Numerical Solution of Non-linear Delay Differential Equations Using Semi Analytic Iterative Method

Asmaa A. Aswhad[1] **Samaher M. Yassein[2*]**

[1,2]Department of Mathematics, College of Education for pure Sciences /Ibn Al - Haitham , University of Baghdad, Iraq .

Email:assmaaaswhad@gmail.com, samamarez@yahoo.com

Abstract:

We present a reliable algorithm for solving, homogeneous or inhomogeneous, nonlinear ordinary delay differential equations with initial conditions. The form of the solution is calculated as a series with easily computable components. Four examples are considered for the numerical illustrations of this method. The results reveal that the semi analytic iterative method (SAIM) is very effective, simple and very close to the exact solution demonstrate reliability and efficiency of this method for such problems.

Keywords: Semi analytic iterative method, Ordinary delay differential equations, Initial conditions.

1. Introduction:

In many practical problems, we come across with a differential equation which cannot be solved by one of the standard known methods .Various methods have introduced for obtaining numerical solution to these problems with a high degree of accuracy. One of these problems is the delay differential equations DDEs. Real life problems of delay differential equations are of great importance .One reason for this importance is that they describe processes with “after effects”, that is, time lag (if the independent variable denotes time), In fact, many applications in the field of mechanics, physics, biology, economics, ecology models and others were expressed in terms of delay differential equations such as control systems, mixing of liquids, population growth, etc. see [1].

The delay differential equation DDE could be written in the form [2],

$$\left.\begin{array}{ll} L\,y(t) = F(t, y(t), y(g(t)))\,, & t \geq 0 \\ y^{(i)}(0) = y_0^{(i)} & i = 0,1,2,\ldots,n-1 \\ y(t) = \emptyset(t)\,, & t < t_0 \end{array}\right\} \quad \ldots(1)$$

As equivalently:

$$L\,y(t) = F(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_k))$$

The tardiness τ_i are always non negative, it may be positive constants or functions of t or functions of t and y itself. And L is a differential operator given by $L(\cdot) = \frac{d^n(\cdot)}{dt^n}$.

Basically, delay differential equations can be classified according to the complexity of the phenomenon, namely,

(i) DDE with constant delay

(ii) DDE with variable or time dependent delay

(iii) DDE with case dependent delay

DDEs were solved analytically and numerically using several methods see [3, 4, 5, 6, 7]. In this paper, we solve some DDEs of type (ii) with initial conditions like Pantograph Equation using semi-analytical iterative method SAIM which is a new technique developed by Temimi and Ansari to solve an ordinary differential equations with boundary conditions [8] and it has been proved that its reliable, promising and powerful .The attractive property of this method is directly executed in a manner straightforward without utilizing linearization, discretization and restrictive assumptions. The approximate solution we obtain is more accurate than the other solutions that can be obtained using other approximate methods.

2. Description of The Method:

This section describes the main steps of the semi analytical iterative method. It is known that any differential equation can be written as :

$$L(y(t)) + N(y(t)) + f(t) = 0 \,, \qquad (2)$$

with boundary conditions $B\left(y, \frac{dy}{dt}\right) = 0$,

where t is the dependent variable, $y(t)$ is an unknown function, f(t) is a known function, L is the differential operator that we have already mentioned, N is a non-linear operator and B is a boundary operator.

To find out how this method works, the following steps as follows:

Step 1 : solving

$$L(y_0(t)) + f(t) = 0 \quad , \quad B(y_0, \tfrac{dy_0}{dt}) = 0 \qquad (3)$$

to get $y_0(t)$

Step 2 : The next iterate is

$$L(y_1(t)) + f(t) + N(y_0(t)) = 0 \,, \quad B(y_1, \tfrac{dy_1}{dt}) = 0 \qquad (4)$$

By solving this equation, we get $y_1(t)$

Step (3) : we construct a simple iterative to get the solution of (2) , which is

$$L(y_{n+1}(t)) + f(t) + N(y_n(t)) = 0 \quad , B(y_{n+1}, \tfrac{dy_{n+1}}{dt}) = 0 \qquad (5)$$

The solution of this equation gives the approximate solution of (2).

This method is characterized by being easy in application, economical in time and gives accurate results compared with other numerical methods mentioned in [3, 4, 5, 6] as we shall see next section.

3. Numerical Examples:

In this section, we will solve some examples of delay differential equations of type (ii) were the delay is a variable and their formula as follows

$$L\left(u(t)\right) = F\left(t, u(t), u(kt)\right) \quad , \quad u(t) \in R^d$$

where $kt = t - \tau(t)$, the delay $\tau(t) \geq 0$

is a given function and $k \in (0,1)$ we use SAIM and MATLAB for a very accurate approximate solution and to demonstrate the efficiency of the method .

Example 1 :

Consider the non - linear delay differential equation of first- order [3,5]:

$y'(x) = 1 - 2y^2\left(\frac{x}{2}\right)$, $0 \leq x \leq 1$, with initial condtion (IC) $y(0) = 0$, $x \leq 0$.

