Study of Chaotic Behaviour with G-Bi-Shadowing Property

Mohammed Hussein Obaid Ajam[1,2]*, Iftichar Mudhar Talb Al-Shara'a[3]
[1,3]Department of Mathematics, Collage of Education for Pure Science, University of Babylon, Babylon, Iraq [2] Presidency of the University of Baghdad, University of Baghdad, Baghdad, Iraq

Abstract:
The aim of this work is to study the $\mathbb{G}$-bi-shadowing property on the metric $\mathbb{G}$-space. We generalized the results to the metric $\mathbb{G}$-space and studied the chaotic properties with introducing new definition of chaos which we call $\mathbb{G}$-(ε, k)-chaotic in a neighborhood of a set $\mathbb{Y}$ and comparing it with the definition of Li-York of chaos in $\mathbb{G}$-space. We will study the above definitions with $\mathbb{G}$-homoclinic orbit and
$\mathbb{G}$-chain components.
The main results that we obtained in this paper, for some conditions, if $\mathbf{x}$ is $\mathbb{G}$-homoclinic orbit of φ, and φ is both $\mathbb{G}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing and $\mathbb{G}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing on $\{\mathbf{x}\}$ (when $\{\mathbf{x}\}$ be an unordered set), then any action ψ which satisfying some conditions is $\mathbb{G}$-(ε, k)-chaotic on a neighborhood of $\{\mathbf{x}\}$. Second, for some conditions, if an action φ is $\mathbb{G}$-c-expansive and both $\mathbb{G}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing and $\mathbb{G}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing with respect to an actions ψ and $\mathbf{x}$ is a
$\mathbb{G}$-homoclinic orbit of φ contained, then every action ψ satisfying some conditions is $\mathbb{G}$-(ε, k)-chaotic on a neighborhood of $\{\mathbf{x}\}$. Third, for some conditions, if $\boldsymbol{C}$ be a $\mathbb{G}$-chain component of an action φ, and φ is both $\mathbb{G}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing and $\mathbb{G}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing on a $\mathbb{G}$-chain recurrent set
$\mathbb{G}$-$CR(\varphi)$, Then every action ψ which satisfying some conditions is $\mathbb{G}$-(ε, k)-chaotic on a neighborhood of $\boldsymbol{C}$.

1. Introduction

The concept of shadowing is of great importance in studying and understanding dynamical systems because it often accounts for the accuracy of a computer simulation of the system being used. Work on it began to be developed by many researchers in recent years as an important link for dynamical systems with stability and chaos. The map with has shadowing property is assumed to have a true orbit fairly close to each pseudo orbit of

* Corresponding author: goomuhhus1994@gmail.com

this map. The researcher who gave the concept of shadowing is Walters in [1], see more [2,3,4]

Some researchers have evolved the shadowing into the bi-shadowing by assuming that the true orbit be on another maps under specific conditions. The researcher who gave the concept of

bi-shadowing is Diamond et al. in [5]. Later in [6,7] the researchers studied many relations between the bi-shadowing and other concept. Ajam in [8] introduce types of bi-shadowing concept.

The chaotic behavior of the map on a discrete dynamical system is very restricted. We present another definition that is more broadly studied and preserves the concept of chaos map by

Li and Yorke [9].

In this paper we present the generalization of the bi-shadowing property and Li-Yorke definition of chaos to the $\mathbb{G}$-space. And also introduced the concepte $\mathbb{G}$-(ε, k)-chaotic in a neighborhood of a set $\mathbb{Y}$.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}^+$ be natural, integer, real, and positive real respectively, and let $\mathbb{N}_0$, and $\mathbb{Z}_0^-$ be $\mathbb{N} \cup \{0\}$, and $\mathbb{Z} - \mathbb{N}$ respectively.

Let $\mathbb{G}$ be a group, $\mathbb{X}$ be a Hausdorff topological space and φ be a map. Then the triple $(\mathbb{G}, \mathbb{X}, \varphi)$ is called topological transformation group.

Definition 2.1. [10] The map $\varphi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ which satisfying:

1. $\varphi(g,.)$ is a homeomorphism of for any $g \in \mathbb{G}$,
2. $\varphi(e,x) = x$ for all $x \in \mathbb{X}$ where e is the identity of the group $\mathbb{G}$,
3. $\varphi\left(g_1, \varphi(g_2, x)\right) = \varphi(g_1 g_2, x)$, for all $g_1, g_2 \in \mathbb{G}$, $x \in \mathbb{X}$.

is called an **action** of a group $\mathbb{G}$ on $\mathbb{X}$. And $\mathbb{X}$ is called a **$\mathbb{G}$-space**.

