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Least Square Method For Solving The Nonlinear Delay Fourth Order Eigenvalue Problems Consists Delay Ordinary Differential Equations

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Abstract:

This paper has been devoted to study the nonlinear delay fourth order eigen-value problems consists delay ordinary differential equations, one of the expansion methods that called the least square method will be developed to solve this type of problems.

Keywords: Nonlinear Fourth Order Sturm-Liouville Problems, Delay Eigen-Value Problems

1.Introduction

The nonlinear delay fourth order eigenvalue problems consist of delay nonlinear ordinary differential equations with the boundary conditions defined on some interval, have many applications in different scientific fields, physics, biology and engineering science. One of the most important application referred to as a delay nonlinear eigenvalue problem, [1].

The delay eigenvalue problem belongs to a wide class of problems whose eigenvalues and eigen-functions have particularly nice properties, [2].

In this paper we study and solve this type of problems using the least square method.

2. Preliminaries

In this section some basic definitions and remarks that needed in this work are recalled. We start with the following definition.

2.1 Definition

A delay differential equation is an equation in which the unknown function and some of its derivatives, evaluated at arguments which are different by any of fixed number or function of values.

Consider the n-th order delay differential equation:

$$K(t, h(t), h(t-\tau_1),\ldots, h(t-\tau_m), h'(t), h'(t-\tau_1),\ldots, h'(x-\tau_m), h^{(n)}(t),\ldots, h^{(n)}(t-\tau_m)) = j(t) \qquad (2.1)$$

where K is a given function and $\tau_1, \tau_2, \ldots, \tau_m$ are given fixed positive numbers called the time delays, [1].

We say that equation (2.1) is homogenous delay differential equation in case $j(t)= 0$, which we handle in this paper, otherwise this equation is called non-homogenous delay differential equation,[2].

2.2 Definition

The delay differential equation is said to be nonlinear when it is nonlinear with respect to the unknown function that enter with different arguments and their derivatives that appeared in it, [1].

Hence, the new concepts of this work is given by the following definition.

2.3 Definition

The delay eigen-value problem consist of delay ordinary differential equation is said to be nonlinear when it is nonlinear with respect to the unknown eigen-function enter with different arguments and their derivatives that appeared in it.

Next, consider the following nonlinear delay fourth order eigen-value problem:

$$-(p(t)h''(x))'' + (q(t)h'(t-\tau))' - j(t, \lambda, h(t-\tau), h'(t-\boldsymbol{\tau}), h''(t-\boldsymbol{\tau}), h'''(t-\boldsymbol{\tau})) = 0$$

(2.2)

with the associated boundary conditions:

$a_1 h(a) + a_2 h'(a) = 0 \quad , t \in [a-\tau, a]$

$b_1 h(a) + b_2 h''(a) = 0 \quad , t \in [a-\tau, a]$

(2.3)

$e_1 h(b) + e_2 h'(b) = 0 \quad , t \in [b-\tau, b]$

$f_1 h(b) + f_2 h''(b) = 0 \quad , t \in [b-\tau, b]$

$h(t-\tau) = \varphi(t-\tau), \text{ if } \ t-\tau < a$

where p, p' and q are given real-valued continuous functions defined on the interval $[a,b]$, p is positive, not both coefficients in one condition are zero, $\tau > 0$ is the time delay, f is a known nonlinear function with respect to y. φ is the initial function defined on $t \in [t_0 - \tau, t_0]$. The problem here is to determine the eigen-value λ in which a nontrivial solution y for the problem given by equations (2.2)-(2.3) occurs. In this case λ is said to be a delay eigenvalue and y is the associated delay eigen-function.

In other words y is an eigen function for the variable x and the nonlinear function $j(t, \lambda, h(-\tau), h'(t-\tau), h''(t-\tau), h'''(t-\tau))$ with respect to the eigen-value λ.

Like the linear fourth order eigenvalue problems, the problem given by equations (2.2)-(2.3) satisfies the following remarks, [2].

