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Observability of fractional order differential impulsive multi control problem with fractional integral nonlocal initial condition

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Abstract :

In this paper ,the obsevability of fractional differential impulsive multi control abstract problem with fractional integral nonlocal initial condition have been studied as abstract Cauchy problem for using Banach fixed point which defined on space of presented problem which is piecewise continuous space and proprieties of the initial observable condition for their problem.

1.Introduction:

The observability of linear and linear impulsive abstract problems which defined in infinite dimensional continues or piecewise continuous appearing in many researches [5][3].The observability depended on nonlinear part and many methods of certain fixed point theorems depened on their problems. The impulsive fractional order abstract control problems with general nonlocal initial condition have been appeared in limited classes with different approach such as, [4],[6],[11],[8], [9].

Consider the following impulsive multi control fractional differential abstract problem with fractional integral nonlocal initial condition:

$$\begin{cases} D^{\alpha}x(t)=A(t)x(t)+f_1\left(t,x(t),\int_0^t h(t,s,x(s))ds\right)f_2(t,x(t),D^{\alpha}x(t))+\sum_{i=1}^{2}B_iu_i(t),t\neq t_k & (1)\\ u_1(t)=-Kx(t),\ u_2(t)\leq v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(t)d\tau, & (2)\\ \Delta x(t_k)=x(t_k^+)-x(t_k^-), & (3)\\ x(0)+I^{\beta}g(x)=x_0, & (4)\\ y(t)=C(t)x(t)C^*(t). & (5)\end{cases}$$

where $t \in J = [0,T], k = 1,2,\dots,m$, $0 < \alpha \leq 1$, D^{α} is the caputo fractional derivative. $v(t), a(t)$ are nonnegative functions. Assume a bounded operator A(t): $X \rightarrow X$ (X Banach space), $f_1, f_2 : J \times X \times X \rightarrow X$, $h: t \times s \times X \rightarrow X$, $0 \leq s \leq t \leq T$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$, $x(t_k^-)$, denoted the left and the right limit of x at t_k , respectively g:PC([0,T]; X) $\rightarrow X$ is a given function. y(.) is referred to as the output which is belong to Banach space Y. C: $X \rightarrow Y$, $C^*: Y \rightarrow X$ is a bounded linear operator.

Our aim to study and present the obsevability of fractional differential impulsive multi control abstract problem with fractional integral nonlocal initial condition (1-5) with necessary and sufficient conditions that which guaranty the problem initial observable.

2. Preliminaries:

The following definitions and results are need it later on for investigate the initial observable for problem(1-3).

Definition(2.1), [7]:

The Riemann- Liounille fractional integral of a function f with order $\alpha > 0$, is $I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$, $t > 0, \alpha > 0$.(4)

Definition(2.2), [2]:

The Caputa fractional derivative of a function f with order $\alpha > 0$, where $n-1 < \alpha \leq n$,and $n \in N$, is defined by:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{n}(s) ds , t > 0, \alpha > 0. \quad (5)$$

Where f is absolutely continuous derivative up to $n-1$. If $0 < \alpha \leq 1$ then

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds .$$

Lemma (2.3), [7]:

Let $\alpha > 0$ and f be a suitable function. Then we have

$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) - f(0)$,where$0 < \alpha \leq 1$. (6)

Definition (2.4), [10]:

The function $x(.) \in PC([0,T]; X)$ is a mild solution of abstract problem (1-5) which is equivalent to

$x(t) =$

$$\begin{cases} x_0 - I^{\beta}g(x) + \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\int_0^t[\,[A(s)-K]x(s) + \\ \quad f_1\left(s, x(s), \int_0^t h(t,\tau,x(\tau))d\tau\right) f_2(s, x(s), D^{\alpha}x(s)) + B_2u_2(s)]ds, \quad t \in [0,t_1] \\ x_0 - I^{\beta}g(x) + \sum_{i=1}^{k} I_i\left(y(t_i)\right) + \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\int_0^t[\,[A(s)-K]x(s) + \\ \quad f_1\left(s, x(s), \int_0^t h(t,\tau,x(\tau))d\tau\right) f_2(s, x(s), D^{\alpha}x(s)) + B_2u_2(s)]ds, \quad t \in (t_k, t_{k+1}] \\ \quad , k = 1,2,\dots,m. \end{cases}$$

.If satisfies the integral (7).

Lemma (2.1.1), [10]:

A Mittage-Leffler function $E_{\alpha,\beta}(At^{\alpha})$, satisfies the following

1. $E_{\alpha,1}(At^{\alpha}) \le K_{E_{\alpha,1}}\|e^{At^{\alpha}}\|$, $\alpha > 1$. $K_{E_{\alpha,1}} > 1$, such that
2. $E_{\alpha,\alpha}(At^{\alpha}) \le K_{E_{\alpha,\alpha}}\|e^{At^{\alpha}}\|$, $\alpha > 1$, $K_{E_{\alpha,\alpha}} > 1$,where $A \in R^{nxn}$.

Lemma (2.1.2), [10]:

Let $\alpha > 0$ v(t) is a nonnegative function locally integrable on [0,T] and a(t) a nonnegative, nondecreasing continuous function defined on [0,T], a(t)<M and suppose z(t) is nonnegative and locally integrable on [0,T] with z(t) $\le$v(t)+a(t)$\int_0^t(t-\tau)^{\alpha-1}z(\tau)d\tau$. If $v(t)$ is a nondecreasing function on [0,T),we have z(t) $\le$v(t) $E_{\alpha}(\Gamma(\alpha)a(t)t^{\alpha})$.

Hypothesis :

Let $B_r = \{x \in X: \|x\| \le r\}$ is a neighborhood of zero, $t \in$[0,T].

(h1): $\|A(t)\|_{B(X)} \le M$ where $A: J \to B(X)$ is bounded linear operator and $\|B_1\| \le \tilde{\tilde{M}}, \|K\| \le \tilde{M}$,$\tilde{M}, M > 0$.

(h2): $f_1 f_2, : J \times X \times X \to X$ is continuous and there exist constant N_1>0 and $N_2 > 0$ such that

$\|f_1(t,x,u) - f_1(t,y,v)\| \le N_1[\|x-y\| + \|u-v\|]$, $x,y,u,v \in B_r$, $N_2 = max_{t\in J}\|f_1(t,x,u)\|$

$\|f_2(t,x,u) - f_2(t,y,v)\| \le \tilde{N}_1[\|x-y\| + \|u-v\|]$, $x,y,u,v \in B_r$, $\tilde{N}_2 = max_{t\in J}\|f_2(t,0,0)\|$

$\|f_2(t,x,u)\| \le K_{f_2}(t)\Omega_{f_2}(\|x\| + \|u\|) \le K_{f_2}(t)\Omega_{f_2}(r + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|u(t)\|)$

(h3): A continuous function $h: \Delta \times X \to X$ with fixed constants $H_1 > 0$ and $H_2 > 0$ satisfy

$\|h(t,s,x_1) - h(t,s,x_2)\| < H_1\|x_1 - x_2\|$, $x_1, x_2 \in B_r$ and $H_2 = max_{t\in J}\|h(t,0,0)\|$

(h4): $\|I_k(x_1)-I_k(x_2)\| \le \ell_1\|x_1-x_2\|$ and $\|I_k(x)\| < \ell_2$ for each $x_1, x_2, x \in X$ and k=1,…,m, $\ell_1, \ell_2 > 0$ (h5):
$\|I^\beta g(x_1)-I^\beta g(x_2)\| \le \left\|\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}d\tau\right\| \|g(x_1)-g(x_2)\| \le \frac{T^\beta G}{\Gamma(\beta+1)}\|x_1-x_2\|$,
for $x_1, x_2 \in PC([0,T];X)$. (h6): $\|I^{1-\alpha}y(t)\| \le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|y(t)\|$.
From the following system:

$$\begin{cases} D^\alpha x(t) = A(t)x(t) + f_1\left(t, x(t), \int_0^t h(t,s,x(s))ds\right) f_2(t, x(t), D^\alpha x(t)) + \sum_{i=1}^2 B_i u_i(t), t \ne t_k \\ u_1(t) = -Kx(t), \ \ u_2(t) \le v(t) + a(t)\int_0^t (t-\tau)^{\alpha-1} u_2(t)d\tau, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-), \\ x(0) + I^\beta g(x) = x_0, \\ y(t) = C(t)x(t)C^*(t). \end{cases}$$

