Numerical Treatment of First Kind Fredholm Integro_Differential Equations Via Spline Functions

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Abstract:

In this paper some algorithms of numerical treatment for Fred holm integro-differential equations [$FIDE_S$] of the first kind three types of Spline functions: Linear, Quadratic and Cubic are introduced. The values of the involved integrals in each algorithm are evaluated numerically using trapezoidal rule. The program for each method is written in Matlab (V.6) language. A comparison amonge the three types has been done depending on the last squares errors and running time.

1- Introduction:

Much attention has been given [5,6,8,9] on the numerical solution of Fred holm integro_differential equations [FIDE$_S$] of first kind. This kind of equation as the form:

$$u'(x) + p(x)u(x) = g(x) + \int_a^b k(x,y)u(y)\,dy, \quad x \in I = [a,b] \qquad \ldots\ldots(1)$$

with the initial condition $u(a)=u_0$, where the functions f and p are assumed to be continuous on I, k denotes given continuous function, were the interval [a,b] is divided in to n equal subintervals, such that $h=(b-a)/n$, $y_0=a$, $y_n=b$ and $y_j= a+jh$, $j= 0,1,..,n$. We set $x_i=y_j$, $i=0,1,..,n$, $u'(x_i)=u_i$, $p(x_i)=p_i$, $f(x_i)=f_i$, $u(x_i)=u_i$ and $k(x_i,y_j)=k_{ij}$.

And $$\int_0^b K(x,t)\,f(t)\,dt = g(x), \quad 0 \le x \le b$$

……(2)

is the Fredholm integral equation of first kind, where $k(x,t)$ and $g(x)$ are known functions, $f(t)$ is the unknown function. We consider the problem is obtaining $f(t)$ values at discrete point in [10].

The numerical solution of the Fredholm integral Eq.(2) has been studied via orthogonal functions in [7] and weighted residual methods in [1]. Taylor series is approached in [2]. The main feature of these methods is its reduction the Fredholm integral Eq.(2) to algebraic forms which are proceeded by forward substitution in order to obtain the results in [10].

In this work, spline functions are used to treat Eq.(2). Here the interval $0 \le x \le b$ is divided into a collection of subintervals, which is followed by a construction of different approximating polynomial on each subinterval.

2- Brief Review of Spline Functions:

To facilitate the presentation of material, a brief review of spline functions is given in this section. The three types of spline function are considered in [3,4]

2.1 Frist Degree Spline (L(x)):

The spline of 1^{st} degree is a simple example for the spline functions whose pieces are linear polynomials.

The linear spline function $L(x)$ consists of $n-1$ piecewise linear function, in an interval $[t_i, t_{i+1}]$, i.e.

$$L(x)=\begin{cases} L_1(x) & x \in [t_1, t_2] \\ L_2(x) & x \in [t_2, t_3] \\ \vdots & \\ L_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

where $L_i(x) = A_i(x) + B_i \qquad i = 1,2,3,...n-1$

If the 1^{st} degree function $L(x)$ is continuous, which is called a spline, then

(i) The domain of L is an interval $[t_1, t_n]$

(ii) L is continuous on $[t_1, t_n]$

(iii)There are a partition of the interval $t_1 < t_2 < ... < t_n$.

2.2 Quadratic Spline Q(x):

The quadratic spline function Q(x) consists of I piecewise quadratic foundation as follows:

$$Q(x)=\begin{cases} Q_1(x) & x \in [t_1, t_2] \\ Q_2(x) & x \in [t_2, t_3] \\ \vdots & \\ Q_I(x) & x \in [t_{n-1}, t_n] \end{cases}$$

which is continuously differentiable on the interval $[t_i, t_{i+1}]$ and $Q_i(t_i) = y_i \quad , \quad i = 1,2,......n$

put $z_i = Q'(t_i)$

Hence, the formula for Q_i on $[t_i, t_{i+1}]$ is:

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(x - t_i)^2 + z_i(x - t_i) + y_i$$

……(3)

with the three conditions :

$Q_I(t_i) = y_i$, $Q'_i(t_i) = z_i$ and $Q'_i(t_{i+1}) = z_{i+1}$

The necessary and sufficient condition in Eq.(3) is

$$Q_i(t_{i+1}) = y_{i+1} \; , \quad i = 1,2,...,n-1$$

Therefore, the result is :

$$z_{i+1} = -z_i + 2\left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right), \qquad i = 1,2,..., n-1$$

……(4)

where z_1 is arbitrary.

