Numerical Solution of Linear Volterra Integral Equations by using Hermite Polynomials

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Abstract:

In this research, Hermite series method is used to find the convergent numerical solutions, this method is tested on linear Volterra integral equation of the second kind, the result was highly accurate. The algorithm and example is given to explain the solution procedures. The comparison of numerical solutions was compatible with exact solutions.

Keywords: Hermite polynomials, Linear Volterra integral equation,Approximate Solution.

1. Introduction:

The Volterra integral equations are a special type of integral equations, and they are divided into the first kind and second kind Alinear Volterra equation of the first kind has the form:

$$f(t) = \beta \int_a^t M(t,s)u(s)ds \qquad a \leq t,s \leq b \qquad (1)$$

A linear Volterra equation of the second kind has the form of

$$u(t) = f(t) + \beta \int_a^t M(t,s)u(s)ds \qquad (2)$$

Where f and M are known continuous functions, β is a constant parameter and u is unknown function. This kind has been described analytically and numerically [1]and[6].

The central objective of this survey is to secure and survey the approximate solution for (2) in the space of continuous maps on the interval [a,b] by means of using function series and we assume that the considered equations have at least one solution.

In the study of solution of Volterra integral equations, different methods of solving Volterra integral equations has been developed in recent years, like Bernestein approximation ,which has been pressented to solve Volterra integral equation[7], Elzaki Transform,applied to find solutions for Volterra integral equation of the second kind[8],Non-polynomial spline functions,used to find solution of integral equation[2],Sinc-collocation method to solve a Volterra integral equation[3],Taylor-series expansion method, which has been used to solve Volterra integral equation of the second kind with convolution kernel[4].

This paper is organized as follows.In section two the definition of Hermite polynomials is given In section three a matrix impersonation for Hermite series is stated.

In section four, solution of linear Volterra integral equation with Hermite polynomials. In section five algorithm. Applications is given in part six. lastly, Abrief of the conclusion and discussion is stated in the final section.

2.Hermite Polynomials

Definition: The Hermite polynomials of nth degree over the interval [a,b] is a set of orthogonal polynomials and it is defined by the recurrence relation[5]:

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right), \quad n = 0,1,2,\ldots. -\infty < t < \infty \qquad (3)$$

$H_0(t) = 1$

$H_1(t) = 2t$

$H_2(t) = -2 + 4t^2$

$H_3(t) = -(12 - 8t^3)$

$H_4(t) = (12 - 48t^2 + 16t^4)$

$H_5(t) = (120t - 160t^2 + 32t^5)$

$H_6(t) = -(120 - 72t^2 + 480t^4 - 64t^6)$

3. A matrix Impersonation For Hermite Series

On much enforcement a matrix shape to the Hermite series is helpful. This is straight forward to develop if only looking at a linear combination in expression of dot products.Given a series written as a linear combination of the Hermite standard function:

$$H(s) = a_0 H_{0,N}(s) + a_1 H_{1,N}(s) + a_2 H_{2,N}(s) + \cdots + a_{N,N} H_N(s) \qquad (4)$$

It is easy to write this as a dot product of two vectors:

$$[H_{0,n}(s)\ \ H_{1,n}(s)\ H_{2,n} \ldots H_{n,n}(s)] \begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

(5)

This can transform to the next form:

$$\text{H(s)}=[1\ \ s\ \ s^2 \ldots s^n] \begin{bmatrix} h_{00} & 0 & 0 \ldots 0 \\ h_{01} & h_{11} & 0 \ldots 0 \\ \vdots & \vdots & \vdots \cdots \vdots \\ h_{0n} & h_{1n} & h_{2n} \end{bmatrix}$$

(6)

where h_{nn} are the coefficients of the force standard that are applied to limit the particular Hermite series.

We note that the matrix in this case is lower triangular.