The exact solution is $y(x) = \sin(x)$.

Firstly, to find y_0 we take

$$L(y_0) = 1 \quad \text{with } y_0(0) = 0 \quad \text{where } f(x) = -1 , \quad L(y) = \frac{dy}{dx} , \ N(y) = 2y^2$$

By applying SAIM we get: $y_0(x) = x$. The next step is finding y_1 as follows:

$y_1'(x) = 1 - 2y_0^2\left(\frac{x}{2}\right)$, where $y_1(0) = 0$ and it has the solution $y_1(x) = x - \frac{1}{6}x^3$ .Next, to find y_2 we start with $y_2'(x) = 1 - 2y_1^2\left(\frac{x}{2}\right)$, thus $y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{8064}x^7$

Continuing by the same way and with using MATLAB we get y_3, y_4, ..., and so on. In this example, we found the solution till y_5 .

$$\begin{aligned} \mathbf{y_5}(X) = (&-6.095896929966372 \times 10^{(-93)} \times X^{63} + 2.671770599188598 \times 10^{(-88)} \times X^{61} \\ &- 5.605853743421019 \times 10^{(-84)} \times X^{59} + 7.497413466328212 \times 10^{(-80)} \times X^{57} - 7.180072888325093 \\ &\times 10^{(-76)} \times X^{55} + 5.243784704578997 \times 10^{(-72)} \times X^{53} - 3.037079991690984 \\ &\times 10^{(-68)} \times X^{51} + 1.43219101490978 \times 10^{(-64)} \times X^{49} - 5.602220517779528 \times 10^{(-61)} \times X^{47} \\ &+ 1.842499296894744 \times 10^{(-57)} \times X^{45} - 5.145889127563266 \times 10^{(-54)} \times X^{43} + 1.229346977727265 \\ &\times 10^{(-50)} \times X^{41} - 2.525135268948063 \times 10^{(-47)} \times X^{39} + 4.474636418016095 \\ &\times 10^{(-44)} \times X^{37} - 6.853033979710528 \times 10^{(-41)} \times X^{35} + 9.074404837606141 \\ &\times 10^{(-38)} \times X^{33} - 1.037773515059482 \times 10^{(-34)} \times X^{31} + 1.022438458957909 \\ &\times 10^{(-31)} \times X^{29} - 8.642417253090528 \times 10^{(-29)} \times X^{27} + 6.2307289373411 \\ &\times 10^{(-26)} \times X^{25} - 3.800974898952346 \times 10^{(-23)} \times X^{23} + 1.941689121991593 \\ &\times 10^{(-20)} \times X^{21} - 8.195071420305914 \times 10^{(-18)} \times X^{19} + 2.808754947461774 \\ &\times 10^{(-15)} \times X^{17} - 7.645590344845466 \times 10^{(-13)} \times X^{15} + 1.605869924712442 \times 10^{(-10)} \\ &\times X^{13} - 2.505210838560021 \times 10^{(-8)} \times X^{11} + 2.755731922398589 \times 10^{(-6)} \\ &\times X^9 - 1.984126984133638 \times 10^{(-4)} \times X^7 + 0.008333333333333333 X^5 - 0.1666666666678793 X^3 + X). \end{aligned}$$

TABLE I: Comparison of numerical results for Example 1

x	Exact solution	SAIM	Error SAIM	Error ADM
0	0	0	0	-

0.1	0.099833416646828	0.099833416646828	0	-
0.2	0.198669330795061	0.198669330795061	0	$-1.E^{-10}$
0.3	0.295520206661340	0.295520206661340	0	-
0.4	0.389418342308651	0.389418342308651	0	$1.E^{-10}$
0.5	0.479425538604203	0.479425538604203	0	-
0.6	0.564642473395035	0.564642473395035	0	$9.99999E^{-05}$
0.7	0.644217687237691	0.644217687237691	0	-
0.8	0.717356090899523	0.717356090899523	0	$-3.E^{-09}$
0.9	0.783326909627483	0.783326909627483	$1.E^{-016}$	-
1	0.841470984807897	0.841470984807893	$3. E^{-016}$	-

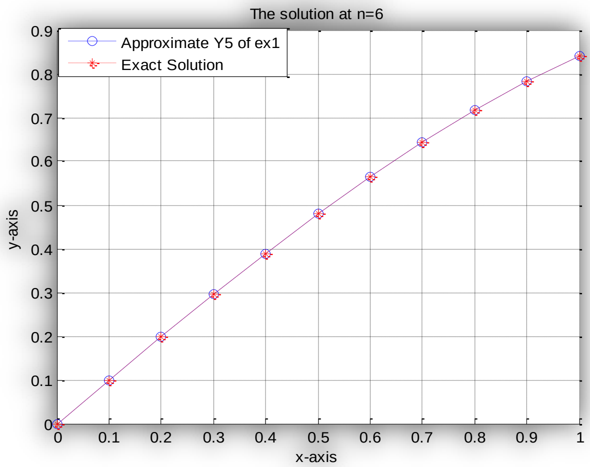

Figure 1: Comparison between the exact and semi analytic iterative solution y_5

Example 2 :

Consider the non – linear delay differential equation of second – order [6]:

$$y''(x) = 5y^2\left(\frac{x}{2}\right) - y(x) \;, \qquad x \geq 0 \quad , \quad \text{with ICs} \quad y(0) = 1 \,,\, y'(0) = -2 \quad .$$

The exact solution is $y(x) = e^{-2x}$.