Definition 2.2. Let $\mathbb{G}$ be a group, then

1- The group $\mathbb{G}$ is called **generated by** $\mathbb{S}$ if $\langle \mathbb{S} \rangle = \mathbb{G}$ [11].

2- The group $\mathbb{G}$ is called **finitely generated** if a generating set $\mathbb{S}$ is finite [10].

3- The set $\mathbb{S}$ is called **symmetric** if for any $\mathbb{s} \in \mathbb{S}$ then $\mathbb{s}^{-1} \in \mathbb{S}$ [10].

In this paper we let $\mathbb{G}$ be a finitely generated group, $\mathbb{X}$ be a compact open bounded subset of $\mathbb{R}^d$, $\mathbb{Y}$ be a compact subset of $\mathbb{X}$ while $\mathbb{X}$ is metric $\mathbb{G}$-space with metric $\mathbb{d}$, and $\varphi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be an action. And we fix a finite symmetric generating set $\mathbb{S}$ of $\mathbb{G}$.

Remark 2.3. For $x \in \mathbb{X}$ and $n \in \mathbb{N}$, we have:

1- By definition 2.1 the image of x by φ is $\varphi(\mathbb{s}, x)$ for $\mathbb{s} \in \mathbb{S}$.

2- We denote to the inverse image of φ by φ^{-1}, and the inverse image of x by φ is $\varphi^{-1}(\mathbb{s},x) = \varphi(\mathbb{s}^{-1},x)$ for $\mathbb{s} \in \mathbb{S}$.

3- The n-iterate of x by φ is
$$\underbrace{\varphi(\mathbb{s},\dots\varphi(\mathbb{s},x))}_{n-\text{iterate}} = \varphi\left(\underbrace{\mathbb{s}\dots\mathbb{s}}_{n-\text{items}},x\right) = \varphi(n\mathbb{s},x) \text{ for } \mathbb{s} \in \mathbb{S},$$
so we denote to n iterate of φ by φ^n and $\varphi^n(\mathbb{s},.) = \varphi(n\mathbb{s},.)$ for $\mathbb{s} \in \mathbb{S}$.

4- The n inverse iterate of x by φ is
$$\underbrace{\varphi(\mathbb{s}^{-1}\dots\varphi(\mathbb{s}^{-1},x))}_{n-\text{iterate}} = \varphi\left(\underbrace{\mathbb{s}^{-1}\dots\mathbb{s}^{-1}}_{n-\text{items}},x\right) = \varphi(n\mathbb{s}^{-1},x) \text{ for } \mathbb{s} \in \mathbb{S},$$
so we denote to n inverse iterate of φ by φ^{-n} and $\varphi^{-n}(\mathbb{s},.) = \varphi(n\mathbb{s}^{-1},.)$ for $\mathbb{s} \in \mathbb{S}$.

Definition 2.4. A sequence $\mathbf{x} = \{x_g \in \mathbb{X} \,|g \in \mathbb{G}\}$ is called $\mathbb{G}_{\mathbb{S}}$**-orbit** for φ if satisfying

$x_{sg} = \varphi(\mathbb{s},x_g)$ for $\mathbb{s} \in \mathbb{S}$ and $g \in \mathbb{G}$. (1)

Remark 2.5:

1- We can rewrite a $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{x} = \{x_g \in \mathbb{X} \,|g \in \mathbb{G}\}$ in Definition 2.4 as a sequence associated with a subset of integer numbers, then a sequence $\mathbf{x}$ is became $\mathbf{x} = \{x_n \in \mathbb{X} \,|n \in \mathbb{I} \subseteq \mathbb{Z}\}$, when the length of an interval $\mathbb{I} \subseteq \mathbb{Z}$ is depend on the members of a group $\mathbb{G}$. We can reformulated the condition (1) as follows $x_{n+1} = \varphi(\mathbb{s},x_n)$ for $n \in \mathbb{Z}, \mathbb{s} \in \mathbb{S}$.

2- A finite $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{x} = \{x_n \in \mathbb{X} \,|n = 0,\dots,N\}$ for φ is called a $\mathbb{G}_{\mathbb{S}}$- **periodic orbit of period N** if $x_0 = x_N$ and $x_0 \neq x_j$ for $j \in \{1,\dots,N-1\}$.

3- An infinite $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{x} = \{x_n \in \mathbb{X} \,|n \in \mathbb{Z}\}$ is called $\mathbb{G}_{\mathbb{S}}$**-homoclinic orbit**, if x_n are not identical and there exists $x_* \in \mathbb{X}$ such that $\lim_{n\to\infty} x_n = x_* = \lim_{n\to-\infty} x_n$.