C: $X \to Y$, $C^*: Y \to X$ is a bounded linear operator, the homogenous part is:

$$x(t) = \begin{cases} [x_0 - I^\beta g(x)] + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[[A(s)-K]x(s) + B_2u_2(s)]\,ds, t \in [0,t_1] \\ [x_0 - I^\beta g(x)] + \sum_{i=1}^k I_i\left(x(t_i)\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[[A(s)-K]x(s) + B_2u_2(s)]\,ds, \\ \qquad t \in (t_k, t_{k+1}], k = 1,2,\dots,m. \end{cases}$$

(8) and

$$y(t) = \begin{cases} C(t)[x_0 - I^\beta g(x)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[[A(s)-K]x(s) + B_2u_2(s)]ds\right] C^*(t), \\ C[x_0 - I^\beta g(x)] + C\sum_{i=1}^k I_i\left(x(t_i)\right) + \frac{C}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[[A(s)-K]x(s) + B_2u_2(s)]\, c \right. \\ \qquad t \in (t_k, t_{k+1}], k = 1,2,\dots,m \end{cases}$$

(9) Let $\Omega = PC(J{:}Y)$, now assume the operator H:$X \to Y$ as

$$H[x_0 - g(x(t))] = \begin{cases} C(t)[x_0 - I^\beta g(x)]C^*(t) \\ \quad + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[[A(s)-K]x(s) + B_2u_2(s)]ds\right] C^*(t), \ \ t \in [0,t_1] \\ C(t)[x_0 - I^\beta g(x)]C^*(t) + C(t)\sum_{i=1}^k I_i\left(x(t_i)\right)C^*(t) \\ + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[A(s)-K]x(s)\,ds\right] C^*(t), t \in (t_k, t_{k+1}], k = 1,2,\dots,m. \end{cases}$$

(10) As the same proving of results in [4], we can prove the following,

Remarks(3.1):

1. The system in (8) is initial observable if kernel={0}.
2. The system in (8) is continuously initially observable if
   $\|H[x_0 - I^{\beta} g(x(t))]\| = \|x_0 - I^{\beta} g(x(t))\|$.
3. If a system in (8) is initially observable which implies the map H is injective but not surjective.
4. when system in (8) is continuous initially observable implies that H^{-1} :Y→ X exists and bounded that is there exists $\tilde{K}$>0 Such that $\|H^{-1}v\| \le \tilde{K}\|v\|$ for all $v \in Y$.

As the same proving of result in [1], we can prove the following,:

Lemma(3.2):

The system in (8) is continuously initially observable on [0,T] if and only if the system

$$\boldsymbol{x(t)} = \begin{cases} \boldsymbol{x_0 - I^{\beta} g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_2 u_2(s)\, ds}\ , \boldsymbol{t \in [0, t_1]} \\ \boldsymbol{x_0 - g(x) + \sum_{i=1}^{k} I_i\left(x(t_i)\right) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_2 u_2(s)\, ds, t \in (t_k, t_{k+1}], k = 1,2,\dots,} \end{cases}$$

is exactly controllable on [0,T].

Concluding remarks(3.3):

If the linear part equation

$$x(t) = \begin{cases} x_0 - I^{\beta} g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_2 u_2(s)\, ds\ , t \in [0, t_1] \\ x_0 - I^{\beta} g(x) + \sum_{i=1}^{k} I_i\left(x(t_i)\right) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_2 u_2(s)\, ds, t \in (t_k, t_{k+1}], k = 1,2,\dots, \end{cases}$$

is exactly controllable on [0.T] .then by remark(3.1)), is continuously initially observable on [0,T], thus remark(3.1)(4) is also satisfied.

Since the system in (8) is continuously initially observable so that the initial state $x_0 - I^{\beta} g(x)$ of the system (8) can be obtained as follow:

$$H^{-1}y(t) = \begin{cases} H^{-1}\left[C(t)[x_0 - I^{\beta} g(x)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[\,A(s) - K]\, x(s) ds\right] C^*(t)\right], t \in [0, t \\ H^{-1}[C(t)[x_0 - I^{\beta} g(x)]C^*(t) + C(t) \sum_{i=1}^{k} I_i\left(x(t_i)\right)C^*(t), \\ \quad + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[\,A(s) - K]\, x(s) ds\right] C^*(t)\Big], \quad t \in (t_k, t_{k+1}], k = 1,2,\dots,m \end{cases}$$

From (10), we have that

$$H^{-1}y(t) = \begin{cases} [x_0 - I^{\beta} g(x)] + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [A(s) - K]\, x(s)\, ds, t \in [0, t_1] \\ [x_0 - I^{\beta} g(x)] + \sum_{i=1}^{k} I_i(x(t_i)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [A(s) - K]\, x(s)\, ds, \\ t \in (t_k, t_{k+1}], k = 1, 2, \dots, m \end{cases}$$

(11) From (11) the equation (8) become

$$x(t)) = H^{-1}y(t) = \begin{cases} [x_0 - I^{\beta} g(x)] + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [A(s) - K] x(s)\, ds, \ t \in [0, t_1] \\ [x_0 - I^{\beta} g(x)] + \sum_{i=1}^{k} I_i(x(t_i)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [A(s) - K] x(s)\, ds, \\ t \in (t_k, t_{k+1}], k = 1,2, \dots, m \end{cases}$$

(12) In the following formulation, we generalization of concluding remark (3.3).

4.The problem formulation :

Consider the the fractional differential impulsive multi control abstract problem with fractional integral nonlocal initial conditions(1-5) and let the output $y_1(t) = C(t)x(t)C^*(t)$. now substitutes (7) in $y_1(t)$

$$y_1(t) = \begin{cases} C(t)[x_0 - I^{\beta} g(x)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1} \big[[A(s) - K] x(s) \\ \quad + f_1\left(s, x(s), \int_0^t h(t, \tau, x(\tau)) d\tau\right) f_2(s, x(s), D^{\alpha} x(s)) + B_2 u_2(s) \big] ds \Big] C^*(t), t \in [0, t_1] \\ C(t)[x_0 - g(x)]C^*(t) + C(t) \sum_{i=1}^{k} I_i(x(t_i)) C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1} \big[[A(s) - K] x(s \\ \quad + f_1\left(s, x(s), \int_0^t h(t, \tau, x(\tau)) d\tau\right) f_2(s, x(s), D^{\alpha} x(s)) + B_2 u_2(s) \big] ds \Big] C^*(t), \\ t \in (t_k, t_{k+1}], k = 1,2, \dots, m \end{cases}$$

(13) For $u_1, u_2 \in L^2(J, U)$, to calculate the finite time observer , need to construct the initial state implicitly function x(.) for arbitrary control function $u_1, u_2 \in L^2(J, U)$, the nonlocal initial state $x_0 - I^{\beta} g(x)$ of the problem (13) can be obtain by:

$$\begin{cases} C(t)[x_0 - I^{\beta} g(x)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [A(s) - K] x(s)\, ds C^*(t) = y_1(t) - \frac{C(t)}{\Gamma(\alpha)} \\ \Big[\int_0^t (t-s)^{\alpha-1} f_1\left(s, x(s), \int_0^t h(t, \tau, x(\tau)) d\tau\right) f_2(s, x(s), D^{\alpha} x(s)) ds + B_2 u_2(s) ds \Big] C^*(t), t \in [0, t_1] \\ C(t)[x_0 - I^{\beta} g(x)]C^*(t) + C(t) \sum_{i=1}^{k} I_i(x(t_i)) C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1} [A(s) - K] x(s) ds \Big] C^*(t) = \\ y_1(t) - \frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1} \big[f_1\left(s, x(s), \int_0^t h(t, \tau, x(\tau)) d\tau\right) f_2(s, x(s), D^{\alpha} x(s)) + B_2 u_2(s) \big] ds \Big] C^*(t), \\ t \in (t_k, t_{k+1}], k = 1,2, \dots, m \end{cases}$$