By the same way used in the previous example, to find y_0 we take

$$L(y_0) = 0 \quad \text{with} \;\; y_0(0) = 1, \quad \text{where } f(x) = 0 \,, L(y) = \frac{d^2y}{dx^2} \,, N(y) = -5y^2\left(\frac{x}{2}\right)$$

By applying SAIM we get: $y_0(x) = 1 - 2x$. The next step is finding y_1 as follows:

$y_1''(x) = 5y_0^2\left(\frac{x}{2}\right) - y_0(x)$ where $y_1(0) = 1$ and its solution is

$y_1(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{5}{12}x^4$ and

$$y_2(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \frac{53}{576}x^6 - \frac{5}{192}x^7 + \frac{155}{32256}x^8 - \frac{25}{41472}x^9 + \frac{25}{663552}x^{10}$$

.
.
.

$$y_5(x) = (1.003210644856261\times 10^{-130}\times x^{94} - 1.207063047898491\times 10^{-127}\times x^{93} + 7.272496886461207\times 10^{-125}\times x^{92} - 2.926659905296487\times 10^{-122}\times x^{91} + 8.8506741031058\times 10^{-120}\times x^{90} - 2.145128837514115\times 10^{-117}\times x^{89} + 4.339139585186412\times 10^{-115}\times x^{88} - 7.531908846102301\times 10^{-113}\times x^{87} + 1.144848164247287\times 10^{-110}\times x^{86} - 1.547432766920442\times 10^{-108}\times x^{85} + 1.882544407574024\times 10^{-106}\times x^{84} - 2.081539860544082\times 10^{-104}\times x^{83} + 2.108759112778537\times 10^{-102}\times x^{82} - 1.970663402743142\times 10^{-100}\times x^{81} + 1.708650401145774\times 10^{-98}\times x^{80} - 1.381413562819964\times 10^{-96}\times x^{79} + + 1.0459878605186\times 10^{-94}\times x^{78} - 7.446313016582143\times 10^{-93}\times x^{77} + 5.001032124803115\times 10^{-91}\times x^{76} - 3.178438766360305\times 10^{-89}\times x^{75} + 1.916891540861769\times 10^{-87}\times x^{74} - 1.099722242736953\times 10^{-85}\times x^{73} + 6.014997481475331\times 10^{-84}\times x^{72} - 3.142861302756151\times 10^{-82}\times x^{71} + 1.571587689141244\times 10^{-80}\times x^{70} - 7.533340708775093\times 10^{-79}\times x^{69} + 3.46671800347169\times 10^{-77}\times x^{68} - 1.533627556488312\times 10^{-75}\times x^{67} + 6.530247040904384\times 10^{-74}\times x^{66} - 2.679420902507135\times 10^{-72}\times x^{65} + 1.060493425739695\times 10^{-70}\times x^{64} - 4.052751841390618\times 10^{-69}\times x^{63} + 1.496769680189692\times 10^{-67}\times x^{62} - 5.346652965861534\times 10^{-66}\times x^{61} + 1.848685251753015\times 10^{-64}\times x^{60} - 6.191607449834464\times 10^{-63}\times x^{59} + 2.009937510807733\times 10^{-61}\times x^{58} - 6.327806209265962\times 10^{-60}\times x^{57} + 1.933041954276341\times 10^{-58}\times x^{56} - 5.732528578638099\times 10^{-57}\times x^{55} + 1.65097866935846\times 10^{-55}\times x^{54} - 4.619300982444045\times 10^{-54}\times x^{53} + 1.255967756039612\times 10^{-52}\times x^{52} - 3.319389588358212\times 10^{-51}\times x^{51} + 8.529270489084957\times 10^{-50}\times x^{50} - 2.13121059323183\times 10^{-48}\times x^{49} + 5.179439805645478\times 10^{-47}\times x^{48} - 1.224501854465245\times 10^{-45}\times x^{47} + 2.816649635279159\times 10^{-44}\times x^{46} - 6.30484178378797\times 10^{-43}\times x^{45} + 1.373528874368371\times 10^{-41}\times x^{44} - 2.912314614436646\times 10^{-40}\times x^{43} + 6.009254051649558\times 10^{-39}\times x^{42} - 1.206238869564025\times 10^{-37}\times x^{41} + 2.354085963546244\times 10^{-36}\times x^{40} - 4.463281926644799\times 10^{-35}\times x^{39} + 8.214293982076312\times 10^{-34}\times x^{38} - 1.466410387243683\times 10^{-32}\times x^{37} + 2.538140847311299\times 10^{-31}\times x^{36} - 4.259444046671711\times 10^{-30}\times x^{35} + 6.934863133632889\times 10^{-29}\times x^{34} - 1.096871624449234\times 10^{-27}\times x^{33} + 1.688824279207416\times 10^{-26}\times x^{32} - 2.53728526132076\times 10^{-25}\times x^{31} + 3.728315647445131\times 10^{-24}\times x^{30} - 5.366944611380232\times 10^{-23}\times x^{29} + 7.571337265238722\times 10^{-22}\times x^{28} - 1.045344404374061\times 10^{-20}\times x^{27} + 1.408527706267635\times 10^{-19}\times x^{26} - 1.846019669655569\times 10^{-18}\times x^{25} + 2.345827020722545\times 10^{-17}\times x^{24} - 2.881294060331896\times 10^{-16}\times x^{23} + 3.406701881054071\times 10^{-15}\times x^{22} - 3.856534018262755\times 10^{-14}\times x^{21} + 4.15605314005823\times 10^{-13}\times x^{20} - 4.243012304357213\times 10^{-12}\times x^{19} + 4.091218551092708\times 10^{-11}\times x^{18} - 3.717492363475381\times 10^{-10}\times x^{17} + 3.169772681157372\times 10^{-9}\times x^{16} - 2.529546837401128\times 10^{-8}\times x^{15} + 1.888018260006569\times 10^{-7}\times x^{14} - 1.315611148466900\times 10^{-6}\times x^{13} + 0.000008547348350083008x^{12} - 0.00005130671797370923x^{11} + 0.0002821869488536155x^{10} - 0.00141093474425702x^{9} + 0.006349206349206349x^{8} - 0.02539682539691057x^{7} + 0.08888888888888889x^{6} - 0.2666666666700621x^{5} + 0.6666666666666667x^{4} - 1.333333333343035x^{3} + 2x^{2} - 2x + 1).$$