2.3 Cubic Spline S(x):

Cubic spline is the most common piecewise polynomial approximation which consists of *n-1* cubic polynomial pieces as follows:

$$S(x) = \begin{cases} S_1(x) & x \in [t_1, t_2] \\ S_2(x) & x \in [t_2, t_3] \\ & \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

subject to :

$S_i(t_i)=S_i$, $i = 1,2,...,n$

$S_{i+1}(t_{i+1})=S_i(t_{i+1})$

$S'_{i+1}(t_{i+1})=S_i'(t_{i+1})$, $i = 1,2,..n-1$

$S''_{i+1}(t_{i+1})=S_i''(t_{i+1})$

where S, S', S'' are all continuous functions on $[t_1, t_n]$.

The formula for S_i can be obtained by using Newton form and divided difference table for S_i.

$S_i(x) = c_{1,i}+c_{2,i}(x-t_i)+c_{3,i}(x-t_i)^2+c_{4,i}(x-t_i)^3$ ……(4.a)

with

$C_{1,i} = S_i(t_i)=S_i$

$C_{2,i} = S'_i(t_i)=S'_i$

$C_{3,i}=S_i''(t_i)/2=([t_i, t_{i+1}]S_i-S'_i)/h-c_{4,i}h$ ……(4.b)

$C_{4,i}=S_i'''(t_i)/6=(S'_i+S'_{i+1}-2[t_i, t_{i+1}]S_i)/h^2$

Substituting (*4.b*) in (*4.a*) and let $S_i(t_{i+1})=S_{i+1}$, we have

$S_i(x)=S_i+S'_i(x-t_i)+(([t_i,t_{i+1}]S_i-S'_i)/h-c_{4,i}h)(x-t_i)^2+((S_i'+S_i'_{+1}-2[t_i,t_{i+1}]S_i)/h^2)(x-t_i)^3$

……(5)

Since S'' is continuous function hence

$S_{i-1}''(t_i) = S_i''(t_i)$

or

$hS'_{i-1}+ 4hS'_i+h S'_{i+1} = b_i$

with

$b_i = 3h([t_{i-1}, t_i]S_i+[t_i, t_{i+1}]S_i)$, $i=2, ... n-1$

3- Reduction to Integral Equation:

The reduction of integro-differential equations [IDEs] to integral equations can be used for the analysis of variety of fredholm integro-differential equations [FIDEs].

Reduction theorm[1]

Let g, k be iterated L_2 integrable function on interval [a,b] and $P_I \in c^n[a,b]$ then Eq.(5) is:

$$[D^n + \sum_{i=1}^{n-1} P_i(t) D^i] f(t) = g(t) + \int_a^t K(t,s) f(s) ds, \qquad t \ \epsilon \ [a,b]$$

……(6)

with initial the conditions $f(a) = f_0, f'(a) = f_1, \ldots\ldots, f^{(n)}(a) = f_n$

can be reduced to linear fred holm integral equation [FIE] in the form :

$$f(t) = G_n(t) + \int_a^b K_n(t,s) f(s) ds$$

where

$$G_n(t) = \sum_{i=0}^{n-1} \frac{f_i t^i}{i!} + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s) ds + \sum_{k=0}^{n-2} \sum_{j=k+1}^{n-1} \sum_{i=n+k-j}^{n-1} (-1)^k P_j^{(k)}(a) A_{k,n-i-1} \frac{f_{i+j-n-k} t^i}{i!}$$

and

$$K_n(t,s) = \frac{1}{(n-1)!} \int_t^x (t-z_1)^{n-1} k(z_1,s) dz_1 - \sum_{k=0}^{n-1} \sum_{j=0}^{k} (-1)^j P_{j+n-k-1}^{(k)}(s) B_{n-k-1,j} \frac{(t-s)^k}{k!}$$

W

here A and B are two special constant matrices of dimensions *n-2×n-2* and *n-1×n-1*,respectively.