(14) Now from H is invertible operator, then,

$$\begin{cases} H^{-1}\left[C(t)[x_0 - I^{\beta}g(x)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[A(s)-K]x(s)ds\right]C^*(t)\right] = H^{-1}[y_1(t) - \frac{C(t)}{\Gamma(\alpha)} \\ \quad \left[\int_0^t (t-s)^{\alpha-1}\left[f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^{\alpha}x(s)) + B\,u(s)\right]ds\right]C^*(t), t\in[0,t_1] \\ H^{-1}\left[C(t)[x_0 - I^{\beta}g(x)]C^*(t) + C(t)\sum_{i=1}^{k} I_i(x(t_i))C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[A(s)-K]x(s)ds\right]C^*(t)\right] \\ \quad = H^{-1}[y_1(t) - \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\left[f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^{\alpha}x(s)) + B\,u(s)\right]ds\right] \\ \qquad C^*(t), \quad t\in(t_k,t_{k+1}], k=1,2,\dots,m \end{cases}$$

(15) From equations (10) and (15) and Substituting in (7), we get: $\boldsymbol{x_0} - g(x) =$

$$\begin{cases} H^{-1}[y_1(t) - \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\left[f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^{\alpha}x(s))\right.\right. \\ \quad +B_2u_2(s)]ds]C^*(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[[A(s)-K]x(s)ds, \\ \quad +f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^{\alpha}x(s)) + B_2u_2(s)\Big], \; t\in[0, \\ H^{-1}[y_1(t) - C(t)\sum_{i=1}^{k} I_i(x(t_i))C^*(t) - \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\left[f_1\left(s,x(s),\int_0^t h(t,\tau,x\right.\right.\right. \\ \quad f_2(s,x(s),D^{\alpha}x(s)) + B_2u_2(s)]ds]C^*(t) + \sum_{i=1}^{k} I_i(y(t_i)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[[A(\\ +f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^{\alpha}x(s)) + B_2u_2(s)\Big]ds, t\in(t_k,t_{k+1}], k=1 \end{cases}$$

<u>Remark (4.1):</u>

The equation in (16) is a finite time observer which provide the mild solution $x(.)\in PC([0,T]:X)$ to have a fixed point ,for all control functions $u_1, u_2 \in L^2([0,T]:U)$.

3. <u>Main results:</u>

Consider the abstract control problem (1-5) and consider their mild solution (7) with hypothesis(h1-h5) and we needs the following adopted in the main result:

(g1) Let : X $\to Y$, C^* : Y $\to X$ is bounded Linear operators , there exist L_1, L_2 >0 such that

$$\|C(t)xC^*(t)\|_Y \le L_1L_2\|x\|_X, x\in X .$$

(g2) If T$\in R^+$ where R^+ is the set of positive numbers.

(g3) (i) $\delta < 1$

$$\tilde{K}\tilde{K}_3 + \left[\frac{2\tilde{K}L_1L_2}{\Gamma(\alpha)}\frac{t_1^{\ \alpha}}{\Gamma(\alpha+1)}\left(N_2\tilde{N}_1\left(r + \frac{t_1^{\ 1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|y(s)\|\right) + N_2\right) + N_1\tilde{N}_2(r + t_1(H_1r + H_1)) + K_1K_2K_{E_{\alpha,1}}e^{\Gamma(\alpha)a(t_1)t_1^{\ \alpha}}\right)\ \right] + \frac{\tilde{K}L_1L_2}{\Gamma(\alpha)}\frac{t_1^{\ \alpha}}{\Gamma(\alpha+1)}\left(M + \tilde{\tilde{M}}\tilde{M}\right) < r, t\in[0,t_1].$$

(g3)(ii)

$\delta < 1, \quad \tilde{K}_4 + \left[\frac{2\tilde{K}L_1L_2}{\Gamma(\alpha)} \frac{t_{k+1}{}^{\alpha}}{\Gamma(\alpha+1)} \left(N_2\tilde{N}_1\left(r + \frac{t_{k+1}{}^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\le t\le T}\|y(s)\|\right) + N_2\right) + N_1\tilde{N}_2(r + t_{k+1}(H_1 r + H_1)) + K_1K_2K_{E_{\alpha,1}} e^{\Gamma(\alpha)a(t_{k+1})t_{k+1}{}^{\alpha}})\right] + \tilde{K}m\ell_2 + \frac{\tilde{K}L_1L_2}{\Gamma(\alpha)} \frac{t_{k+1}{}^{\alpha}}{\Gamma(\alpha+1)}\left(M + \tilde{\tilde{M}}\tilde{M}\right) < r, t \in (t_k, t_{k+1}]$.

Lemma (5.1):

Consider the abstract control problem (1-5) and let $\tilde{M} = PC([0,T]:B_r)$ be a nonempty subset $Z = PC([0,T]:X)$ where X a complex Banach space with norm $\|x\|_{PC} = \sup\{\|x(t)\|, t \in [0,T]\}$. then M is a closed set.

Proof:

Let $w^n \in \tilde{M}$ be a sequence, $w^n \to w$, as $n \to \infty$, where w^n is a continuous sequence functions at t≠ t_k , k=1,2,…m and only left continuous at t= t_k and only right limit x(t_k^+)exists with $u_1, u_2 \in L^2(J,U)$ these sequence is pointwis convergent to w , now we need to prove $w \in \tilde{M}$, so our aim that $w \in Z$ and $\|w(t)\| \le r$,with $u_1, u_2 \in L^2(J,U)$. Now to prove that $w \in Z$, since $w^n \in \tilde{M}$ pointwise converges , thus sequence w^n is uniformly convergence to w, hence $w \in Z$. now to show that $\|w(t)\| \le r$, from above sequence w^n is uniformly convergent to w and $\|w^n - w\|_{PC} = sup_{t\in[0,T]}\|w^n(t) - w(t)\|$ in a complex Banach space Z then $sup_{t\in[0,T]}\|wx^n(t) - w(t)\| \to 0$, as $n \to \infty$ for all 0≤ t ≤ T, we have that

$\|w(t)\|_{PC} = \left\|\lim_{n\to\infty} w^n(t)\right\|_{PC} = \lim_{n\to\infty}\|w^n(t)\|_{PC} \le \lim_{n\to\infty} r$. Therefore $\tilde{M}$ is a closed subset of Z.

