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On The Analytical Solutions of Nonhomogeneous Heat equations

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Abstract:

In this paper, we consider the analytical solution of Nonhomogeneous mixed problem of Heat equation. Namely, we use separation of variable and Duhamel's principle to derive the formula of the general solution to this problem. Moreover, we find the solution for a special case of this problem as an application to our result.

Keywords: Heat equations, Mixed problem, Separation of variables and Duhamel's principle.

1. Introduction

Over the last decades, the solutions of partial differential equations have been considered by many authors; see for instance [1-4].
The nonhomogeneous heat equation with n space variables takes the form [1]:

$$\frac{\partial u(x,t)}{\partial t} = c^2 \Delta u(x,t) + h(x,t), \qquad (t > 0, x \in R^n).$$

It is well known that heat equation is a special case of a class of equations called parabolic type equations. Heat equation can model heat conduction, diffusion, etc. In this paper, we will study the analytical solution of the case when $n = 1$. (With non-zero initial – boundary conditions)

2. Mixed problem: Separation of variables

Consider the following mixed problem [2]:

$$\begin{cases} u_t = c^2 u_{xx} + h(x,t), (t > 0.0 < x < l), \\ u(0,x) = f(x), (0 < x < l), \\ u(0,t) = \emptyset(t), u(l,t) = \psi(t), (t > 0). \end{cases} \qquad (2.1)$$

Since the equation and the initial and boundary conditions are linear, we may split the problem into three simpler problems and then use superposition's to get a solution of the original problem. We have

$$\begin{cases} u_t = c^2 u_{xx,} \ (t > 0, 0 < x < l), \\ u(x,0) = f(x), (0 < x < l), \\ u(0,t) = 0, u(l,t) = 0, (t > 0). \end{cases} \qquad (I)$$

$$\begin{cases} u_t = c^2 u_{xx} + h(x,t), (t > 0, 0 < x < l), \\ u(x,0) = 0, (0 < x < l), \\ u(0,t) = 0, \ \ u(l,t) = 0, (t > 0). \end{cases} \qquad (II)$$

$$\begin{cases} u_t = c^2 u_{xx,(t>0,0<x<l)}, \\ u(x,0) = 0, (0 < x < l), \\ u(0,t) = \phi(t), \ \ u(l,t) = \psi(t), (t > 0). \end{cases} \qquad (III)$$

Suppose u_1, u_2 and u_3 are solutions of (I), (II) and (III) respectively, $u_1+u_2+u_3$ is a solution of (2.1). We can see that (III) can be reduced to (I) and (II). Indeed, letting

$$U(x,t) = \emptyset(t) + \frac{x}{l}(\psi(t),$$

We see that U (0, t) = Ø (t) and U (L, t) = ψ(t).

Let u be a solution of (III) and define V(x,t) = u (x,t) – U (x,t). Then V (x,t) is a solution of the following problem

$$\begin{cases} V_t - c^2 V_{xx} = -\left(\phi'(t) + \frac{x}{l}(\psi'(t) - \phi'(t)\right), (t > 0, 0 < x < l), \\ V(x,0) = -\left(\phi(0) + \frac{x}{l}(\psi(0) - \phi(0)\right), (0 < x < L) \\ V(0,t) = 0, \qquad V(l,t) = 0, \qquad (t > 0). \end{cases} \qquad (III)$$

We have u (x,t)= V (x,t) is a solution of (III). Therefore, if we can solve (I) and (II), we can solve (I) first. We can solve this problem by using separation of variables.

Solution of (I) We first find a solution of the equation in the form u (x,t) = X(x) T (t) satisfying the boundary conditions u (0,t)= u (L,t)= 0. Then we use superpositions to find a solution of the original problem by matching the initial condition. Substitute the function in this special form to the equation and the boundary condition,

$$X(x)T'(t) = c^2 X''(x)T(t), \qquad X(0) = X(l) = 0$$

Rewrite the equation in the form

$$\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

(Because the left hand side is a function of l while the right hand side is a function of x).