4- Solution of Fredholm integro_differential Equation:

Recall eq. (1)

$$\int_a^b k\ (x,t) f(t) dt = g(x)\ ,\ a \le x \le b$$

In this section, the spline functions Linear, quadratic and cubic are used to find the numerical solution of Eq.(2).

4.1 Using Linear Spline Functions L(x):

Linear interpolation in the interval $[t_i, t_{i+1}]$ gives the formula in [5]:

$L_i(x) = A_i(x)y_i + B_i(x)y_{i+1}$ ……(7)

where $A_i(x)=\frac{t_{i+1}-x}{h}$, $B_i(x)=1-A_i(x)=\frac{x-t_i}{h}$ and $h = t_{i+1}-t_i$

let $y_i = L_i(t_i)$, $i = 1,2,....,n$

substituting Eq.(7) into Eq.(2) for f(y), we obtain

$$g_r = \sum_{j=0}^{r} \int_{x_j}^{x_{j+1}} k(x_r, y)\left[A_j(y)y_j + B_j(y)y_{j+1}\right]dy, \quad r = 1,2,... \qquad ......(8)$$

where $g_r = g(x_r)$,

Eq.(8) is solved by iteration and the integrals which are approximated by trapezoidal rule.

4-2 Using Quadratic Spline Functions Q(x):

The formula of a quadratic Spline functions $Q(x)$ in the interval $[t_i, t_{i+1}]$ is :

$Q_i(x) = A_i(x)\ Q_i + B_i(x)\ Q'_i + C_i(x)\ Q'_{i+1}$ ……(9)

where

$$A_i(x)=1, \quad B(x)=\left((x-t_i)-\frac{(x-t_i)^2}{2h}\right), \quad C_i=\frac{(x-t)^2}{2h}$$

Let $Q_i(t_i)=Q_i$, $Q'_i(t_i)=Q'_i$ and $Q'_i(t_{i+1})=Q'_{i+1}$

Substitute Eq.(9) in Eq.(2) and obtain :

$$g_r = \sum_{j=0}^{r-2} \int_{X_j}^{x_{j+1}} k(x_r, y)\,(A_i(y)Q_i + B_i(y)\,Q'_i + C_i(y)\,Q'_{i+1})\,dy + \int_{x_{r-1}}^{x_r} k\,(x_r, y)(A_{r-1}(y)\,Q_{r-1} + B_r(y)Q'_{r-1} + C_r(y)\,Q'_r\,)dy$$

……(10)

Note that, for the continuity of Q', the requirement should be :

$$Q'_{i+1} = -\,Q_i + \tfrac{2}{h}\,(Q_{i+1} - Q_i) \quad , \qquad i = 0,1,....$$

……(11)

Eq.(10) is solved by iteration using Eq.(11).

4-3 Using Cubic Spline Functions S(x):

The cubic spline functions $S(x)$ in the interval $[t_i,t_{i+1}]$ of Eq.(5) Can be written as :

$$S_i(x)=A_i(x)S_i+B_i(x)S_{i+1}+C_i(x)S'_i+D_i(x)S'_{i+1} \quad ,$$

...... (12)

where

$$A_i(x) = 1 - 3\left(\frac{x-t_i}{h}\right)^2 + 2\left(\frac{x-t_i}{h}\right)^3 ,$$

$$B_i(x) = 1 - A_i(x),$$

$$C_i(x) = (x-t_i)\left(\frac{x-t_{i+1}}{h}\right)^2 \quad \text{and} \quad D_i(x) = \left(\frac{x-t_i}{h}\right)^2 (x - t_{i+1})$$