4. Solution for Linear Volterra integral Equation with Hermite polynomials

In this section, Hermite series are applied to find the numerical solution for linear Volterra integral equation, as follows.: Rendering linear Volterra integral equation of the second kind.

$$u(t) = f(t) + \beta \int_a^t M(t,s)u(s)ds \quad t \in [a,b] \qquad (7)$$

$$u(s) = H(s)$$

Let $=\left[H_{0,n}(s)\ \ H_{1,n}(s)\ H_{2,n}(s) \dots H_{n,n}(s)\ \right]\begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix}$

By using equation (3) and applying the Hermite polynomials method for equation(5), we obtain the next form.

$$\left[H_{0,n}(t)\ H_{1,n}(t)H_{2,n}(t) \dots H_{n,n}(t)\right]\begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix} =$$

$$f(t) + \beta \int_a^t M(t,s)\left[H_{0,n}(s)\ H_{1,n}(s) \dots\ H_{n,n}(s)\right]\begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix} ds$$

(8)

By using equation(4),which can be transform to the next formula :

$$[1\ \ t\ \ t^2 \dots t^n]\begin{bmatrix} h_{00} & 0 & 0 \dots 0 \\ h_{10} & h_{11} & 0 \dots 0 \\ \vdots & \vdots & \vdots \ \dots \ \vdots \\ h_{n0} & h_{n1} & \dots\ h_{nn} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} =$$

$$f(t) + \beta \int_a^t M(t,s)[1\ \ s\ \ s^2 \dots s^n]\begin{bmatrix} h_{00} & 0 & 0 \dots 0 \\ h_{10} & h_{11} & 0 \dots 0 \\ \vdots & \vdots & \vdots \\ h_{n0} & h_{n1} & \dots\ h_{nn} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} ds$$

(9)

Now, to find all integration in equation (7).

Thereafter in order to locate $a_0, a_1, \dots, a_n$, we want n equations.

Now, select $t_i, i = 0,1,2,\dots,n$ in the interval [a,b], whose give (n) equations.

Solve the (n) equations by Gauss elimination to find the values $a_0, a_1, \dots, a_n$ and substitute in numerical method we get the numerical solution.

The following algorithm summarises the proceedings for finding the numerical solution for linear Volterra integral equation of the second kind.

5. Algorithm

Step1: select n the degree of Hermite polynomials

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right), n = 0,1,2,\dots - \infty < t < \infty$$

Step2: Put the Hermite polynomials in linear Volterra integral equation of the second kind.

$$\sum_{i=0}^{n} a_i H_{i,n}(t) = f(t) + \beta \int_0^t M(t,s) \sum_{i=0}^{n} a_i H_{i,n}(s) ds$$

Step3: Calculate the integration

$$\int_0^t M(t,s) \sum_{i=0}^{n} a_i H_{i,n}(s) ds$$

Step4: select $t_i, i = 0,1,2,\dots,n, t_i \in [a,b]$

Step5: Solve the linear system by Gauss elimination to find the unknown coefficients $(a_0, a_1, \dots, a_n)$.

Step6 : Substitute $a_0, a_1, \dots, a_n$ in $u(t) = \sum_{i=0}^{n} a_i H_{i,n}$ we get the numerical solution u(t).

6. Application

Consider the following (Linear Volterra Integral Equation)of the second kind on[0,1]

$$u(t) = t + \int_0^t (s-t)u(s)ds \ ,$$

Where $f(t) = t,\ \beta = 1, and\ M(t,s) = (s-t)$ and the exact solution of this linear Volterra integral equation is triangular function u(t)=sin(t)

1. Selecting the degree of Hermite series n=1, we have

$$u(t) = a_0 + 2a_1 t$$

$$a_0 + 2a_1 t = t + \int_0^t (s-t)[a_0 + 2a_1 s]ds$$

Next

$$a_0 + 2a_1 t = t + \int_0^t a_0(s-t)ds + \int_0^t 2a_1 s(s-t)ds$$

And after performing integration

$$a_0 + 2a_1 t = t + a_0\left[\frac{-t^2}{2}\right] + 2a_1\left[\frac{-t^3}{6}\right]$$

Then in order to determine $a_0\ and\ a_1$ we need two equations .