4- The point $x \in \mathbb{X}$ is called a $\mathbb{G}_{\mathbb{S}}$**-periodic point** for φ with period n if $\varphi(n\mathbb{s},x) = x$ and $\varphi(k\mathbb{s},x) \neq x$ for $1 \leq k < n$, and $\mathbb{s} \in \mathbb{S}$.

5- Let $\mathbb{G}$-$\mathbb{P}(\varphi)$ be denote to the set of all $\mathbb{G}$-periodic points for φ with any period.

Definition 2.6. [10] For $\delta > 0$, a sequence $\mathbf{y} = \{y_g \in \mathbb{X} \,|\, g \in \mathbb{G}\}$ is called $\mathbb{G}_{\mathbb{S}}$**-δ-pseudo orbit** for φ if satisfying

$\mathbb{d}\left(y_{sg},\varphi(\mathbb{s},y_g)\right) \leq \delta$, for $\mathbb{s} \in \mathbb{S}$ and $g \in \mathbb{G}$. (2)

Remark 2.7:

1- As in Remark 2.5 we can reformulated $\mathbf{y} = \{y_n \in \mathbb{X}|n \in \mathbb{I} \subseteq \mathbb{Z}\}$ and the condition (2) as follows $\mathbb{d}(y_{n+1},\varphi(\mathbb{s},y_n)) \leq \delta$ for $n \in \mathbb{Z}, \mathbb{s} \in \mathbb{S}$.

2- A finite $\mathbb{G}_{\mathbb{S}}$-δ-pseudo orbit $\mathbf{y} = \{y_n \in \mathbb{X} \,|n = 0,\dots,N\}$ for φ is called a $\mathbb{G}_{\mathbb{S}}$-**periodic**
δ-pseudo orbit of period N if $\mathbb{d}(y_N,y_0) \leq \delta$.

3- A $\mathbb{G}_{\mathbb{S}}$-δ-pseudo orbit $\mathbf{y} = \{y_n \in \mathbb{X} | n \in \mathbb{I} \subseteq \mathbb{Z}\}$ is called a **$\mathbb{G}_{\mathbb{S}}$-δ-pseudo equilibrium** if the y_n are identical for all n under consideration.

4- Let $\mathbb{O}(\varphi)$, $\mathbb{O}_p(\varphi)$, $\mathbb{O}(\varphi,\delta)$ and $\mathbb{O}_p(\varphi,\delta)$ denote to the sets of all finite or infinite $\mathbb{G}_{\mathbb{S}}$-orbits for φ, $\mathbb{G}_{\mathbb{S}}$-periodic-orbits for φ of any period, $\mathbb{G}_{\mathbb{S}}$-δ-pseudo orbits, and
$\mathbb{G}_{\mathbb{S}}$-periodic δ-pseudo orbits of any period.

Definition 2.8. Let $\varphi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ and $\psi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be an actions, the **$\mathbb{G}_{\mathbb{S}}$-distance** between φ and ψ is given by

$$\mathbb{d}_0(\varphi,\psi) = \sup_{x\in\mathbb{X}}\{\mathbb{d}(\varphi(\mathbb{s},x),\psi(\mathbb{s},x))\} \text{ for } \mathbb{s} \in \mathbb{S}.$$

Definition 2.9. The action φ is called **$\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing on** $\mathbb{Y} \subseteq \mathbb{X}$ if there exists $0 < \delta \leq \mathbb{b}$ such that for any $\mathbb{G}_{\mathbb{S}}$-δ-pseudo orbit (finite or infinite) $\mathbf{y} = \{y_n \in \mathbb{Y} | n \in \mathbb{I} \subseteq \mathbb{Z}\} \in \mathbb{O}(\varphi,\mathbb{Y},\delta)$ and any action $\psi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ satisfying $\mathbb{d}_0(\varphi,\psi) \leq \mathbb{b} - \delta$ then there exists an $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{x} = \{x_n \in \mathbb{X} | n \in \mathbb{I} \subseteq \mathbb{Z}\} \in \mathbb{O}(\psi,\mathbb{X})$ such that
$\mathbb{d}(x_n,y_n) \leq \mathbb{a}(\delta + \mathbb{d}_0(\varphi,\psi)) \leq \mathbb{ab}$ for all n as define in $\mathbf{y}$.
If $\mathbb{Y} = \mathbb{X}$, then an action φ is called **$\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing**.