TABLE II: Comparison of numerical results for Example 2

x	Exact solution	SAIM	Error
0	1.000000000000000	1.000000000000000	0
0.1	0.818730753077982	0.818730753077982	0
0.2	0.670320046035639	0.670320046035639	0
0.3	0.548811636094027	0.548811636094025	$2.E^{-015}$

0.4	0.449328964117222	0.449328964117160	6.2 E^{-014}
0.5	0.367879441171442	0.367879441170561	8.81 E^{-013}
0.6	0.301194211912202	0.301194211904498	7.704 E^{-012}
0.7	0.246596963941606	0.246596963893739	4.7867 E^{-011}
0.8	0.201896517994655	0.201896517763193	2.31462 E^{-010}
0.9	0.165298888221587	0.165298887298137	9.23450 E^{-009}
1	0.135335283236613	0.135335280073181	3.163432E^{-008}

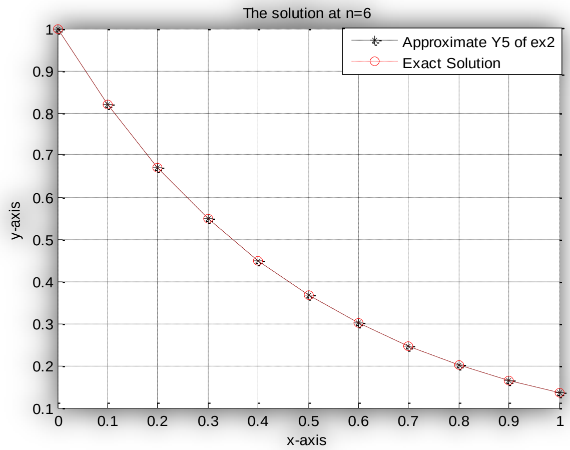

Figure 2: Comparison between the exact and semi analytic iterative solution y_5

Example 3:

Consider the non - linear singular delay differential equation of second- order[6]:

$$y''(x) = y(x) - \frac{8}{x^2}\, y^2\left(\frac{x}{2}\right) \quad , \qquad x \geq 0 \quad , \quad \text{with ICs} \quad y(0) = 0\,, \quad y'(0) = 1 \quad .$$

The exact solution is $y(x) = Xe^{-x}$.

To find y_0 we take $L(y_0) = 0$ with $y_0(0) = 0$ where $f(x) = 0$, $L(y) = \frac{d^2y}{dx^2}$, $N(y) = \frac{8}{x^2} y^2 \left(\frac{x}{2}\right)$

By applying SAIM we get: $y_0(x) = x$, the next step is finding y_1 as follows:

$y_1''(x) = y_0(x) - \frac{8}{x^2}\, y_0^2\left(\frac{x}{2}\right)$ where $y_1(0) = 0$ and has $y_1(x) = x - x^2 + \frac{1}{6}\, x^3$ and

$$y_2(x) = x - x^2 + \frac{1}{2}\, x^3 - \frac{5}{36}\, x^4 + \frac{1}{80}\, x^5 - \frac{1}{8640}\, x^6$$

.