Therefore we have the following two problems

$$(A) \qquad \begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(l) = 0, \end{cases} \qquad\qquad (B) T'(t) + \lambda c^2 T(t) = 0,$$

To find an non-trivial solution of (A) we must have $\lambda > 0$ and then we may find that $\lambda_k = \frac{k^2\pi^2}{l^2}$ for k = 1,2,... and the solution of (A) is in the form $X_k(x) = C_k \sin\frac{k\pi\pi}{l}$, where C_k is a constant to be determined. Substitute λ_k into (B) we can solve it and the solution is $T_k(t) = B_k e - \frac{c^2k^2\pi^2 t}{l^2}$, where B_k is a constant. Hence we have

$$u_k(x,t) = A_k e - \frac{c^2k^2\pi^2 t}{l^2} \sin\frac{k_{\pi x}}{l},$$

Where A_k is a constant to be determined. Using superpositions, we obtain

$$u(x,t) = \sum_{k=1}^{\infty} A_k e - \frac{c^2k^2\pi^2 t}{l} \sin\frac{k_{\pi x}}{l}. \qquad (2.2)$$

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi x}{l} = f(x) \qquad (2.3)$$

To make (2.3) an equality, A_k must be the Fourier coefficient of the Fourier-sine series expansion of f, i.e.,

$$A_k = \frac{2}{l}\int_0^l \int(x)\sin\frac{k\pi x}{l}\,dx \qquad (2.4)$$

Theorem 2.1. Assume that $f \in C^1$, f (0) = f (l) = 0. Then the solution of (I) is given by (2.2) with A_k given by (2.4), [2]

Example 2.1. (a) Find all values of μ for which the problem

$$\begin{cases} X''(x) - \mu X(x) = 0, & 0 < X < 1, \\ X(0) = 0 = X(1) \end{cases}$$

Has non-trivial solutions (i.e. $X \not\equiv 0$). Find these non-trivial solutions.

(b) Using the result of (a) solve the following initial-boundary value problem by the method of separation of variables

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, & 0 < x < 1, 0 < t < +\infty, \\ u(0,t) - 0 - u(1,t), & 0 \leq t < +\infty, \\ u(x,0) = 9\sin 2\pi x + 7\sin 5\pi x, & 0 < x < 1. \end{cases}$$

(c) Using the result of (b) solve the following initial-boundary value problem

$$\begin{cases} \frac{\partial \omega(x,t)}{\partial t} - \frac{\partial^2 \omega(x,t)}{\partial x^2} = 0, & 0 < x < 1, 0 < t < +\infty, \\ \omega(0,t) = 1, \quad \omega(1,t) = 0, & 0 \leq t < +\infty, \\ \omega(x,0) = 1 - x9\sin 2\pi x + 7\sin 5\pi x, & 0 < x < 1. \end{cases}$$

Solution (a) If $\mu = 0$ the general solution of the equation is $X(x) = a_1+a_2 x$. The boundary conditions are equivalent to $a_1 = 0$, $a_1 + a_2 = 0$, i.e. to $a_1 = 0 = a_2$. Therefore $X(x) \equiv 0$, i.e. there are no non-trivial solutions.

If $\mu > 0$ the general solution of the equation is

$$X(x) = a_1 e\sqrt{\mu x} + a_2 e - \sqrt{\mu x}.$$

The boundary conditions are equivalent to

$$\begin{cases} a_1 + a_2 = 0 \\ a_1 e\sqrt{\mu} + a_2 e - \sqrt{\mu} = 0 \end{cases}$$

i.e. to

$$\begin{cases} a_1 = -a_2 \\ a_1\left(e\sqrt{\mu} - e - \sqrt{\mu}\right) = 0 \end{cases}$$

This system has only the trivial solution.