Definition 2.10. The action φ is called **$\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing on** $\mathbb{Y} \subseteq \mathbb{X}$ if there exists $0 < \delta \leq \mathbb{b}$ such that for any finite $\mathbb{G}_{\mathbb{S}}$-periodic δ-pseudo orbit
$\mathbf{y} = \{y_n \in \mathbb{Y} | n = 0,\dots,N\} \in \mathbb{O}_p(\varphi,\mathbb{Y},\delta)$ and any action $\psi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ satisfying $\mathbb{d}_0(\varphi,\psi) \leq \mathbb{b} - \delta$ then there exists an $\mathbb{G}_{\mathbb{S}}$-periodic orbit $\mathbf{x} = \{x_n \in \mathbb{X} | n = 0,\dots,N\} \in \mathbb{O}(\psi,\mathbb{X})$ of period N equal to that of $\mathbf{y}$ such that $\mathbb{d}(x_n,y_n) \leq \mathbb{a}(\delta + \mathbb{d}_0(\varphi,\psi)) \leq \mathbb{ab}$ for all n as define in $\mathbf{y}$.
If $\mathbb{Y} = \mathbb{X}$, then an action φ is called **$\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing**.

Definition 2.11. [10] An action φ is called $\mathbb{G}$-c-expansive in $\mathbb{Y}$ if for any infinite orbits
$\mathbf{x},\mathbf{y} \in \mathbb{O}(\varphi,\mathbb{Y})$ either $\mathbf{x} = \mathbf{y}$ or $\sup_{g\in\mathbb{G}} \mathbb{d}(x_g,y_g) \geq c$, and the number c is called a $\mathbb{G}$-expansive constant for φ.

Definition 2.12. An action φ is called **$\mathbb{G}$-chaotic in the sense of Li-Yorke** if satisfying

L1. There exists $N \in \mathbb{N}$ such that φ has a $\mathbb{G}$-p-periodic point for any $p \geq N$, $p \in \mathbb{Z}$.

L2. There exists an uncountable φ-invariant set $S \subseteq \mathbb{X}$ containing no $\mathbb{G}$-periodic points, called a scrambled set, such that $\limsup_{n\to\infty} \mathbb{d}(\varphi(n\mathbb{s},x),\varphi(n\mathbb{s},y)) > 0$ for every $x,y \in S$ with $x \neq y$, and for every $x \in S$ and any $\mathbb{G}$-periodic point y,

L3. There exists an uncountable subset S_0 of S such that $\liminf_{n\to\infty} \mathbb{d}(\varphi(n\mathbb{s},x),\varphi(n\mathbb{s},y)) = 0$ for every $x,y \in S_0$.

Definition 2.13. Let $\mathbb{Z}^m$ be denote to the set of all sequences $\mathbf{b} = \{b_n \in \{1,\dots,m\}: \text{for } n \in \mathbb{Z}\}$ which associated with a group $\mathbb{G}$, and let $V = \{v_1,\dots,v_m \mid v_i \in \mathbb{X}, v_i \neq v_j\}$ be an unordered. We used some sequences in $\mathbb{Z}^m$ to describe the order in which some disjoint balls of the form $U_i = \{z \in \mathbb{X}: \mathbb{d}(z,v_i) < \varepsilon\ ;\ i = 1,\dots,m\}$ are to be visited.

Definition 2.14. Let $\varepsilon > 0$ and $k \in \mathbb{N}$, and let $\mathbb{Y}$ for which $\max_{x,y\in\mathbb{Y}} \mathbb{d}(x,y) \geq 2\varepsilon$. Action φ is called $\mathbb{G}_{\mathbb{S}}$-(ε,k)**-chaotic in a neighborhood of** $\mathbb{Y}$ if for each finite subset $V = \{v_1,\dots,v_m\}$ of $\mathbb{Y}$ with $\min_{i\neq j} \mathbb{d}(v_i,v_j) \geq 2\varepsilon$ there exists an action $Z_\varphi: \mathbb{Z}^m \to \mathbb{O}(\varphi)$ such that

S1. For each $\mathbf{b} = \{b_n \in \{1,\dots,m\}: \text{for } n \in \mathbb{Z}\} \in \mathbb{Z}^m$ the $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{z} = Z_\varphi(\mathbf{b}) = \{z_n \in \mathbb{X}: \text{ for } n \in \mathbb{Z}\}$ satisfies $z_{kn} \in U_{b_n}$,

S2. The action $\mathbf{b} \to Z_\varphi(\mathbf{b})$ is shift invariant (that is a k-shift Sh^k of $\mathbf{b} \in \mathbb{Z}^m$ is corresponding to $\mathbb{G}_{\mathbb{S}}$-orbit $Z_\varphi(\mathbf{b})$),

