial J}{\partial \lambda} \geq \frac{\partial J}{\partial c_i} \geq 0$, $i = 1,2,\ldots,n$, to get a system of $n-1$ nonlinear inequalities with $n-1$ unknowns which can be solved by any suitable method to get the values of λ and $\vec{c}$,[9],[10].

To illustrate this method, consider the following example:

2.1 Example

Consider the following nonlinear delay eigenvalue problem:

$$-(xy'(x))' + 2x\, y(x-1) - \lambda(y^2(x-1) - 0.5) \geq 0 \qquad [2.4]$$

with the associated boundary conditions:

$$y(1) \geq y'(1) \quad , \; x \in [0,1]$$
$$y(2) \geq 2\, y'(2) \, , \; x \in [1,2] \qquad [2.5]$$
$$y(x-1) \geq x-1$$

we use the least-square method to solve this problem. To do this, we approximate the unknown function y as a polynomial of degree three, that is, write

$$y(x) \geq \sum_{i=1}^{4} c_i x^{i-1}$$

But, this approximated solution must satisfy the boundary conditions given by inequalities [2.5], thus this approximated solution reduces to

$$y(x) \geq c_1 + c_1 x - 6c_4 x^2 + c_4 x^3$$

From which, we have

$$y(x-1) \geq c_1 + c_1(x-1) - 6c_4(x-1)^2 + c_4(x-1)^3$$

By substituting this approximated solution into inequality [2.4], we obtain

$$\begin{aligned} R(x,\lambda,c_1,c_4) \geq & -x(-18c_4 + 6c_4 x) - (c_1 - 18c_4 x - 9c_4 + 3c_4 x^2) \\ & + 2x(c_1 + c_1(x-1) - 6c_4(x-1)^2 + c_4(x-1)^3) \\ & - \lambda[(c_1 + c_1(x-1) - 6c_4(x-1)^2 + c_4(x-1)^3)^2 - \frac{1}{2}] \end{aligned}$$

Thus, if we minimize the functional:

$$J(\lambda,c_1,c_4) \geq \int_1^2 (R(x,\lambda,c_1,c_4))^2\,dx$$

set $\frac{\partial J}{\partial \lambda} \geq \frac{\partial J}{\partial c_1} \geq \frac{\partial J}{\partial c_4} \geq 0$ to get the following system of nonlinear inequalities:

$$\frac{\partial}{\partial \lambda}\int_1^2 (R(x,\lambda,c_1,c_4))^2\,dx \geq 0$$

$$\frac{\partial}{\partial c_1}\int_1^2 (R(x,\lambda,c_1,c_4))^2\,dx \geq 0$$

$$\frac{\partial}{\partial c_4}\int_1^2 (R(x,\lambda,c_1,c_4))^2\,dx \geq 0$$

Solving The above system by any suitable method, to find that the nontrivial solution is $\lambda \geq 2$, $c_1 \geq 1$ and $c_4 \geq 0$. Therefore 2 is delay eigenvalue with the corresponding delay eigen-function

$$y(x-1) \geq 1 + (x-1),\ x \in [1,2].$$

Generally, if $y(x) \geq \sum_{i=1}^{n} c_i x^{i-1}$, then the same result can be obtained for all values of n, $n \in N$.

That is if $y(x-\tau) \geq \sum_{i=1}^{n} c_i (x-\tau)^{i-1}$, then $y(x-1) \geq 1 + (x-1)$, $x \in [1,2]$ corresponding to the same delay eigenvalue.

Hint:

The above problem has been solved with the help of Math-Cad software package program, programs are so easy that omitted.

Results

1. The nonlinear second order delay eigenvalue problems consist of delay nonlinear ordinary differential inequalities satisfies the same properties as those consist of delay nonlinear ordinary differential equations with or with out delay.
2. The least square method can be developed to solve the above type of problems.

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مســـائل القيم الذاتية التباطؤيه اللاخطية المحتوية على متباينات تفاضلية اعتيادية تباطؤية ثنائية الرتبة

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الخلاصـــــــة

الهدف من هذا البحث هو دراسة مسائل القيم الذاتية التباطؤيه اللاخطية المحتوية على متباينات تفاضلية

اعتيادية تباطؤية ثنائية الرتبة و تطوير واحدة من الطـــرق التوســـعية وهـــي طريقـــة المربعات الصغرى لحل هذا النوع من المسائل.