Substituting Eq.(12) into Eq.(2), this yields :

$$g_r = \sum_{j=0}^{r-2} \int_{x_j}^{x_{j+1}} k(x_r, y)\,[\,A_j(y)\,S_j + B_j(y)\,S_{j+1} + C_j(y)\,S'_j + D_j(y)\,S'_{j+1}]\,dy + \int_{x_{r-1}}^{x_r} k\,(x_r, y)[A_{r-1}(y)\,S_{r-1} + B_{r-1}(y)S_r + C_r(y)\,S'_{r-1} + D_{r-1}(y)\,S'_r]\,dy$$

……. (13)

with

$S'_{i-1} + 4S'_i + S'_{i+1} = \frac{3}{h}(S_{i+1} - S_{i-1})$, $i=1,2,.... n-1$

Since the central difference for the first derivative is :

$$S'_i \approx \frac{S_{i+1} - S_{i-1}}{2h} \quad ......(14)$$

So

$$S'_{i+1} \approx 2S'_i - S'_{i-1} \quad , \quad i = 1,2,...... n-1 \quad ....(15)$$

Finally, eq. (13) is solved by central difference in Eq.(15) to get
an approximate solution for Eq.(2).

5- Numerical Examples:

Example (1):

Consider the following problem after reducing the [FIDE], we get:-

$$\int_a^b \cos(x-y)f(y)dy = x \quad , \quad 0 \le x \le 1$$

where the exact solation $f(x) = 1 + \frac{x^2}{2}$

The solution of $f(x)$ for $0 \le x \le 1$ is required.

Take $h=0.1$, and use the spline functions: Linear, quadratic and cubic discussed in section 3, the results are obtained as shown in table (1) with the exact solution, least square errors (L.S.E.) and running time (R.T.) are also listed for comparison.

Table (1) (S.F.)

Spline Functions				
x	Linear Spline (LS)	Quadratic Spline (QS)	cubic Spline (CS)	Exact
0	1	1	1	1
0.1	1.004996	1.005016	1.005036	1.005
0.2	1.019983	1.019964	1.019984	1.02
0.3	1.044963	1.044983	1.045003	1.045
0.4	1.079933	1.079914	1.079935	1.08
0.5	1.124896	1.124961	1.124938	1..125
0.6	1.179850	1.179831	1.179854	1.18
0.7	1.244796	1.244861	1.244840	1.245
0.8	1.319733	1.319714	1.319740	1.32
0.9	1.404663	1.404683	1.404710	1.405
1	1.499583	1.499565	1.499593	1.5
L.S.E	0.000000	0.000000	0.000000	
R.T.	0: 0 : 0 :4	0: 0 : 0 :4	0: 0 : 0 :4	

Table (2) gives the least square error and running time for different values **b** and **h**.

Table (2) (L.S.,Q.S., C.S.)

b \ h	S.F.	0.1		0.2		0.5	
		L.S.E	R.T	L.S.E	R.T	L.S.E	R.T
1	LS	0.000000	0 : 0 :4	0.000004	0 : 0 :4	0.000113	0 : 0 : 4
	QS	0.000000	0 : 0 :4	0.000004	0 : 0 :4	0.000114	0 : 0 : 4
	CS	0.000000	0 : 0 :4	0.000002	0 : 0 :4	0.000099	0 : 0 : 4
5	LS	0.001139	0: 0 : 0 :16	0.009546	0: 0 : 0 :10	0.168962	0 : 0 :5
	QS	0.001130	0: 0 : 0 :16	0.008724	0: 0 : 0 :10	0.132592	0 : 0 :5
	CS	0.000053	0: 0 : 0 :16	0.000389	0: 0 : 0 :10	0.005481	0 : 0 :5
10	LS	0.035572	0: 0 : 0 :43	0.291075	0: 0 : 0 :21	4.819912	0: 0 : 0 :10
	QS	0.033983	0: 0 : 0 :43	0.229988	0: 0 : 0 :21	2.967439	0: 0 : 0 :10
	CS	0.000099	0: 0 : 0 :43	0.000997	0: 0 : 0 :21	0.108899	0: 0 : 0 :10

Example (2):

Consider the following problem after reducing the [FIDE], we get:-

$$\int_a^b e^{x+y}\ f\,(y)\,dy = x\,e^{2x} \quad , \qquad 0 \le x \le 1$$

Where the exact solution is $f(x) = 1+x$.