Finally, consider $\mu < 0$. Then the general solution is

$X(x) = a_1 \cos\left(\sqrt{-\mu x}\right) + a_2 \sin\left(\sqrt{-\mu x}\right).$

The boundary conditions are equivalent to

$$\begin{cases} a_1 = 0 \\ a_1 cos\sqrt{-\mu} + a_2 sin\sqrt{-\mu} = 0, \end{cases}$$

i.e. to

$$\begin{cases} a_1 = 0, \\ a_2 sin\sqrt{-\mu} = 0. \end{cases}$$

This system has a non-trivial solution if and only if sin $\sqrt{-\mu} = 0,$ i.e. $\mu = -\pi^2 k^2, k \in \mathbb{N}$. The corresponding non-trivial solution of our boundary-value problem is X (x) = const sinπkx.Thus, the non-trivial solutions are $\mu_k = -\pi^2 k^2, X_k(x) = c_k sin\pi kx, k \in \mathbb{N}.$

(b) Let us find a non-trivial solution of the heat equation which satisfies the boundary condition u (0,t) = 0 = u (1,t), 0 ≤ t < + ∞ and has the form X (x) M (t). From the heat equation we have

$$M'(t)X(x) - M(t)X''(x) = 0 \Longrightarrow \frac{X''(x)}{X(x)} = \frac{M'(t)}{M(t)} \Longrightarrow$$

$$\frac{X''(x)}{X(x)} = \frac{M'(t)}{M(t)} = const := \mu \Longrightarrow$$

$$X''(x) - \mu X(x) = 0, \qquad M'(t) - \mu M(t) = 0.$$

The boundary condition is equivalent to

$M(t)X(0) = 0, M(t)X(1) = 0.$

Since M $\not\equiv$ 0, we obtain for X the problem from (a). Its non-trivial solutions are found above.

The general solution of M `(t)- μ M(t) = 0 is M (t) = const $e^{\mu t}$.

$c_k e^{-\pi^2 k^2 t} sin\ \pi kx, k \in \mathbb{N},$

Are non-trivial solutions of the heat equation which satisfy the boundary condition. Hence, a linear combination

$u(x,t) = \sum_k c_k e^{-\pi^2 k^{2t}} \sin \pi kx,$

Is also such a solution

Now the initial condition implies

$$\sum_k ck \sin \pi kx = 9sin2\pi x + 7sin5\pi x,$$
i.e.

And all other coefficients c_k equal 0. Thus
$u(x,t) = 9e^{-4\pi^{2t}} \sin 2\pi x + 7e^{-25\pi^{2t}} \sin 5\pi x$
Is the solution of the initial- boundary value problem?
(c) It is clear that the solution w has the form w = u+v , where u is the solution from (b) and v is the solution of the following initial-boundary value problem

$$\begin{cases} \frac{\partial v(x,l)}{\partial t} - \frac{\partial^2 v(x,l)}{\partial x^2} = 0, 0 < x < 1, \quad 0 < t < +\infty, \\ \quad v(0,t) = 1, \quad v(1,t) = 0, \quad 0 \le t < +\infty \\ \quad\quad v(x,0) = 1 - x, \quad 0 < x < 1. \end{cases}$$

It is easy to see that $v(x,t) := 1 - x$ solves the last problem. Therefore
$\omega(x,t) = 1 - x + 9e^{-4\pi^{2t}} \sin 2\pi x + 7e^{-25\pi^{2t}} \sin 5\pi x.$

3. Non-homogeneous problem: Duhamel's principle

Theorem 3.1. [2] (Duhamel) assume that
$h(x,t) in\ (II) satisfies\ h \in C^1\ and\ h(0,t) = h(l,0) = 0.$ Let W (t,x,τ) be a solution of

$$\begin{cases} W_t = c^2 W_{xx,} (t > \tau, 0 < x < l) \\ W|_{t=\tau} = h(x,\tau), (0 < x < l), \\ W|_{x=0} = 0\ w|_{x=l} = 0, (t > \tau). \end{cases} \quad (3.1)$$

Then

$$u(x,t) = \int_0^l W(t,x,\tau) d\tau$$

Is a solution of (II)
Proof the solution of (3.1) is

$$W(t,x,\tau) = \sum_{k=1}^{\infty} A_{k(\tau)} e^{-\frac{c^2 k^2 \pi^2 (t-\tau)}{l^2}} sin \frac{k\pi x}{l}, \quad (3.2)$$