.

.

$$
\begin{aligned}
&y_5(x) = (-2.331260475928636518800603194035 2 \times 10^{-240} \times x^{96} + 8.99189265384883868589437612030 57 \times \\
&10^{-200} \times x^{95} - 1.65295240303758165077185124448 61 \times 10^{-195} \times x^{94} + 1.92700973380713028695276244971 66 \times \\
&10^{-191} \times x^{93} - 1.59942511048644471594022425336 32 \times 10^{-187} \times x^{92} + 1.00597636686150789321318037757 3 \times \\
&10^{-183} \times x^{91} - 4.98336053300381551105236128458 29 \times 10^{-180} \times x^{90} + 1.99542342837072938580301916538 04 \times \\
&10^{-176} \times x^{89} - 6.57858072677058979109960332030 02 \times 10^{-173} \times x^{88} + 1.81031950842307272034519676171 14 \times \\
&10^{-169} \times x^{87} - 4.20238658949188620219936852228 7 \times 10^{-166} \times x^{86} + 8.29979936159746651307583524443 79 \times \\
&10^{-163} \times x^{85} - 1.40489048480012750952763859941 92 \times 10^{-159} \times x^{84} + 2.05168935303340001699281367484 35 \times \\
&10^{-156} \times x^{83} - 2.60187504630515951742684913623 91 \times 10^{-153} \times x^{82} + 2.88433370190243467411459972367 3 \times \\
&10^{-150} \times x^{81} - 2.81459504716594757478987165818 59 \times 10^{-147} \times x^{80} + 2.43544944791698705515884566741 21 \times \\
&10^{-144} \times x^{79} - 1.88279603926069053955817679580 79 \times 10^{-141} \times x^{78} + 1.31016743712070295229343985507 69 \times \\
&10^{-138} \times x^{77} - 8.26483745259956173323020160021 91 \times 10^{-136} \times x^{76} + 4.75732848199108401665964117113 9 \times \\
&10^{-133} \times x^{75} - 2.51336720849516066484799938950 77 \times 10^{-130} \times x^{74} + 1.22504463520221469861190370444 11 \times \\
&10^{-127} \times x^{73} - 5.53350338783581138651289968067 14 \times 10^{-125} \times x^{72} + 2.32532473585864367115833336677 4 \times \\
&10^{-122} \times x^{71} - 9.12108522829983994802527204891 44 \times 10^{-120} \times x^{70} + 3.34905613240396308112075765301 72 \times \\
&10^{-117} \times x^{69} - 1.15387791469689083050411909394 36 \times 10^{-114} \times x^{68} + 3.73805324437188001456352909336 08 \times \\
&10^{-112} \times x^{67} - 1.14057738732164402405460888290 21 \times 10^{-109} \times x^{66} + 3.28261688168827878784074838696 54 \times \\
&10^{-107} \times x^{65} - 8.92170261138041707701666384584 15 \times 10^{-105} \times x^{64} + 2.29209048368371037123127520857 91 \times \\
&10^{-102} \times x^{63} - 5.57082025772659129838648318125 78 \times 10^{-100} \times x^{62} + 1.28171975729996453705897050695 34 \times \\
&10^{-97} \times x^{61} - 2.79310231562299372406094016332 17 \times 10^{-95} \times x^{60} + 5.76764001140629457518984272832 59 \times \\
&10^{-93} \times x^{59} - 1.12902645388652701329490945780 32 \times 10^{-90} \times x^{58} + 2.09593133085297509064766336523 55 \times \\
&10^{-88} \times x^{57} - 3.69147893381313516769667977564 08 \times 10^{-86} \times x^{56} + 6.17140647005801843492495739098 32 \times 10^{-84} \times \\
&x^{55} - 9.79891554706636123199561371887 62 \times 10^{-82} \times x^{54} + 1.47870009585856255578132039625 15 \times 10^{-79} \times x^{53} - \\
&2.12245633407373257287037944025 52 \times 10^{-77} \times x^{52} + 2.90031750586266972661858173297 46 \times 10^{-75} \times x^{51} - \\
&3.77675725122357912065282646179 52 \times 10^{-73} \times x^{50} + 4.69118449059939934088634616754 63 \times 10^{-71} \times x^{49} - \\
&5.56339871539359120844957493728 23 \times 10^{-69} \times x^{48} + 6.30463437002376416349576708915 95 \times 10^{-67} \times x^{47} - \\
&6.83226233033064545221349313674 39 \times 10^{-65} \times x^{46} + 7.08482898782355148099180970950 34 \times 10^{-63} \times x^{45} - \\
&7.03358078714002998418659162302 39 \times 10^{-61} \times x^{44} + 6.68746116470995906592677262368 9 \times 10^{-59} \times x^{43} - \\
&6.09058097151474294988256438962 01 \times 10^{-57} \times x^{42} + 5.31328703114369681312699309414 53 \times 10^{-55} \times x^{41} - \\
&4.43915355621474408088807593249 46 \times 10^{-53} \times x^{40} + 3.55076596072321070459059855016 9 \times 10^{-51} \times x^{39} - \\
&2.71757794242387962275718019042 62 \times 10^{-49} \times x^{38} + 1.98847193425817173234403663915 67 \times 10^{-47} \times x^{37} - \\
&1.38966098839541029480135781893 03 \times 10^{-45} \times x^{36} + 9.26637940387302514129447219896 93 \times 10^{-44} \times x^{35} - \\
&5.88890940703739924526283190687 78 \times 10^{-42} \times x^{34} + 3.56221032164213909970724483245 98 \times 10^{-40} \times x^{33} - \\
&2.04848118071054312436432215926 93 \times 10^{-38} \times x^{32} + 1.11908102491046018765561478726 35 \times 10^{-36} \times x^{31} - \\
&5.80853083033005196542774263826 92 \times 10^{-35} \times x^{30} + 2.86819405814258983015162215531 28 \times 10^{-33} \times x^{29} - \\
&1.350035614030679 \times 10^{-31} \times x^{28} + 6.062432200757834 \times 10^{-30} \times x^{27} - 2.593792399490007 \times 10^{-28} \times x^{26} + \\
&1.054834483968515 \times 10^{-26} \times x^{25} - 4.069021649462145 \times 10^{-25} \times x^{24} + 1.486105572627995 \times 10^{-23} \times x^{23} - \\
&5.152383300709261 \times 10^{-22} \times x^{22} + 1.688367760570223 \times 10^{-20} \times x^{21} - 5.142891682318646 \times 10^{-19} \times x^{20} + \\
&1.439484266469809 \times 10^{-17} \times x^{19} - 3.692325957950766 \times 10^{-16} \times x^{18} + \times 8.673466067293682\, 10^{-15} \times x^{17} - \\
&1.870738151660065 \times 10^{-13} \times x^{16} + 3.717976933385329 \times 10^{-12} \times x^{15} - 6.852237984108448 \times 10^{-11} \times x^{14} + \\
&1.336194064505420 \times 10^{-9} \times x^{13} - 2.261519923456648 \times 10^{-8} \times x^{12} + 2.724351410340337 \times 10^{-7} \times x^{11} - \\
&2.754145672079211 \times 10^{-6} \times x^{10} + 2.480128192810839 \times 10^{-5} \times x^{9} - 0.00019841267918963012562016956508 16x^{8} + \\
&0.00138888888888888888888888888888 89x^{7} - 0.00833333333343944104854017496109 01x^{6} + \\
&0.04166666666666666666666666666666 67x^{5} - 0.16666666666787932626903057098389x^{4} + 0.5x^{3} - x^{2} + x) .
\end{aligned}
$$