2.4 Remarks

1. The linear delay operator:

$L = -\frac{d^4}{dt^4} p(t) - \frac{d^3}{dt^3} p'(t) - \frac{d^2}{dt^2} p''(t) - \frac{d}{dt} p'''(t) + A(t)q(t),$ where $A(x)$ is an operator defined by $A(t)h(t) = h(t-\tau)$, is self- adjoint, [3] .

2. The delay eigen-functions are orthogonal.

3. There are infinite number of delay eigenvalues forming a monotone increasing sequence with $\lambda_j \to \infty$ as $j \to \infty$**.** Moreover, the delay eigen-functions corresponding to the delay eigenvalues has exactly j roots on the interval (a,b)**.**

4. The delay eigen-functions are complete and normal in $L^2[a,b]$.

5. Each delay eigenvalue corresponds only one delay eigen-function in $L^2[a,b]$.

To prove remarks (2-5), see [4]**.**

3. The Least-Square Method

This method is one of the expansion methods that used to solve the linear (nonlinear) differential equations with or without delays, [5], [6].

Here we develop this method to solve the problem given by equations(2.1)-(2.2).

The method is based on approximating the unknown function y as a linear combination of n linearly independent functions $\{\phi_i\}^n{}_{i=1}$, that is write

$$h=\sum_{i=1}^{n}\phi_i(t) \tag{3.1}$$

which implies that, $h(t-\tau)=\sum_{i=1}^{n}\phi_i(t-\tau)$

this approximated solution must satisfy the boundary conditions given by equation (2.2) to get a new approximated solution. By substituting this approximated solution into equation(2.1) one can get:

$$R(t,\lambda,\vec{c})=-((p''(t))(\sum_{i=1}^{n}\phi_i(t)'')''+q'(t)(\sum_{i=1}^{n}\phi_i(t-\tau))'$$
$$-\,j(t,\lambda,\sum_{i=1}^{n}\phi_i(t-\tau),(\sum_{i=1}^{n}\phi_i(t-\tau))',(\sum_{i=1}^{n}\phi_i(t-\tau))'',(\sum_{i=1}^{n}\phi_i(t-\tau))''')$$

(3.2)

where R is the error in the approximation of equation(3.2)and $\vec{c}$ is the vector of $n-2$ elements of c_i, $i=1,2,\dots,n$, [7], [8]

Thus, to minimize the functional:

$$J(\lambda,\vec{c})=\int_a^b (R(t,\lambda,\vec{c}))^2\,dx \tag{3.3}$$

put $\frac{\partial J}{\partial\lambda}=\frac{\partial J}{\partial c_i}=0$, $i = 1,2,\dots,n$, to get a system of $n-1$ nonlinear equations with $n-1$ unknowns which can be solved by any suitable method to get the values of λ and $\vec{c}$,[9],[10].

To illustrate this method, consider the following example:

Example 3.1

Consider the following nonlinear delay eigenvalue problem:

$$(t-1)h^{(4)}(t)=-1+h'(t)-\frac{t}{3}h''(t)+\lambda h'''(t-1),\quad t\in[1,2] \tag{3.4}$$

with the associated boundary conditions:

$h(1)+h'(1)=1 \qquad ,t\in[0,1]$

(3.5)

$yh''(2)-h'''(2)=-12 \qquad ,t\in[1,2]$

we use the least-square method to solve this problem. To do this, we approximate the unknown function *y* as a polynomial of degree four, that is, write

$$h(t)=\sum_{i=1}^{5} c_i t^{i-1}=c_1+c_2t+c_3t^2+c_4t^3+c_5t^4$$

Therefore,

$$h(t-1)=c_1+c_2(t-1)+c_3(t-1)^2+c_4(t-1)^3+c_5(t-1)^4$$

Thus, if we satisfy the boundary conditions by the above approximated solution. We get:

$$c_1=1-c_2$$
$$c_3=6c_5-6$$
$$c_4=0$$

By substituting this approximated solution into equation (3.4), we obtain

$$R(t,\lambda,c_2,c_5)=1+8t-c_2-24c_5+16c_5t-\lambda(24c_5t-24c_5)$$

Thus, if we minimize the functional

$$J(\lambda,c_2,c_5)=\int_1^2 (R(t,\lambda,c_2,c_5))^2\,dx$$

set $\frac{\partial J}{\partial\lambda}=\frac{\partial J}{\partial c_2}=\frac{\partial J}{\partial c_5}=0$ to get the following system of nonlinear equations:

$$\frac{\partial}{\partial\lambda}\int_1^2 (R(t,\lambda,c_2,c_5))^2\,dt = 0$$

$$\frac{\partial}{\partial c_2}\int_1^2 (R(t,\lambda,c_2,c_5))^2\,dt = 0$$

$$\frac{\partial}{\partial c_5}\int_1^2 (R(t,\lambda,c_2,c_5))^2\,dt = 0$$

Solving the above system by any suitable method, to find that the nontrivial solution we got $\lambda = c_2 = c_5 = 1$ and $c_1 = c_3 = 0$. Therefore the delay eigenvalue $\lambda=1$ with the corresponding delay eigen-function $h(t) = t + t^4$, $t\in[1,2]$ is the nontrivial solution to equation (3.4) with the associated boundary conditions given by equations(3.5).

Which implies that $h(t-1) = (t-1) + (t-1)^4$, $x\in[1,2]$

Generally, if $h(t) = \sum_{i=1}^{n} c_i t^{i-1}$, then the same result can be obtained for all values of n, $n\in N$.

That is if $h(t-\tau) = \sum_{i=1}^{n} c_i (t-\tau)^{i-1}$, then $h(t-1) = (t-1) + (t-1)^4$, $t \in [1,2]$ corresponding to the same delay eigenvalue.

Example 3.2

Consider the following nonlinear delay eigenvalue problem:

$$3h^{(4)}(t) = t^2 h''(t-\frac{\pi}{2}) + t^2 h(t-\frac{\pi}{2}) + \lambda h'(t-\frac{\pi}{2}),\quad t\in[\frac{\pi}{2},\pi] \qquad (3.6)$$

with the associated boundary conditions:

$$\left.\begin{aligned} &h(\frac{\pi}{2}) + h'(\frac{\pi}{2}) = 1 \\ &h''(\frac{\pi}{2}) - h'''(\frac{\pi}{2}) = -1 \end{aligned}\right\} ,t\in[0,\frac{\pi}{2}]$$

$$\left.\begin{aligned} &2h(\pi) + h'(\pi) = 2 \\ &-h''(\pi) + h'''(\pi) = 1 \end{aligned}\right\} ,t\in[\frac{\pi}{2},\pi] \qquad (3.7)$$

Using the same steps in example (3.1) we find that the nontrivial solution is $\lambda = 2$. The delay eigenvalue with the corresponding delay eigen-function

$$h(t-\frac{\pi}{2}) = \cos(t-\frac{\pi}{2}), \qquad t \in [\frac{\pi}{2}, \pi].$$

4. **conclusions**

From this study we can point out the following conclusions

1. The nonlinear fourth order delay eigenvalue problems consist of delay nonlinear ordinary differential equations satisfies the same properties as those consist of delay nonlinear ordinary differential equations with or without delay.

2. The least square method can be developed to solve the above source of problems.

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طريقة المربعات الصغرى لحل مسـائل القيم الذاتية التباطؤيه اللاخطية رباعية المحتوية على معادلات تفاضلية اعتيادية تباطؤية رباعية الرتبة

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مستخلص البحث

كرس هذا البحث لدراسة مسـائل القيم الذاتية التباطؤيه اللاخطية المحتوية على معادلات تفاضلية اعتيادية تباطؤية رباعية الرتبة و تطوير واحدة من الطرق التوسعية وهي طريقة المربعات الصغرى لحل هذا النوع من المسائل.