Lemma (5.2):

Assume that the hypotheses (h1-h3) and (h5) holds. From (13) we defined $y_1(t)$ and $y_2(t)$ as a nonlinear observations such that

$$y_1(t) = \begin{cases} C(t)[x_0 - I^{\beta}g(x_1)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[A(s) - K]x_1(s) \right. \\ \quad \left. + f_1\left(s, x(s), \int_0^t h(t,\tau,x(\tau))d\tau\right) f_2\left(s, x(s), D^{\alpha}x(s)\right) + B_2u_2(s)\right] ds\Big] C^*(t), \quad t \in [0,t \\ C(t)[x_0 - g(x_1)]C^*(t) + C(t)\sum_{i=1}^{k} I_i\left(x_1(t_i)\right)C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[A(s) - K]x_1 \right. \\ \quad \left. + f_1\left(s, x(s), \int_0^t h(t,\tau,x(\tau))d\tau\right) f_2\left(s, x(s), D^{\alpha}x(s)\right) + B_2u_2(s)\right] ds\Big] C^*(t), \quad t \in (t_k, t_k \end{cases}$$

$y_2(t) =$

$$\begin{cases} C(t)[x_0 - I^{\beta} g(x_2)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\Big[\int_0^t (t-s)^{\alpha-1} [\,A(s) - K]x_2(s) \\ \quad +f_1\left(s, x(s), \int_0^t h(t,\tau,x(\tau))d\tau\right) f_2\left(s, x(s), D^{\alpha}x(s)\right) + B_2 u_2(s)\Big] ds\Big] C^*(t), t \in [0, t_1] \\ C(t)[x_0 - g(x_2)]C^*(t) + C(t)\sum_{i=1}^{k} I_i\left(x_2(t_i)\right)C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\Big[\int_0^t (t-s)^{\alpha-1} [\,A(s) - K]x_2 \\ \quad +f_1\left(s, x(s), \int_0^t h(t,\tau,x(\tau))d\tau\right) f_2\left(s, x(s), D^{\alpha}x(s)\right) + B_2 u_2(s)\Big] ds\Big] C^*(t), t \in (t_k, t_{k+1}] \end{cases}$$

If $\quad \widetilde{K}_3 = L_1 L_2 \left(r + (r + (\frac{T^{\beta} G}{\Gamma(\beta+1)} r + \|g(0)\| \ \right) \ + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^{\alpha}}{\Gamma(\alpha+1)} (\left(M + \widetilde{\widetilde{M}}\widetilde{M}\right) r$

$+N_2 \widetilde{N}_1\left(r + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\le t\le T} \|y(s)\|\right) + N_2) + N_1 \widetilde{N}_2 (r + T(H_1 r + H_1)) +$
$K_1 K_2 K_{E_{\alpha,1}} e^{\Gamma(\alpha) a(T) T^{\alpha}})$

,

$\widetilde{K}_4 = L_1 L_2 \left(r + (r + (\frac{T^{\beta} G}{\Gamma(\beta+1)} r + \|g(0)\| \ \right) \ + Lm\ell_2 + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^{\alpha}}{\Gamma(\alpha+1)} (\left(M + \widetilde{\widetilde{M}}\widetilde{M}\right) r$

$+N_2 \widetilde{N}_1\left(r + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\le t\le T} \|y(s)\|\right) + N_2) + N_1 \widetilde{N}_2 (r + T(H_1 r + H_1)) +$
$K_1 K_2 K_{E_{\alpha,1}} e^{\Gamma(\alpha) a(T) T^{\alpha}})$

$$\tilde{\ell}_3 = \Big[L_1 L_2 \frac{T^{\beta} G}{\Gamma(\beta+1)} + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^{\alpha}}{\Gamma(\alpha+1)} \Big[\left(M + \widetilde{\widetilde{M}}\widetilde{M}\right) + N_2 \widetilde{N}_1 \left(1 + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\le t\le T}\right)$$
$$+N_1[+T H_1 K_{f_2}(t) \Omega_{f_2}(r + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\le t\le T} \|x_2(s)\|) \ \Big]\Big]$$
$$\tilde{\ell}_4 = \Big[L_1 L_2 \frac{T^{\beta} G}{\Gamma(\beta+1)} + L_1 L_2\, m\ell_2 + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^{\alpha}}{\Gamma(\alpha+1)} \Big[\left(M + \widetilde{\widetilde{M}}\widetilde{M}\right) + N_2 \widetilde{N}_1 (1 + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\le t\le T})$$
$$+N_1[+T H_1 K_{f_2}(t) \Omega_{f_2}(r + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\le t\le T} \|x_2(s)\|) \ \Big]\Big]$$

Then

1. $\quad \|y(t)\| \le \begin{cases} \widetilde{K}_3, t \in [0, t_1] \\ \widetilde{K}_4, \ t \in (t_k, t_{k+1}], k = 1,2,\dots,m \end{cases}$

2. $\quad \|y_1(t) - y_2(t)\| \le \begin{cases} \tilde{\ell}_3, t \in [0, t_1] \\ \tilde{\ell}_4, \ t \in (t_k, t_{k+1}], k = 1,2,\dots,m \end{cases}$

<u>Proof:</u>

$$\|y(t)\|_Y = \Big\| C(t)[x_0 - I^{\beta} g(x)]C^*(t) + \frac{C(t)}{\Gamma(\alpha)}\Big[\int_0^t (t-s)^{\alpha-1}[[\,A(s) - K]x(s)$$
$$+f_1\left(s, x(s), \int_0^t h(t,\tau,x(\tau))d\tau\right) f_2\left(s, x(s), D^{\alpha}x(s)\right)$$
$$+B_2\ [\mathrm{v}(t) + \mathrm{a}(t)\int_0^t (t-\tau)^{\alpha-1} u_2(\tau)d\tau]\Big] ds\Big] C^*(t)\Big\|_Y,\ t \in [0, t_1]$$
$$\|y(t)\|_Y \le \|C(t)[x_0 - I^{\beta} g(x)]C^*(t)\|_Y + \frac{1}{\Gamma(\alpha)} \Big\| C(t) \Big[\int_0^t (t-s)^{\alpha-1}[[\,A(s) - K]x(s)$$

$$+f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^\alpha x(s))$$
$$+B_2\left[\mathrm{v}(t)+\mathrm{a}(t)\int_0^t(t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]ds\Big]C^*(t)\Big\|_Y,$$
$$\|y(t)\|_Y\le L_1L_2\left\|x_0-\boldsymbol{I}^{\boldsymbol{\beta}}g(x)\right\|_Y+\frac{L_1L_2}{\Gamma(\alpha)}\left\|\left[\int_0^t(t-s)^{\alpha-1}[A(s)-K]x(s)\right.\right.$$

$$\left.\left.+f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^\alpha x(s))+B_2\,\mathrm{v}(t)\,\mathrm{E}_\alpha(\Gamma(\alpha)\mathrm{a}(t)\mathrm{t}^\alpha)\right\|_Y$$
$$\|y(t)\|_Y\le L_1L_2\left\|x_0-\boldsymbol{I}^{\boldsymbol{\beta}}g(x)\right\|_Y+\frac{L_1L_2}{\Gamma(\alpha)}\left\|\left[\int_0^t(t-s)^{\alpha-1}[A(s)-K]x(s)\right.\right.$$

$$\left.+f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^\alpha x(s))+B_2\,\mathrm{v}(t)\,\mathrm{E}_\alpha(\Gamma(\alpha)\mathrm{a}(t)\mathrm{t}^\alpha)]\right\|_Y$$

$$\|y(t)\|_Y\le L_1L_2\|x_0\|+\left\|I^\beta(g(x)-g(0)+g(0))\right\|_Y+\frac{L_1L_2}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds(\|A(s)-B_1K\|\,\|x(s)\|$$

$$+\left\|f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^\alpha x(s))-f_1(s,0,0)f_2(s,0,0)\right\|$$
$$+\|B_2\|\,\|\mathrm{v}(t)\|\,\|\,\mathrm{E}_\alpha(\Gamma(\alpha)\mathrm{a}(t)\mathrm{t}^\alpha)]\|)$$

$$\le L_1L_2\|x_0\|+\left\|I^\beta(g(x)-g(0)+g(0))\right\|_Y+\frac{L_1L_2}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds(\|A(s)-B_1K\|\,\|x(s)\|$$

$$+\left\|f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^\alpha x(s))-f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)\right.$$
$$\left.f_2(s,0,0)+f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,0,0)-f_1(s,0,0)f_2(s,0,0)\right\|$$
$$+\|B_2\|\,\|\mathrm{v}(t)\|\,\mathrm{K}_{\mathrm{E}_{\alpha,1}}\mathrm{e}^{\Gamma(\alpha)\mathrm{a}(\mathrm{T})\mathrm{T}^\alpha}),$$