In order to solve the problem for $0 \le x \le 1$.The results are listed in table (3) against the exact solution. The least square error (L.S.E.) and running time (R.T.). are also listed for comparison.

Table (3) (S.F.)

Spline Functions				
x	Linear Spline (LS)	Quadratic Spline (QS)	cubic Spline (CS)	Exact
0	1	1	1	1
0.1	1.095163	1.098782	1.098782	1.1
0.2	1.199381	1.196451	1.20007	1.2
0.3	1.295405	1.298401	1.299090	1.3
0.4	1.398844	1.396478	1.400163	1.4
0.5	1.495574	1.498060	1.499378	1.5
0.6	1.598374	1.596470	1.600274	1.6
0.7	1.695682	1.697750	1.699469	1.7
0.8	1.797960	1.796433	1.800400	1.8
0.9	1.895740	1.897466	1.899907	1.9
1	1.997590	1.996373	2.000539	2
L.S.E	0.000115	0.000083	0.000003	
R.T.	0: 0 : 0 :5	0: 0 : 0 :5	0: 0 : 0 :5	

In table (4), the least square error (L.S.E.) and running time (R.T.) for different values of **b** and **h**.

Table (4) (L.S.,Q.S., C.S.)

b \ h	S.F.	0.1		0.2		0.5	
		L.S.E	R.T	L.S.E	R.T	L.S.E	R.T
1	LS	0.000115	0 : 0 :5	0.001026	0 : 0 :5	0.014741	0 : 0 :5
	QS	0.000083	0 : 0 :5	0.000549	0 : 0 :5	0.007169	0 : 0 :5
	CS	0.000003	0 : 0 :5	0.000046	0 : 0 :5	0.000419	0 : 0 :5
5	LS	0.001173	0: 0 : 0 :21	0.009485	0: 0 : 0 :10	0.147848	0 : 0 :5
	QS	0.001057	0: 0 : 0 :21	0.007528	0: 0 : 0 :10	0.072899	0 : 0 :5
	CS	0.000029	0: 0 : 0 :21	0.001167	0: 0 : 0 :10	0.003396	0 : 0 :5
10	LS	0.005101	0: 0 : 0 :43	0.040973	0: 0 : 0 :21	0.629511	0: 0 : 0 :10
	QS	0.004689	0: 0 : 0 :43	0.032901	0: 0 : 0 :21	0.285691	0: 0 : 0 :10
	CS	0.000066	0: 0 : 0 :43	0.002567	0: 0 : 0 :21	0.011088	0: 0 : 0 :10

6- Conclusion:

In this paper, a method of using spline functions has been presented for solving Fred holm integro-differential equations [FIDEs] of the first kind. It has been shown that the proposed method is convenient for computer programming.

For the comparison of computing accuracy and the speed, the last square error and running time are given in tables conclude that: the cubic spline gives a better accuracy than quadratic and linear spline.

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Numerical Treatment of First Kind Fredholm Integro_Differential Equations Via Spline Functions

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المعالجة العددية لمعادلات فريدهولم التكاملية التفاضلية من النوع الاول باستخدام دوال الثلمة

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المستخلص :

قدمت الخوارزميات لمعالجة معادلات فريدهولم التكاملية – التفاضلية من النوع الاول عددياً باستخدام دوال الثلمة: الخطية، التربيعية والتكعيبية، استخدمت طريقة شبه المنحرف لحساب قيم التكاملات الموجودة في الخوارزميات عددياً.

تم كتابة برنامج لكل طريقة بلغة ماتلاب وتمت المقارنة بين الانواع الثلاثة لدوال الثلمة بالاعتماد على مربع مجموع الاخطاء وزمن تنفيذ البرنامج .

تم الحمد لله