TABLE III: Comparison of numerical results for Example 3

x	Exact solution	SAIM	Error
0	0	0	0

0.1	0.090483741803596	0.090483741803596	0
0.2	0.163746150615596	0.163746150615596	0
0.3	0.222245466204515	0.222245466204516	$0.E^{-016}$
0.4	0.268128018414256	0.268128018414259	$3.E^{-015}$
0.5	0.303265329856317	0.303265329856321	$4.E^{-015}$
0.6	0.329286981656416	0.329286981656260	$1.56\ E^{-013}$
0.7	0.347609712653987	0.347609712652568	$1.418E^{-012}$
0.8	0.359463171293777	0.359463171286723	$7.054\ E^{-012}$
0.9	0.365912693766539	0.365912693741649	$2.4890\ E^{-011}$
1	0.367879441171442	0.367879441103774	$6.7668\ E^{-011}$

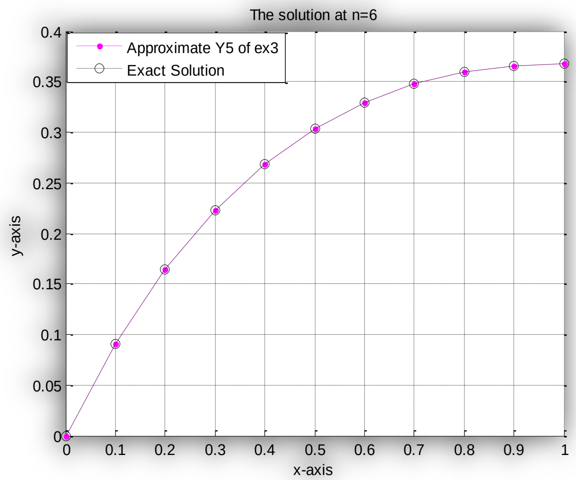

Figure 3: Comparison between the exact and semi analytic iterative solution y_5

Example 4:

Consider the non- linear delay differential equation of third- order [5]:

$y'''(x) = -1 + 2\,y^2(\frac{x}{2})$, $0 \le x \le 1$, with ICs $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$,

To find y_0 we take $L(y_0) = -1$ with $y_0(0) = 0$ where $f(x) = 1$, $L(y) = \frac{d^3y}{dx^3}$, $N(y) = -2y^2$

By applying SAIM we get: $y_0(x) = x - \frac{1}{6}x^3$ The next step is finding y_1 as follows:

$y_1'''(x) = -1 + 2y_0^2(\frac{x}{2})$ where $y_1(0) = 0$ and its solution is:

$y_1(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{580608}x^9$, and

$$y_2(x) = \frac{1}{35259719146782326784 0}x^{21} - \frac{1}{278745919594168320}x^{19} + \frac{83}{40752327426048000}x^{17} - \frac{13}{18728091648000}x^{15} + \frac{1009}{6376469299200}x^{13} - \frac{1}{39916800}x^{11} + \frac{1}{362880}x^{9} - \frac{1}{5040}x^{7} + \frac{1}{120}x^{5} - \frac{1}{6}x^{3} + x$$

$$y_3(x) = \Big(\frac{X^{45}}{2327668693423089057151387493345835971845497328856761945292 8} - \frac{X^{43}}{2000455087145740829454048072362004936311234829093961728} + \frac{319\,X^{41}}{1148196272637198382377734223464477830855661767465369 60} - \frac{17701X^{39}}{179779429407061314838583580871869998561809916965683 2} + \frac{67861007x^{37}}{2710490300819181373173732653570030275059766144270336} - \frac{4256143X^{35}}{874372223073004506893098734851681927895502028 80} + \frac{1027841963x^{33}}{1366665827660326401945697859844161031759095398 4} - \frac{21348683x^{31}}{225550135911489401265051407674800076750848} + \frac{22221173077x^{29}}{225428872397558499411607027535971302244352} - \frac{1785309641x^{27}}{2082163230273015796333573340594700288 0} + \frac{1066828753057x^{25}}{17054755803518291970805543404849594368} - \frac{1061897167x^{23}}{277583916072888882324808516239 36} + \frac{535185221x^{21}}{27429240718664907651809280000} - \frac{123289x^{19}}{15006368565446049792000} + \frac{2097047x^{17}}{745930601206382592000} - \frac{x^{15}}{1307674368000} + \frac{x^{13}}{6227020800} - \frac{x^{11}}{39916800} + \frac{x^{9}}{362880} - \frac{x^{7}}{5040} + \frac{x^{5}}{120} - \frac{x^{3}}{6} + x \Big).$$

TABLE Iv: Comparison of numerical results for Example 4

x	Exact solution	SAIM	Error
0	0	0	0
0.1	0.099833416646828	0.099833416646828	0
0.2	0.198669330795061	0.198669330795061	0
0.3	0.295520206661340	0.295520206661340	0
0.4	0.389418342308651	0.389418342308651	0
0.5	0.479425538604203	0.479425538604203	0
0.6	0.564642473395035	0.564642473395035	0
0.7	0.644217687237691	0.644217687237691	0
0.8	0.717356090899523	0.717356090899523	0
0.9	0.783326909627483	0.783326909627483	0
1	0.841470984807897	0.841470984807897	0

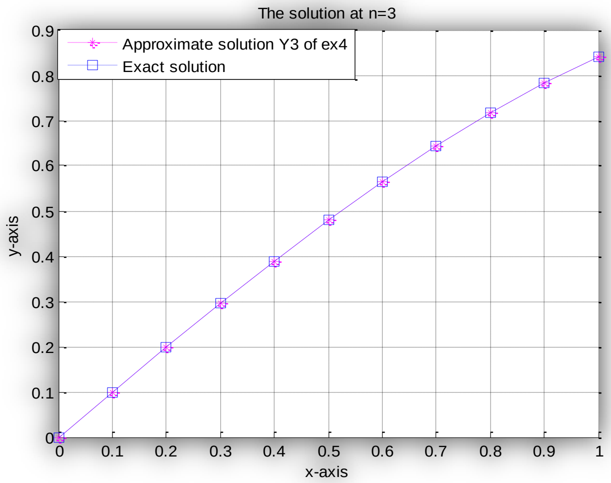

Figure 4: Comparison between the exact and semi analytic iterative solution y_3

Analysis of Error:

The error plays a pivotal role in approximate solutions. It shows that the accuracy and rapidity of the used method .Whenever a lower error is shown, more accurate solution and closer to the exact solution is obtained.