condition (h2), given

$$\le\ L_1L_2rI^\beta(\|g(x)-g(0)\|+\|g(0)\|\ \ )+\frac{L_1L_2}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds((M+\tilde{\tilde{M}}\tilde{M})r$$
$$+\left\|f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)\right\|\,\|f_2(s,x(s),D^\alpha x(s))-f_2(s,0,0)\|$$

$$+\left\|f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)-f_1(s,0,0)\right\|\,\|f_2(s,0,0)\|+K_1K_2\mathrm{K}_{\mathrm{E}_{\alpha,1}}\mathrm{e}^{\Gamma(\alpha)\mathrm{a}(\mathrm{T})\mathrm{T}^\alpha})$$

condition (h5) given that,

$$\le\ L_1L_2\Big(r+\Big(\frac{T^\beta G}{\Gamma(\beta+1)}r+\|g(0)\|\ \ \Big)+\frac{L_1L_2}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\Big(\Big(M+\tilde{\tilde{M}}\tilde{M}\Big)r$$

$$+N_2\tilde{N}_1(r+\|D^\alpha x(s)\|)+N_1\tilde{N}_2\left(r+\left\|\int_0^t\Big(h(t,\tau,x(\tau))-h(t,\tau,0)+h(t,\tau,0)\Big)d\tau\right\|\right)$$
$$+K_1K_2\mathrm{K}_{\mathrm{E}_{\alpha,1}}\mathrm{e}^{\Gamma(\alpha)\mathrm{a}(\mathrm{T})\mathrm{T}^\alpha})$$

, from condition (h3),

$$\leq \ L_1L_2(r+(r+(\frac{T^\beta G}{\Gamma(\beta+1)}r+\|g(0)\| \ ) \ +\frac{L_1L_2}{\Gamma(\alpha)}\frac{T^\alpha}{\Gamma(\alpha+1)}((M+\tilde{\tilde{M}}\tilde{M})r$$
$$+N_2\tilde{N}_1\left(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\leq t\leq T}\|y(s)\|\right)+N_2)+N_1\tilde{N}_2(r+T(H_1r+H_1))$$
$$+K_1K_2K_{E_{\alpha,1}}e^{\Gamma(\alpha)a(T)T^\alpha})$$

Now, $\|y(t)\|_Y\leq \tilde{K}_3$,

where

$$\tilde{K}_3=L_1L_2(r+(r+(\frac{T^\beta G}{\Gamma(\beta+1)}r+\|g(0)\| \ ) \ +\frac{L_1L_2}{\Gamma(\alpha)}\frac{T^\alpha}{\Gamma(\alpha+1)}((M+\tilde{\tilde{M}}\tilde{M})r+N_2\tilde{N}_1\Big(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\leq t\leq T}\|y(s)\|\Big)+N_2)+N_1\tilde{N}_2(r+T(H_1r+H_1))+K_1K_2K_{E_{\alpha,1}}e^{\Gamma(\alpha)a(T)T^\alpha})$$

For $\in (t_k,t_k], k=1,2,\dots,m$,

$$\|y(t)\|_Y=\Big\|C(t)[x_0-I^\beta g(x)]C^*(t)+C(t)\textstyle\sum_{i=1}^k I_i\left(x(t_i)\right)C^*(t)+\frac{C(t)}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}$$
$$[A(s)-B_1K]x(s)+f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)f_2(s,x(s),D^\alpha x(s))$$
$$+B_2\left[v(t)+a(t)\int_0^t(t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]ds\Big]C^*(t)\Big\|_Y$$
$$\leq \|C(t)[x_0-I^\beta g(x)]C^*(t)\| \ +\|C(t)\textstyle\sum_{i=1}^k I_i\left(x(t_i)\right)C^*(t)\|$$

$$+\frac{1}{\Gamma(\alpha)}\Big\|C(t)\int_0^t(t-s)^{\alpha-1}\Big[[A(s)-B_1K]x(s)+f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)$$
$$f_2(s,x(s),D^\alpha x(s))+B_2\left[v(t)+a(t)\int_0^t(t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]ds\,C^*(t)\Big\| \ ,$$

condition (g1), yield

$$\|y(t)\|_Y\leq L_1L_2\|[x_0-I^\beta g(x)]\| \ +L_1L_2\|\textstyle\sum_{i=1}^k I_i\left(x(t_i)\right)\|$$

$$+\frac{L_1L_2}{\Gamma(\alpha)}\Big\|\int_0^t(t-s)^{\alpha-1}\int_0^t(t-s)^{\alpha-1}[[A(s)-B_1K]x(s) \ +f_1\left(s,x(s),\int_0^t h(t,\tau,x(\tau))d\tau\right)$$
$$f_2(s,x(s),D^\alpha x(s))+B_2\left[v(t)+a(t)\int_0^t(t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]ds\Big\|$$

conditions (h1-h5) and first part of proving, we get

$$\|y(t)\|_Y\leq L_1L_2\left(r+(r+(\frac{T^\beta G}{\Gamma(\beta+1)}r+\|g(0)\|\right) \ +L_1L_2m\ell_2$$
$$+\frac{L_1L_2}{\Gamma(\alpha)}\frac{T^\alpha}{\Gamma(\alpha+1)}((M+\tilde{\tilde{M}}\tilde{M})r$$
$$+N_2\tilde{N}_1\left(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\leq t\leq T}\|y(s)\|\right)+N_2)$$
$$+N_1\tilde{N}_2(r+T(H_1r+H_1))+K_1K_2K_{E_{\alpha,1}}e^{\Gamma(\alpha)a(T)T^\alpha}),\text{ hence,}$$

$\|y(t)\|_Y\leq \tilde{K}_4$,

where

$\widetilde{K}_4 \;\; = L_1 L_2 (r + (r + (\frac{T^\beta G}{\Gamma(\beta+1)} r + \|g(0)\| \;\;) \;\; + L_1 L_2 m\ell_2 + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^\alpha}{\Gamma(\alpha+1)} ((M + \widetilde{\widetilde{M}}\widetilde{M}) r + N_2 \widetilde{N}_1 \left(r + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0 \le t \le T} \|y(s)\|\right) + N_2) + N_1 \widetilde{N}_2 (r + T(H_1 r + H_1)) + K_1 K_2 K_{E_{\alpha,1}} e^{\Gamma(\alpha) a(T) T^\alpha})$

For $t \in [0, t_1]$. To satisfy Lipshtiz property,

$$\|y_1(t) - y_2(t)\|_Y \le \Big\| C(t)[x_0 - I^\beta g(x_1)] C^*(t) + \frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1} [[A(s) - B_1 K] x_1(s)$$
$$+ f_1\left(s, x_1 x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s))$$

$$+ B_2 \left[v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) d\tau\right]\Big] ds\Big] \, C^*(t) - C(t)[x_0 - I^\beta g(x_2)] C^*(t)$$

$$- \frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1} \Big[[A(s) - B_1 K] x_2(s) + f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau)) d\tau\right) f_2(s, x_2(s), D^\alpha x(s))$$

$$- B_2 \left[v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) d\tau\right]\Big] ds\Big] \, C^*(t) \Big\|$$

$$\le \|C(t)\| \|C^*(t)\| \left\|I^\beta g(x_2) - I^\beta g(x_1)\right\| + \frac{\|C(t)\| \|C^*(t)\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [[A(s) - B_1 K][x_1(s) - x_2(s)]$$
$$+ \Big\| f_1\left(s, x_1 x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s))$$
$$- f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau)) d\tau\right) f_2(s, x_2(s), D^\alpha x(s)) \Big\| \Big] ds$$
$$\le L_1 L_2 \frac{T^\beta G}{\Gamma(\beta+1)} \|x_1(s) - x_2(s)\| + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^\alpha}{\Gamma(\alpha+1)} \left[\left(M + \widetilde{\widetilde{M}}\widetilde{M}\right) [x_1(s) - x_2(s)]\right.$$
$$+ \Big\| f_1\left(s, x_1 x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s)) - f_1\left(s, x_1 x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2(s, x_2(s), D^\alpha x(s))$$
$$+ f_1\left(s, x_1 x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2(s, x_2(s), D^\alpha x(s)) - f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau)) d\tau\right) f_2(s, x_2(s), D^\alpha x(s)) \Big\| \Big] d$$
$$\le L_1 L_2 \frac{T^\beta G}{\Gamma(\beta+1)} \|x_1(s) - x_2(s)\| + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^\alpha}{\Gamma(\alpha+1)} \left[\left(M + \widetilde{\widetilde{M}}\widetilde{M}\right) [x_1(s) - x_2(s)]\right.$$

