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Fuzzy Fixed Point of Picard-S Iterative Scheme for Fuzzy Mapping

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Abstract:

In this paper, we define the fuzzy contraction mapping in metric space. Furthermore, we show that the Picard-S iteration method converges to an α-fuzzy invariant point for fuzzy contraction mapping.

Keywords: fuzzy contraction mapping, fuzzy fixed point, Picard-S iterative scheme.

1. Introduction

In the theory of invariant point the contraction is a powerful tool of mathematics for studying the existence of an invariant point. Banach contraction method [1] is an incredibly useful tool via the development of nonlinear equations.

In 2014, Gürsoy introduced his new iterative which known as the Picard-S iterative scheme [2].

Zadeh [7] introduced the concept of fuzzy sets, in 1965.

In 1981, Heilpern [3] achieved an invariant point thm. for fuzzy contraction mapps. On the other hand, in 2012 Wardowski [6] generalizations Banach by suggesting the concept of contraction and proving a fixed-point theorem.

In this search, we suggest a definition of fuzzy contraction mapping in metric space. Also, we prove the convergence of the Picard-S iterative scheme to an α-fuzzy fixed point when applied to fuzzy contraction mapping.

2. Preliminaries

In this section, we state some basic definitions, those will be used in next section to prove the main results.

Definition(2.1),[5]: Let $T: X \to X$ be a mapping known as contraction such that (X,d) is a metric space if there is a real number $0 < \delta < 1$ such that

$d(Tx, Ty) \leq \delta d(x, y)$ for all $x, y \in X$ (2.1)

Definition (2.2),[2]: Let B a Banach space, and C be a nonempty convex subset of B and $T: C \to C$ be a self mapping. Let $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences of real number in $[0,1]$.For arbitrary $u_1 \in C$ define a sequence $\{x_n\}_{n=1}^{\infty}$ in C by

$u_{n+1} = Ty_n$

$v_n = (1 - \beta_n)Tu_n + \beta_n Tw_n$

$w_n = (1 - \gamma_n)u_n + \gamma_n Tu_n, n \in \mathbb{N}$ (2.2)

The sequence $\{u_n\}_{n=1}^{\infty}$ is called **Picard-S iterative scheme**.

Definition (2.3), [4]: Let(X, d)be a Banach space. A fuzzy set D in X is a fun. from X into $[0,1]$. If $x \in X$ then the fun. value $D(x)$ is known as the grade of membership of $x \in D$. The collection of all fuzzy sets in X is $\mathcal{F}(X)$.
For $\alpha \in [0,1]$ and $D \in \mathcal{F}(X)$. The notation $[D]_\alpha$ is called α-level set (or α-cut set) of D and is defined as follows:

$[D]_\propto = \{x: D(x) \geq \propto\}$ if $\propto \in (0,1]$,

and

$[D]_0 = \overline{\{x: D(x) > 0\}}$

whenever the closure of the set B in X is denoted by $\bar{B}$.

Definition (2.4), [4]: Let Xbe any set and Ybe a Banach space.A mapping$T: X \to \mathcal{F}(Y)$ is known a fuzzy mapping over the set Y.

Definition (2.5),[4]: Let $T: X \to \mathcal{F}(X)$ be a fuzzy mapping over a Banach space (X, d). A point u^* in X is called α-fuzzy-fixed point of T if $u^* \in [Tu^*]_{\alpha(u^*)}$.

Definition (2.6): Let $T: X \to \mathcal{F}(X)$ be a mapping known as fuzzy contraction mapping with $\alpha: X \to (0,1]$ in a metric space. (X, d) such that nonempty closed set denotes by $[Tu]_{\propto(u)}$ is a subset of X for all $u \in X$ if there exists constant $\delta \in (0,1]$ such that

$$d\left([Tu]_{\propto(u)}, [Tv]_{\propto(v)}\right) \leq \delta d(u, v) \text{ for all } u, v \in X \quad (2.3)$$

3. Main Results

In this part, in the framework of a Banach space, we state and prove the convergence of fuzzy contraction mapping (2.3) to an α-fuzzy fixed point as follows:

Theorem (3.1): Let X be a Banach space C be a non-void closed convex set such that C is a subset of X and $T: C \to \mathcal{F}(C)$ be a fuzzy contraction mapping (2.3). We have $\alpha: C \to (0,1]$ such that $[Tu]_{\propto(u)}$ is a nonvoid closed convex subset of C for all $u \in C$ with $\{u_n\}_{n=1}^{\infty}$ is a Picard-S iterative scheme defined by (2.2) with real sequences $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty} \beta_k \gamma_k = \infty$ Then $\{u_n\}_{n=1}^{\infty}$ converges to an α-fuzzy invariant point of T.