Where

$$A_k(\tau) = \frac{2}{l} \int_0^l h(x,\tau) sin \frac{k\pi x}{l} dx, (3.3)$$

$$W(t,x,t) = \sum_{k=1}^{\infty} A_k(t) e^{-\frac{c^2 k^2 \pi^2 (t-\tau)}{l^2}} sin \frac{k\pi x}{l} = \sum_{k=1}^{\infty} A_k(t) sin \frac{k\pi x}{l} = h(x,t)$$

Therefore

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial t}\int_0^t W(t,x,\tau)d\tau$$

$$= W(t,x,t) + \int_0^t \frac{\partial}{\partial t}W(t,x,\tau)d\tau = h(x,t) + \int_0^t \frac{\partial}{\partial t}W(t,x,\tau)d\tau;$$

$$c^2\frac{\partial^2 u(x,t)}{\partial x^2} = \int_0^l c^2\frac{\partial^2}{\partial x^2}W(t,x,\tau)d\tau$$

$$u_t - c^2u_{xx} = h(x,l) + \int_0^t [W_t - c^2W_{xx}]d\tau = h(x,l),$$

(W is a solution of (3.1)). The boundary conditions can be easily checked.

4. Application

In this section, we solve the following two mixed problems by separation of variables and Duhamel's principle.

Examples 1:

$$\begin{cases} u_t = c^2u_{xx} + tx(l-x), & (t>0, 0<x<l), \\ u(x,0) = 0, & (0<x<l), \\ u(0,t) = 0, u(l,t) = 0, & (t>0) \end{cases}$$

Since $h(x,t)$ satisfies the hypotheses of Theorem 3.1 we calculate $A_k(\tau)$, noting that

$$\int x^2\sin(ax)\,dx = \frac{-a^2x^2\cos(ax) + 2\cos(ax) + 2axsin(ax)}{a^3}$$

$$\int xsin(ax)dx = \frac{\sin(ax) - axcos(ax)}{a^2} + C.$$

Thus with a=$\frac{k\pi}{l}$ $we\ have \sin(al) = 0\ and \cos(al) = \cos(k\pi).Hence,$

$$A_k(\tau) = \frac{2}{l}\int_0^l \tau x(i-x)sin\frac{k\pi x}{l}dx$$

$$= \frac{2\tau}{la^3}[a^2x^2\cos(ax) - 2\cos(ax) - 2axsin(ax) + al\sin(ax) - a^2lxcos(ax)]\,|_0^l$$

$$= \frac{2\tau}{la^3}[a^2l^2\cos(k\pi) - 2\cos(k\pi) - a^2l^2\cos(k\pi) + 2]$$

$$= \frac{4\tau l^2}{k^3\pi^3}[1-\cos(k\pi)]$$

$$= \begin{cases} \frac{8l^2\tau}{(2n+1)^3\pi^3}, k = 2n+1, n = 0,1,\dots. \\ 0, \quad k = 2n, \quad n = 1,2,\dots. \end{cases}$$

Therefore

$$u(x,t) = \int_0^t \sum_{n=0}^{\infty} A_k(\tau)e^{-\frac{c^2(2n+1)^2\pi^2(t-\tau)}{l^2}} sin\frac{(2n+1)\pi x}{l}$$

$$= \sum_{n=0}^{\infty}\left[\int_0^t e^{-\frac{c^2(2n+1)^2\pi^2(t-\tau)}{l^2}} \frac{8l^2\tau}{(2n+1)^3\pi^3} dr\right] sin\, \frac{(2n+1)\pi x}{l}.$$

$$= \sum_{n=0}^{\infty}\left[\frac{8l^4 t}{c^2(2n+1)^5\pi^5} + \frac{8l^6}{c^4(2n+1)^7\pi^7}(e^{-\frac{c^2(2n+1)^2\pi^2 t}{l^2}} - 1)\right] sin\, \frac{(2n+1)\pi x}{l}.$$