$\| y_1(t) - y_2(t)\|_Y \leq$
$\Big[L_1 L_2 \frac{T^\beta G}{\Gamma(\beta+1)} + \frac{L_1 L_2}{\Gamma(\alpha)} \frac{T^\alpha}{\Gamma(\alpha+1)} \Big[\Big(M + \tilde{\tilde{M}}\tilde{M}\Big) + N_2 \tilde{N}_1 (1 + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\leq t\leq T}) +$
$[T H_1 \|x_1(s) - x_2(s)\| K_{f_2}(t)\Omega_{f_2}(r + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{0\leq t\leq T}\|x_2(s)\|) \Big]\Big] \|x_1(s) - x_2(s)\|$

Hence, $\|y_1(t) - y_2(t) \|_Y \leq \tilde{\ell}_3 \|x_1(t) - x_2(t)\|$. For $t \in [0, t_1]$

For $\in (t_k, t_{k+1}], k = 1,2,\dots,m$. To satisfy Lipshtiz property,

$\| y_1(t) - y_2(t) \|_Y \leq \Big\| C(t)[x_0 - I^\beta g(x_1)] C^*(t) + C(t) \sum_{i=1}^k I_i \left(x_1(t_i)\right) C^*(t) +$
$\frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1}$
$\Big[[A(s) - B_1 K] x_1(s) + f_1\left(s, x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2\left(s, x_1(s), D^\alpha x_1(s)\right)$
$+ B_2 \left[v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) d\tau\right]\Big] ds\Big] C^*(t) - C(t)[x_0 - I^\beta g(x_2)] C^*(t)$
$- C(t) \sum_{i=1}^k I_i \left(x_2(t_i)\right) C^*(t) - \frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1}$
$\Big[[A(s) - B_1 K] x_2(s) + f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau)) d\tau\right) f_2\left(s, x_2(s), D^\alpha x_2(s)\right)$

$- B_2 \left[v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) d\tau\right]\Big] ds\Big] C^*(t) \Big\| \leq L_1 L_2 \|I^\beta g(x_2) - I^\beta g(x_1)\| +$
$L_1 L_2\, m\ell_2 \|x_1(s) - x_2(s)\| + \frac{T^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)} L_1 L_2 \left(M + \tilde{\tilde{M}}\tilde{M}\right) \|x_1(s) - x_2(s)\| +$
$\frac{L_1 L_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big\|\Big[f_1\left(s, x_1 x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2\left(s, x_1(s), D^\alpha x_1(s)\right) -$
$f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau)) d\tau\right) f_2\left(s, x_2(s), D^\alpha x(s)\right)\Big]\Big\| ds$

Conditions (g1), (h1), (h2) and (h5), obtain,

$\| y_1(t) - y_2(t) \|_Y \leq \tilde{\ell}_4 \| x_1(t) - x_2(t) \|$, for $t \in (t_k, t_{k+1}], k = 1,2,\dots,m$.

Theorem (5.3):

Assume that the hypotheses (h1-h5) and conditions g3(i),(ii) are satisfied Then the the impulsive multi control fractional differential abstract problem with fractional integral nonlocal initial condition(1-5) has a unique fixed point $x(.) \in PC([0,T]:X)$ for all control function $u_1(.), u_2(.) \in L^2([0,T]:U)$.

Proof:

Define the nonlinear map: $\varphi: \tilde{M} = PC([0,T]:B_r) \to Z = PC([0,T]:X)$ as follows:

$(\varphi x)(t) =$
$\frac{C(t)}{\Gamma(\alpha)} \Big[\int_0^t (t-s)^{\alpha-1} \Big[f_1\left(s, x_1 x_1(s), \int_0^t h(t,\tau,x_1(\tau)) d\tau\right) f_2\left(s, x_1(s), D^\alpha x_1(s)\right) +$
$B_2 \left[v(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} u_2(\tau) d\tau\right]\Big] ds\Big] C^*(t) \sum_{i=1}^k I_i \left(y(t_i)\right) t \in (t_k, t_{k+1}]$

$$(\varphi x)(t) = \begin{cases} H^{-1}[y_1(t) - \Big[\Big[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2\left(s, x_1(s), D^\alpha x_1(s)\right) \\ \quad +B_2\left[\mathrm{v}(\mathrm{t}) + \mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\right]\Big] ds\Big] C^*(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ \quad [[A(s) - B_1K]x(s) + f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s)) \\ \quad +B_2\left[\mathrm{v}(\mathrm{t}) + \mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\right]\mathrm{ds}, \qquad t \in [0, t_1] \\ H^{-1}[y_1(t) - \Big[\Big[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2\left(s, x_1(s), D^\alpha x_1(s)\right) \\ \quad +B_2\left[\mathrm{v}(\mathrm{t}) + \mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\right]\Big] ds\Big] C^*(t) + \sum_{i=1}^k I_i\left(y(t_i)\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ \quad [[A(s) - B_1K]x(s) + f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s)) \\ \quad +B_2\left[\mathrm{v}(\mathrm{t}) + \mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\right]\mathrm{ds}, \qquad t \in (t_k, t_{k+1}] \end{cases}$$

(17)

for all control function $u_1(.), u_2(.) \in L^2([0,T]:U)$.

Our interest to prove φx has a fixed point. So we need to do the following steps:

Step1: $\widetilde{\mathrm{M}} = PC([0,T]:B_r)$ is a closed subset of Z= $PC([0,T]:X)$.

Step2: $\varphi\widetilde{\mathrm{M}} \subseteq \widetilde{\mathrm{M}}$ with $u_1(.), u_2(.) \in L^2([0,T]:U)$.

Step3: φ is a contraction on $\widetilde{\mathrm{M}}$ for $u_1(.), u_2(.) \in L^2([0,T]:U)$. From Lemma (5.2) .

step (1) have been satisfied. For proving step (2) , we need lemma (5.2), now let $x \in M$

1. $\varphi\mathrm{x} \in Z$ for $u_1(.), u_2(.) \in L^2([0,T]:U)$.
2. $\|\varphi\mathrm{x}(t)\| \leq r$, for $u_1(.), u_2(.) \in L^2([0,T]:U)$. From (17) , it is clear (1) satisfied.