Now choice $t_i, i = 0,1$ in the interval [0,1] we give two equations

$$a_0 = 0$$

$$a_0 + \frac{7}{3}a_1 = 1$$

Solve two equations by Gauss elimination to find the values a_0 *and* a_1 as follows

$a_0 = 0$

$a_1 = 0.4286$

Then the solution is $u(t) = 0.8571t$

2. Selecting the degree of Hermite series n=2, we have

$u(t) = (a_0 - 2a_2) + 2a_1 t + 4a_2 t^2$

$$(a_0 - 2a_2) + 2a_1 t + 4a_2 t^2 = t + \int_0^t (s-t)[(a_0 - 2a_2) + 2a_1 s + 4a_2 s^2]ds$$

Next

$$(a_0 - 2a_2) + 2a_1 t + 4a_2 t^2 = x + \int_0^t (a_0 - 2a_2)(s-t)ds + \int_0^t 2a_1 s(s-t)ds + \int_0^t 4a_2 s^2 (s-t)ds$$

And after performing integration.

$$(a_0 - 2a_2) + 2a_1 t + 4a_2 t^2 = t + (a_0 - 2a_2)\left(\frac{-t^2}{2}\right) - a_1 \frac{t^3}{3} - a_2 \frac{t^4}{3}$$

Then in order to determine $a_0, a_1, and\ a_2$ we need three equations.

Now choice t_i , i=0,1,2 in the interval [0,1], we give three equations.

$a_0 - 2a_2 = 0$

$\frac{9}{8} a_0 + \frac{25}{24} a_1 - \frac{59}{48} a_2 = \frac{1}{2}$

$\frac{3}{2} a_0 + \frac{7}{3} a_1 + \frac{4}{3} a_2 = 1$

Solve three equations by Gauss elimination to find the values $a_0, a_1 and\ a_2$ as follows

$a_0 = -0.1172$

$a_1 = 0.5374$

$a_2 = -0.0586$

Then the solution is: $u(t) = 1.0748t - 0.2344t^2$

7.Conclusion and Discussion

The effect acquired from the application, has shown that Hermite series is a powerful and active technicality in finding approximate solution of linear Volterra integral equation of the second kind .Table (1) and figure(1) present small errors whereas Table (2) and figure(2)show that the numerical solution is almost equal to the exact one. Thus increasing the number of expression in Hermite series makes the approximate solution approach to the exact solution.

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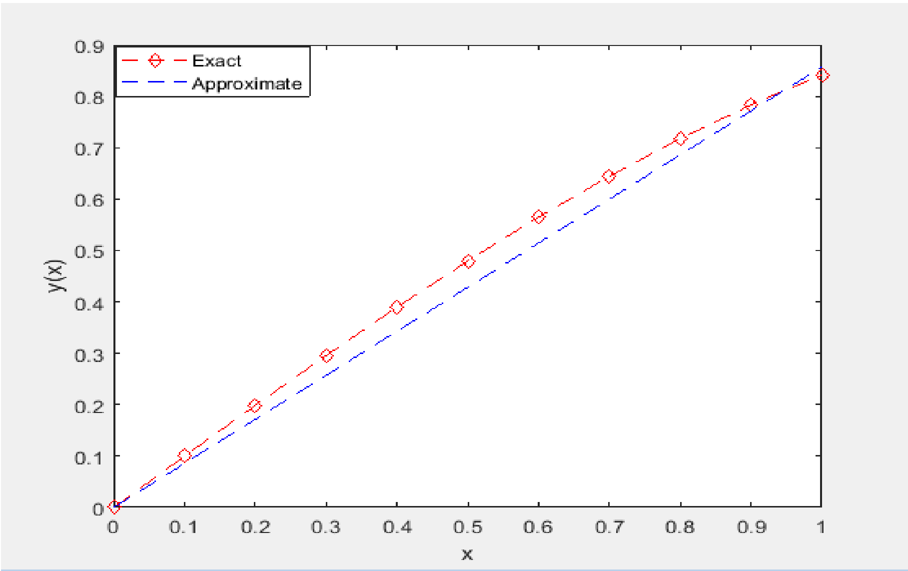
Table(1):The results of numerical example when(n=1)