Approximate solution are usually used to find solutions to some problems that cannot be solved in analytical mathematics. This means there is an error amount we have to calculate. If one can find the exact error, then the exact solution will be found. This means finding the exact error is not possible. Thus, we seek to find an approximation of the error (i.e. a value which is not exceeded by error).

For the method used in this paper, we analyze successive errors that can be expressed as follows:

$$E_n = \|y_{n+1} - y_n\| \quad , \; n= 0, 1, 2 , \ldots$$

where E_n , represent the difference between two successive iterative solutions . $\|\,.\,\|$ represents the standard Euclidean norm .

The Error term $\mathbf{E_n}$ between two consecutive for examples 1, 2, 3 and 4

n	E_4 of Ex1	E_4 of Ex2	E_4 of Ex3	E_2 of Ex4
0	0.0629940788348712	0.5325183919288651	0.3852436870510644	$6.0633E^{-006}$
1	$6.155171977034576E^{-006}$	0.0005474395980499056	0.007030163699362195	$5.3782E^{-014}$
2	$1.861696496487456E^{-005}$	0.0004780723689526526	0.002506769209273337	$1.9378E^{-025}$
3	$2854252132861244\ E^{-008}$	$8.164787440381631E^{-006}$	$6.928149788705148E^{-006}$	
4	$9.183696950584957E^{-012}$	$8.808596182455132E^{-008}$	$1.18332482349006E^{-009}$	

Conclusion:

In this paper, we solved ordinary delay differential equations with initial conditions based on the Semi Analytic Iterative Method SAIM which was provided by Temimi and Ansari to solve ordinary differential equations with boundary conditions, we noticed that this method is characterized by its ease, accuracy, and lack of time consumed when used in comparison with other numerical methods. We have resolved some of the examples in [3, 5], and got better results. For example 1 see table 3.1 in [3], for examples 2, 3 see [6], for example 4 see table Iv in [5].

References:

[1] Y. kuang, " Delay Differential Equations with applications in population Dynamics," 1st ed. Tempe, Academic press .Inc. Newyourk, USA, vol. 191, pp. 163-165, 1993.

[2] B. Liu, X. Zhou and Q. Du, "Differential Transform Method for Some Delay Differential Equations," Applied Mathematics, vol. 6, pp. 585-593, 2015.

[3] O. F. Ogunfiditimi, " Numerical Solution Of Delay Differential Equations Using The Adomian Decomposition Method (ADM)," The International Journal Of Engineering And Science(IJES) : FCT, Nigeria, University of Abuja, vol. 4, Issue.5, pp. 18-23, 2015.

[4] H. Liu, A. Xiao and L. Su, "Convergence of Variational Iteration Method for Second –Order Delay Differential Equations," Journal of Applied Mathematics, Article ID 634670, vol. 2013, pp. 9, 2013.

[5] D. J. Evans and K. R. Raslan, "The Adomian Decomposition Method for solving delay differential equation," International Journal of Computer Mathematics, vol. 82, no. 1, pp. 49-54, 2005.

[6] B. Benhammouda, H. V. Leal and L. H. Martinez, "Procedure for Exact Solutions of Nonlinear Pantograph Delay Differential Equations," British Journal of Science, Education and Culture,, vol.1, no. 2(6), pp. 309-322, 2014.

[7] A. A. Dascioglu and M. Sezer, "Bernoulli collocation method for high -order generalized pantograph equations," NTMSCI 3, no. 2, pp. 96-109, 2015.

[8] H .Temimi and A. R. Ansari, " A semi-analytic iterative technique for solving nonlinear problems," Computer and Mathematics with Applications, no. 61, pp. 203-210, 2011.

الحل العددي للمعادلات التفاضلية التباطؤية غير الخطية بأستخدام الطريقة شبه التحليلية التكرارية

أسماء عبد عصواد[1] سماهر مرز ياسين[2]

[1,2]قسم الرياضيات، كلية التربية \ابن الهيثم، جامعة بغداد، بغداد، العراق.

Numerical Solution of Non-linear Delay Differential Equations Using Semi Analytic Iterative Method

Asmaa A. Aswhad[1] **Samaher M. Yassein[2*]**

الخلاصة:

قدمنا خوارزمية موثوقة لحل ا لمعادلات التفاضلية التباطؤية الاعتيادية غير الخطية متجانسة كانت او غير متجانسة ذات الشروط الابتدائية. يتم حساب الحل بصيغة سلسلة مع عناصر قابلة للحساب بسهولة. اربعة أمثلة تم تدارسها لتوضيح المسار العددي لهذه الطريقة ولقد اثبتت النتائج ان الطريقة شبه التحليلية التكرارية فعالة جداً،بسيطة وقريبة جداًمن الحل المضبوط وتُظهر بوضوح دقة وكفاءة الطريقة لمثل هذه المسائل.

الكلمات المفتاحية: الطريقة شبه التحليلية التكرارية، ا لمعادلات التفاضلية التباطؤية الاعتيادية، الشروط الابتدائية.