Example 2:

$$\begin{cases} u_t = c^2 u_{xx} + tsinx, & (t > 0, 0 < x < \pi), \\ u(x,0) = 0, & (0 < x < \pi), \\ u(0,t) = 0, \quad u(\pi,t) = 0, & (t > 0). \end{cases}$$

Since $h\,(x,t)$ satisfies the hypotheses of Theorem 3.1 we calculate $A_k\,(\tau)$, noting that

$$\int x^2 \sin(ax)\, dx = \frac{-a^2x^2\cos(ax) + 2\cos(ax) + 2axsin(ax)}{a^3}$$

$$\int xsin(ax)dx = \frac{\sin(ax) - axcos(ax)}{a^2} + C.$$

Thus with a=$\frac{k\pi}{l}$ $we\ have \sin(al) = 0\ and \cos(al) = \cos(k\pi)\,.Hence,$

$$A_k(\tau) = \frac{2}{l}\int_0^l \tau \sin x \sin\frac{k\pi x}{l} dx$$

$$= \frac{2\tau}{l}\left[\frac{\sin\left(1-\frac{k\pi}{l}\right)x}{2\left(1-\frac{k\pi}{l}\right)} - \frac{\sin\left(1+\frac{k\pi}{l}\right)x}{2\left(1+\frac{k\pi}{l}\right)}\right]_0^l$$

$$= \frac{\tau}{l}\left[\frac{\sin(1-a)l}{(1-a)} - \frac{\sin(1+a)l}{(1+a)}\right]$$

$$= \frac{2\tau k\pi}{l^2 - k^2\pi^2}[\sin l \cos kl]$$

Therefore

$$u(x,t) = \int_0^t \sum_{n=0}^{\infty} A_k(\tau) e^{-\frac{c^2k^2\pi^2(t-\tau)}{l^2}} sin\frac{k\pi x}{l}$$

$$= \sum_{n=0}^{\infty}\left[\int_0^t e^{-\frac{c^2k^2\pi^2(t-\tau)}{l^2}} \frac{2\tau k\pi}{l^2 - k^2\pi^2}[\sin l \cos kl] d\tau\right] sin\frac{k\pi x}{l}.$$

$$= \sum_{n=0}^{\infty}\left[\left[\frac{l^2}{c^2k^2\pi^2} - \left[\frac{l^4}{c^4k^4\pi^4}\left(e^{-\frac{c^2k^2\pi^2 t}{l^2}} + 1\right)\right] + e^{-\frac{c^2k^2\pi^2 t}{l^2}}\right]\frac{2k\pi}{l^2 - k^2\pi^2}[\sin l \cos kl]\right] sin\,\frac{k\pi x}{l}.$$

5. Conclusion

In this paper, we have used both the separation of variables method and Duhamel's principle to solve the non-homogeneous heat equation with non-zero initial-boundary conditions. We conclude that this technique is robust and it can be used efficiently to obtain the solution on time-depended problems, such as heat equation.

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References

1. L.C. Evans, **"Partial Differential Equations"**, American Mathematical Socity, 1998.

2. P. Giesl, **"Partial Differential Equation"**, Special Lectures, university of Sussex, 2010
3. O. A. Ladyzenskaja, V.A.Solonnikov and N.N.Uralceva, **Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs**, American Mathematical Society, 23, 1968.

4. M. A. Rasheed, Y. H. Mohammed, **On the Analytical and Numerical Solutions of Heat Equation**, J. colle. Basic edu. , 24(101), 49-61. (2018).

On The Analytical Solutions of Nonhomogeneous Heat equations

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حول الحلول التحليلية لمعادلات الحرارة غير المتجانسة

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الملخص:
في هذا البحث، نعتبر الحل التحليلي للمشكلة المختلطة غير المتجانسة لمعادلة الحرارة. ونستخدم طريقة فصل المتغيرات ومبدأ دوهاميل لاستخلاص صيغة الحل العام لهذه المشكلة. علاوة على ذلك، نجد الحل لحالة خاصة من هذه المشكلة كتطبيق لنتائجنا.

الكلمات المفتاحية: معادلات الحرارة، مشكلة مختلطة، فصل المتغيرات ومبدأ دوهاميل.