x	Exact,y(x)	Approximate,y(x) of degree (n=1)	Error	$(y_{Exact}(x) - yApproximation(x))^2$
0	0	0	0	0
0.1	0.0998	0.0857	0.0141	0.00019881
0.2	0.1987	0.1714	0.0272	0.00073984
0.3	0.2955	0.2571	0.0384	0.00147456
0.4	0.3894	0.3428	0.0466	0.00217156
0.5	0.4794	0.4285	0.0509	0.00259081
0.6	0.5646	0.5143	0.0504	0.00254016
0.7	0.6442	0.6000	0.0442	0.00195364
0.8	0.7174	0.6857	0.0317	0.00100489
0.9	0.7833	0.7714	0.0119	0.00014161
1	0.8415	0.8571	0.0156	0.00024336
L.S.E=$\sum_{i=0}^{10}(y_{Exact}(x) - y_{Approximation}(x))^2$				0.01306068

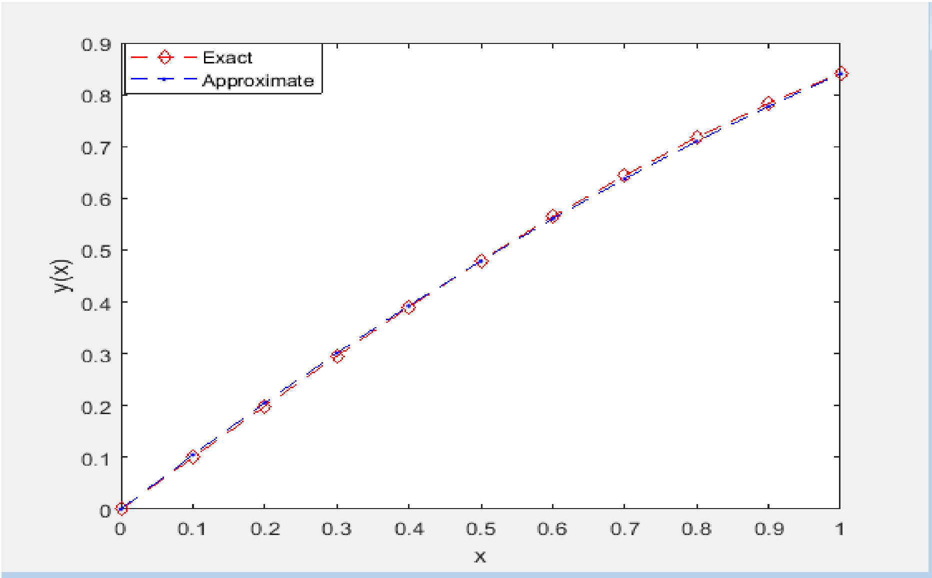
Table(2):The results of numerical example when(n=2)

x	Exact,y(x)	Approximate,y(x) of degree(n=2)	Error	$(y_{Exact}(x) - y_{Approximation}(x))^2$
0	0	0	0	0
0.1	0.0998	0.1051	0.0053	0.00002809
0.2	0.1987	0.2056	0.0069	0.00004761
0.3	0.2955	0.3013	0.0058	0.00003364
0.4	0.3894	0.3924	0.0030	0.000009
0.5	0.4794	0.4788	0.0006	0.00000036
0.6	0.5646	0.5605	0.0041	0.00001681
0.7	0.6442	0.6375	0.0067	0.00004489
0.8	0.7174	0.7098	0.0075	0.00005625
0.9	0.7833	0.7775	0.0059	0.00003481
1	0.8415	0.8404	0.0011	0.00000121
L.S.E=$\sum_{i=0}^{10}(y_{Exact}(x) - y_{Approximation}(x))^2$				0.00027591



Figure(1)

Exact and Approximate solution of linear Volterra integral equation of application when (n=1)



Figure(2)

Exact and Approximate solution of linear Volterra integral equation of application when (n=2)

الخلاصة: في هذا البحث أستخدمت طريقة متعددة حدود هيرميت لايجاد الحلول العددية التقريبية. جربت هذه الطريقة على معادلة فولتيرا الخطية التكاملية من النوع الثاني وأن هذه الطريقة كانت عالية الدقة. كما ان الخوارزمية والمثال المعطاة هي لتوضيح أجراءات الحل. مقارنة الحلول العددية كانت متوافقة مع الحلول الحقيقية.