Proof: Using iterative scheme (2.2) and condition (2.3), we have:

$$\begin{aligned}
\|u_n - u^*\| &= \left\|(1-\gamma_n)u_n + [Tu_n]_{\alpha(u_n)} - u^*\right\| \\
&= \left\|(1-\gamma_n)u_n + [Tu_n]_{\alpha(u_n)} - u^* + \gamma_n u^* - \gamma_n u^*\right\| \\
&= \left\|(1-\gamma_n)u_n + [Tu_n]_{\alpha(u_n)} - u^* + (1-\gamma_n)u^* - \gamma_n [Tu^*]_{\propto(u^*)}\right\| \\
&\leq (1-\gamma_n)\|u_n - u^*\| + \gamma_n \left\|[Tu_n]_{\alpha(u_n)} - [Tu^*]_{\propto(u^*)}\right\| \\
&\leq (1-\gamma_n)\|u_n - u^*\| + \gamma_n \delta \|u_n - u^*\| \\
&\leq [1 - \gamma_n(1-\delta)]\|u_n - u^*\|
\end{aligned}$$

(3.1)

$$\|v_n - u^*\| = \left\|(1-\beta_n)[Tu_n]_{\alpha(u_n)} + \beta_n [Tw_n]_{\alpha(w_n)} - u^*\right\|$$

$$= \left\|(1-\beta_n)[Tu_n]_{\alpha(u_n)} + \beta_n[Tw_n]_{\alpha(w_n)} - [Tu^*]_{\propto(u^*)} + \beta_n[Tu^*]_{\propto(u^*)} - \beta_n[Tu^*]_{\propto(u^*)}\right\|$$

$$\leq \left\|(1-\beta_n)\left\|[Tu_n]_{\alpha(u_n)} - [Tu^*]_{\propto(u^*)}\right\| + \beta_n\left\|[Tw_n]_{\alpha(w_n)} - [Tu^*]_{\propto(u^*)}\right\|\right\|$$

$$\leq (1-\beta_n)\delta\|u_n - u^*\| + \beta_n\delta\|w_n - u^*\|$$

(3.2)

Thus by (3.1) and (3.2), we get:

$$\|u_{n+1} - u^*\| = \left\|[Tv_n]_{\alpha(v_n)} - u^*\right\|$$

$$\|u_{n+1} - u^*\| = \left\|[Tv_n]_{\alpha(v_n)} - [Tu^*]_{\propto(u^*)}\right\|$$

$$\leq \delta\|v_n - u^*\|$$

$$\leq \delta^2(1-\beta_n)\|u_n - u^*\| + \beta_n\delta^2\|w_n - u^*\|$$

$$\leq \delta^2(1-\beta_n)\|u_n - u^*\| + \beta_n\delta^2[1-\gamma_n(1-\delta)]\|u_n - u^*\|$$

$$\leq \delta^2\|u_n - u^*\| - \beta_n\delta^2\|u_n - u^*\| + \beta_n\delta^2\|u_n - u^*\| - \beta_n\gamma_n\delta^2(1-\delta)\|u_n - u^*\|$$

$$= \delta^2[1-\beta_n\gamma_n(1-\delta)]\|u_n - u^*\|$$

Repeating this process n times, we obtain:

$$\|u_{n+1} - u^*\| \leq \delta^{2(n+1)}\prod_{k=0}^{n}[1-\beta_k\gamma_k(1-\delta)]\|u_1 - u^*\|$$

$$\leq \delta^{2(n+1)}\|u_1 - u^*\|e^{-(1-\delta)\sum_{k=1}^{\infty}\beta_k\gamma_k}$$

since $0 < \propto < 1$ and $\sum_{k=1}^{\infty}\beta_k\gamma_k = \infty$ so $\delta^{2(n+1)}e^{-(1-\delta)\sum_{k=1}^{\infty}\beta_k\gamma_k} \to 0$ as $n \to \infty$. Which implies that $\lim_{n\to\infty}\|u_n - u^*\| = 0$. Therefore, $\{u_n\}_{n=1}^{\infty}$ converges to u^*.

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النقطة الثابتة الضبابية لمخطط Picard-S التكراري للدوال الضبابية

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مستخلص البحث:

في هذا البحث سوف نحدد مخطط الانكماش الضبابي في الفضاء المتري ،علاوة على ذلك ، نوضح ان طريقة التكرار picard –s تتقارب مع نقطة ثابتة غير متغيرة -α ضبابية لدوال الانكماش الضبابية.