To prove (2) ,thus

$\|(\varphi x)(t)\| =$

$$\begin{cases} \Big\| H^{-1}[y_1(t) - \Big[\Big[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D \\ \quad +B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]\Big] ds\Big] C^*(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ \quad [[A(s)-B_1K]x(s) + f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D^\alpha x_1(s \\ \quad +B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right] ds\Big\|, \qquad t\in[0,t_1] \\ \Big\| H^{-1}[y_1(t) - \Big[\Big[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D \\ \quad +B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]\Big] ds\Big] C^*(t) + \sum_{i=1}^k I_i\left(y(t_i)\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t- \\ \quad [[A(s)-B_1K]x(s) + f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D^\alpha x_1(s \\ +B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right] ds\Big\|, \qquad t\in(t_k,t_{k+1}] \end{cases}$$

From (18) and boundedness of H^{-1} which given from remark(3.1)(4) , yield

$\leq$

$$\begin{cases} \tilde{K}[\|y_1(t)\| - \Big[\Big[\frac{\|C(t)\|\|C^*(t)\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\Big\|f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s) \\ \quad +\Big\|B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]\Big\|\Big] ds\Big] C^*(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ \quad [\|A(s)-B_1K\|\|x(s)\| + \Big\|f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D^\alpha x_1(\\ \quad +\Big\|B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]\Big\| ds, \qquad t\in[0,t_1] \\ \tilde{K}[\|y_1(t)\| - \Big[\Big[\frac{\|C(t)\|\|C^*(t)\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\Big\|f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s) \\ \quad +\Big\|B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]\Big\|\Big] ds\Big] C^*(t) + \sum_{i=1}^k \|I_i(y(t_i))\| + \frac{1}{\Gamma(\alpha)}\int_0^t (\\ \quad [\|A(s)-B_1K\|\|x(s)\| + \Big\|f_1\left(s, x_1x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D^\alpha x_1(\\ \quad +\Big\|B_2\left[v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1}u_2(\tau)d\tau\right]\Big\| ds, \qquad t\in(t_k,t_{k+1}] \end{cases}$$

$$\leq \begin{cases} \tilde{K}\tilde{K}_3 + \left[\frac{2\tilde{K}L_1L_2}{\Gamma(\alpha)}\frac{t_1{}^{\alpha}}{\Gamma(\alpha+1)}\left(N_2\tilde{N}_1\left(r+\frac{t_1{}^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|y(s)\|\right)+N_2\right)+N_1\tilde{N}_2\left(r+t_1(H_1r+\right.\right. \\ \left.\left.+K_1K_2\mathrm{K}_{\mathrm{E}_{\alpha,1}}\mathrm{e}^{\Gamma(\alpha)a(t_1)t_1{}^{\alpha}}\right)\right]+\frac{\tilde{K}L_1L_2}{\Gamma(\alpha)}\frac{t_1{}^{\alpha}}{\Gamma(\alpha+1)}\left(M+\tilde{\tilde{M}}\tilde{M}\right) \\ \quad , \qquad t\in[0,t_1]. \\ \tilde{K}\tilde{K}_4 + \left[\frac{2\tilde{K}L_1L_2}{\Gamma(\alpha)}\frac{t_{k+1}{}^{\alpha}}{\Gamma(\alpha+1)}\left(N_2\tilde{N}_1\left(r+\frac{t_{k+1}{}^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|y(s)\|\right)+N_2\right)+N_1\tilde{N}_2\left(r+t_{k+1}(H_1r\right.\right. \\ \left.\left.+K_1K_2\mathrm{K}_{\mathrm{E}_{\alpha,1}}\mathrm{e}^{\Gamma(\alpha)a(t_{k+1})t_{k+1}{}^{\alpha}}\right)\right]+\tilde{K}m\ell_2+\frac{\tilde{K}L_1L_2}{\Gamma(\alpha)}\frac{t_{k+1}{}^{\alpha}}{\Gamma(\alpha+1)}\left(M+\tilde{\tilde{M}}\tilde{M}\right) \\ \quad , \qquad t\in(t_k,t_{k+1}]\,. \end{cases}$$

condition (g3)(i), given that,
$\|(\varphi x)(t)\| \leq r$, for $t \in [0,t_1]$ and $t \in (t_k,t_k], k = 1,\dots,m$
To satisfy step (3).

$$\|(\varphi x_1)(\mathrm{t}) - \varphi x_2)(\mathrm{t})\ \| \leq$$

$$\Big\| \ H^{-1}[y_1(t) -$$
$$\left[\left[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s,x_1(s),\int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D^{\alpha}x_1(s)) +\right.\right.$$
$$\left.B_2\left[\mathrm{v(t)}+\mathrm{a(t)}\int_0^{\mathrm{t}}(\mathrm{t}-\tau)^{\alpha-1}u_2(\tau)\mathrm{d}\tau\right]\right]ds\,C^*(t)\Big|$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[[\,A(s)-B_1K]x_1(s)+$$
$$f_1\left(s,x_1x_1(s),\int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_1(s),D^{\alpha}x_1(s)) + B_2\left[\mathrm{v(t)}+\mathrm{a(t)}\int_0^{\mathrm{t}}(\mathrm{t}-\right.$$
$$\left.\tau)^{\alpha-1}u_2(\tau)\mathrm{d}\tau\right]\Big]ds$$
$$-H^{-1}[\,y_2(t)\ -$$
$$\left[\left[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s,x_2(s),\int_0^t h(t,\tau,x_2(\tau))d\tau\right) f_2(s,x_2(s),D^{\alpha}x_1(s)) +\right.\right.$$
$$\left.B_2\left[\mathrm{v(t)}+\mathrm{a(t)}\int_0^{\mathrm{t}}(\mathrm{t}-\tau)^{\alpha-1}u_2(\tau)\mathrm{d}\tau\right]\right]ds\,C^*(t)\Big|$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[[\,A(s)-B_1K]x_2(s)+$$
$$f_1\left(s,x_2(s),\int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s,x_2(s),D^{\alpha}x_2(s)) + B_2\left[\mathrm{v(t)}+\mathrm{a(t)}\int_0^{\mathrm{t}}(\mathrm{t}-\right.$$
$$\left.\tau)^{\alpha-1}u_2(\tau)\mathrm{d}\tau\right]\Big]ds\Big\|\ ,\ t\in[0,t_1]$$

.

$$\|(\varphi x_1)(\mathrm{t}) - \varphi x_2)(\mathrm{t})\ \| \leq$$

$\Big\| H^{-1}[y_1(t) -$
$\Big[\Big[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s, x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s)) +$
$B_2\left[\mathrm{v}(\mathrm{t}) + \mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\right]\Big] ds C^*(t)\Big]$

$+\sum_{i=1}^k I_i\left(x_1(t_i)\right) +$
$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[[A(s) - B_1 K]x_2(s) +$
$f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s, x_2(s), D^\alpha x_1(s))\Big] ds$

$+B_2\left[\mathrm{v}(\mathrm{t}) + \mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\right]\Big] ds - H^{-1}[y_2(t) -$
$\Big[\Big[\frac{C(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau))d\tau\right)$

$f_2(s, x_2(s), D^\alpha x_2(s)) + B_2\left[\mathrm{v}(\mathrm{t}) + \mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\right]\Big] ds C^*(t)\Big] -$
$\sum_{i=1}^k I_i\left(x_2(t_i)\right) -$
$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[[A(s) -$
$B_1 K]x_2(s) + f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s, x_2(s), D^\alpha x_2(s)) + B_2\Big[\mathrm{v}(\mathrm{t}) +$
$\mathrm{a}(\mathrm{t})\int_0^t (\mathrm{t}-\tau)^{\alpha-1} u_2(\tau)\mathrm{d}\tau\Big]\Big] ds\Big\|, t \in (t_k, t_{k+1}] k = 1,2, \dots m$

. Then

$\|(\varphi x_1)(\mathrm{t}) - \varphi x_2)(\mathrm{t})\| \le \|\boldsymbol{H^{-1}}\| \|\boldsymbol{y_1(t)} - \boldsymbol{y_2(t)}\| + \frac{\|H^{-1}\| \|C(t)\| \|C^*(t)\|}{\Gamma(\alpha)}$
$\Big[\Big\| f_1\left(s, x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s)) -$
$f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau))d\tau\right) f_2(s, x_2(s), D^\alpha x_1(s))\Big\|\Big] \boldsymbol{ds}, \int_0^t (\boldsymbol{t}-\boldsymbol{s})^{\alpha-1} +$
$\frac{\|H^{-1}\|}{\Gamma(\alpha)}\int_0^t (\boldsymbol{t}-\boldsymbol{s})^{\alpha-1}\Big[\|A(s) - B_1 K\| \|\boldsymbol{x_1(s)} - \boldsymbol{x_2(s)}\| +$
$\Big\| f_1\left(s, x_1(s), \int_0^t h(t,\tau,x_1(\tau))d\tau\right) f_2(s, x_1(s), D^\alpha x_1(s)) -$
$f_1\left(s, x_2(s), \int_0^t h(t,\tau,x_2(\tau))d\tau\right) f_2(s, x_2(s), D^\alpha x_2(s))\Big\|\Big] \boldsymbol{ds}, \ \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{t_1}]$

.

$\|(\varphi x_1)(\mathrm{t}) - \varphi x_2)(\mathrm{t})\| \le \|\boldsymbol{H^{-1}}\| \ \|\boldsymbol{y_1(t)} - \boldsymbol{y_2(t)}\|$

$+\frac{\|H^{-1}\|\|C(t)\|\|C^*(t)\|}{\Gamma(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left\|f_1\left(s,x_1(s),\int_0^t h(t,\tau,x_1(\tau))d\tau\right)f_2\left(s,x_1(s),D^\alpha x_1(s)\right)-f_1\left(s,x_2(s),\int_0^t h(t,\tau,x_2(\tau))d\tau\right)f_2\left(s,x_2(s),D^\alpha x_2(s)\right)\right\|$
$ds+\sum_{i=1}^{k}\|I_i(x_1(t_i))-I_i(x_2(t_i))\|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[\|A(s)-B_1K\|\|x_1(s)-x_2(s)\|$
$+\left\|f_1\left(s,x_1(s),\int_0^t h(t,\tau,x_1(\tau))d\tau\right)f_2\left(s,x_1(s),D^\alpha x_1(s)\right)-f_1\left(s,x_2(s),\int_0^t h(t,\tau,x_2(\tau))d\tau\right)f_2\left(s,x_2(s),D^\alpha x_2(s)\right)\right\|\ ]ds$
$,t\in(t_k,t_{k+1}]k=1,2,\dots m$

From Lemma(5.2), we obtain:

$\|(\varphi x_1)(t)-\varphi x_2)(t)\ \|\le\tilde{K}\ell_3\|x_1(s)-x_2(s)\|+\frac{\tilde{K}L_1L_2}{\Gamma(\alpha)}\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\left[\left(M+\tilde{\tilde{M}}\tilde{M}\right)+N_2\tilde{N}_1(1+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T})\quad+N_1[TH_1\|x_1(s)-x_2(s)\|K_{f_2}(t)\Omega_{f_2}(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|x_2(s)\|)\right]\right]$
$+\frac{\tilde{K}}{\Gamma(\alpha)}\frac{T^\alpha}{\Gamma(\alpha+1)}\left[\left(M+\tilde{\tilde{M}}\tilde{M}\right)+N_2\tilde{N}_1(1+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T})\quad+N_1[+TH_1\|x_1(s)-x_2(s)\|K_{f_2}(t)\Omega_{f_2}(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|x_2(s)\|)\right],$
$t\in[0,t_1]$.

$\|(\varphi x_1)(t)-\varphi x_2)(t)\ \|\le\ \tilde{K}\ell_3\|x_1(s)-x_2(s)\|$
$+\frac{\tilde{K}L_1L_2}{\Gamma(\alpha)}\frac{T^\alpha}{\Gamma(\alpha+1)}\left[\left(M+\tilde{\tilde{M}}\tilde{M}\right)+N_2\tilde{N}_1(1+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T})\quad+N_1[+TH_1\|x_1(s)-x_2(s)\|K_{f_2}(t)\Omega_{f_2}(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|x_2(s)\|)\right]$

$+L_1L_2\,m\ell_2+\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\left[\left(M+\tilde{\tilde{M}}\tilde{M}\right)+N_2\tilde{N}_1(1+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T})\quad+N_1[TH_1\|x_1(s)-x_2(s)\|K_{f_2}(t)\Omega_{f_2}(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|x_2(s)\|)\right]\right],t\in(t_k,t_{k+1}]k=1,2,\dots m$

Thus,

$\|(\varphi x_1)(t)-\varphi x_2)(t)\ \|\le\left[\tilde{K}\ell_3+\frac{\tilde{K}L_1L_2}{\Gamma(\alpha)}\left[\frac{T^\alpha}{\Gamma(\alpha+1)}+1\right]\right]\left[\left[\frac{\tilde{K}}{\Gamma(\alpha)}\left(M+\tilde{\tilde{M}}\tilde{M}\right)+N_2\tilde{N}_1(1+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T})\quad+N_1[TH_1K_{f_2}(t)\Omega_{f_2}(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|x_2(s)\|)\right]\right]$
$\|x_1(s)-x_2(s)\|<\delta\|x_1(s)-x_2(s)\|,t\in[0,t_1]$

$\|(\varphi x_1)(t)-\varphi x_2)(t)\ \|\le[\tilde{K}\ell_3+\left[\frac{T^\alpha}{\Gamma(\alpha+1)}+1]\right]\frac{\tilde{K}L_1L_2}{\Gamma(\alpha)}\left[\left(M+\tilde{\tilde{M}}\tilde{M}\right)+N_2\tilde{N}_1(1+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T})\quad+N_1[+TH_1K_{f_2}(t)\Omega_{f_2}(r+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|x_2(s)\|)\right]$
$+L_1L_2\,m\ell_2]\|x_1(s)-x_2(s)\|<\delta\|x_1(s)-x_2(s)\|\ t\in(t_k,t_{k+1}],\ k=1,2,\dots m$

From condition(g3)(ii).Hence $\|(\varphi x_1)(t) - \varphi x_2)(t)\| \le \delta \|x_1(s) - x_2(s)\|$
Therefore, $\varphi(x)(.)$ is contraction. Thus $\varphi x = x$ and we had C $(t)(\varphi x)(t)C^*(t) = y(t)$, hence

$$C(t)\, x(t)C^*(t) = y(t).$$

References:

[1] Balakrishnan A.V., and Thoma M.,(1981) Lecture Notes in Control and Information Science , Springer- Verlag , Berlin.

[2] Caputo, M., (1967) Linear model of dissipation whose Q is almost frequency independent. Part II, Geophysical Journal of Royal Astronomical Society, (13)529-539.

[3] Dengguo Xu, Yanmei Li, and Weifeng Zhou,(2014) Controllability and Observability of Fractional Linear Systems with Two Different Orders, The Scientific World Journal Volume 2014, Article ID 618162, 8 pages.

[4] Fazal and Khan,(2003), Controllability and Observability of Non-linear systems using fixed point theory, Appl.Comput. Math., Vol.2,pp.30-41.

[5] Guo T. L., (2012) Controllability and observability of impulsive fractional linear time-invariant system ,Computers and Mathematics with Applications (64) 317.

[6] Iseki, K., (1974), Multi-Valued Contraction Mappings in complet metric spaces, Math Seminar Notes, Vol. 2,pp,45-90.

[7] Kilbas, A.A., Srivastava H.M., Trujillo J.J.,(2006) Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam.

[8] Lukes, D.,(1985) Affine Feedback Controllability of constant Coefficient Differential Equations, SIAM J. on Control and Optimization , Vol 23,pp.952-972.

[9] Magnusson K., Pritchard , A.J., and Quinn M.D., The Application of Fixed Point Theorems to Global Nonlinear Controllability Problems, Second Edition Control Theory Center Report 87, Uni. Of Warwick,(1980).

[10] Shaochun Ji., and Gang Li.,(2012) Existence and Controllability Results for Fractional integrodifferential Equations with Impulsive and Nonlocal Conditions, Int. Journal of Math .Analysis, Vol. no. 50, 2449-2458.

[11] Sussmann, H., (1975), on the number of directions need to archieve controllability , SIAM J. on Control and Optimization, Vol. 13,pp,414-419.