S3. If $\mathbf{b} \in \mathbb{Z}^m$ is $\mathbb{G}_{\mathbb{S}}$-periodic with period p, then the corresponding $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{z} = Z_\varphi(\mathbf{b})$ is $\mathbb{G}$-periodic with period kp,

S4. For each $\eta > 0$ there exists an uncountable subset $\mathbb{Z}_0^\eta$ of $\mathbb{Z}^m$ such that

$$\limsup_{n\to\infty} \mathbb{d}(Z_\varphi(\mathbf{a})_n, Z_\varphi(\mathbf{b})_n) \geq \frac{1}{2}\varepsilon \text{ for all } \mathbf{a},\mathbf{b} \in \mathbb{Z}_0^\eta, \mathbf{a} \neq \mathbf{b},$$

and $\liminf_{n\to\infty} \mathbb{d}(Z_\varphi(\mathbf{a})_n, Z_\varphi(\mathbf{b})_n) < \eta$, for all $\mathbf{a},\mathbf{b} \in \mathbb{Z}_0^\eta$.

3- $\mathbb{G}$**-Chaotic Behavior with** $\mathbb{G}$**-Homoclinic Orbit:**

The above definition of $\mathbb{G}$-chaotic behaviour are similar to those in the Smale transverse homoclinic orbit theorem (see Theorem 16.2 in [12]) with generalize a properties to $\mathbb{G}$-space.

Through the definition we assumed either the uniqueness of the $\mathbb{G}$-orbit $Z_\varphi(\mathbf{b})$ for $\mathbf{b} \in \mathbb{Z}^m$ or a $\mathbb{G}$-continuity of Z_φ.

On the physical side, a Definition 2.14 means a $\mathbb{G}$-orbits of an action φ seems to act chaotic if a mathematical calculations accurately is not less than $k(\varepsilon)$.

Lemma 3.1.[13] Let $\mathbb{X}$ be with the power of the continuum and let $\mathcal{S}$ be the set of sequences $\mathbf{s} = \{s_i \in \mathbb{X} \mid i \in \mathbb{N}\}$. Then for each $\eta > 0$ there exists a subset $\mathcal{S}(\eta)$ of $\mathcal{S}$ with the power of the continuum such that $\liminf_{i\to\infty} \mathbb{d}(s_i,t_i) < \eta$ for any $\mathbf{s},\mathbf{t} \in \mathcal{S}(\eta)$.

Theorem 3.2. Let $\mathbf{x}$ be $\mathbb{G}_{\mathbb{S}}$-homoclinic orbit of φ, suppose that φ is both $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing and $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing on $\{\mathbf{x}\}$ (when $\{\mathbf{x}\}$ be an unordered set), and define $\delta(\varepsilon) = \frac{1}{2}\min\{\mathbb{b}, {}^{\varepsilon}/_{\mathbb{a}}\} > 0$. Then every bounded action
$\psi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ satisfying $\mathbb{d}_0(\varphi,\psi) < \delta(\varepsilon)$ is $\mathbb{G}_{\mathbb{S}}$-(ε, k)-chaotic on a neighborhood of $\{\mathbf{x}\}$ for any $k \geq k(\varepsilon)$, $k \in \mathbb{N}$ where
$k(\varepsilon) = \max\{m|\text{there exist } i_0 \in \mathbb{Z}, \text{with } \mathbb{d}(x_i, x_*) \geq \delta(\varepsilon), i = i_0, i_0+1, \dots, i_0+m\}$. (3)
We must distinguish an orbit $\mathbf{x}$ from the unordered set $\{\mathbf{x}\}$.
Note that: a number $k(\varepsilon)$ in (3) act a maximal of orbits of an element of a set $\{\mathbf{x}\}$ that can stay outside of $\delta(\varepsilon)$-neighborhood of a homoclinic point x_*.
Proof: Let $\varepsilon > 0$ be arbitrary with $\max_{x,y\in\mathbb{X}} \mathbb{d}(x,y) \geq \varepsilon$, $k > k(\varepsilon)$, $k \in \mathbb{N}$, and $V = \{v_1, \dots, v_m\}$, $m > 1$ satisfying $\min_{i \neq j} \mathbb{d}(v_i, v_j) \geq 2\varepsilon$.
Construct an action $Z_\psi: \mathbb{Z}^m \to \mathbb{O}(\psi)$ which satisfying $\boldsymbol{S1} - \boldsymbol{S4}$. Let S be a subset of $\mathbb{R}^d$, then a set of all open ρ-neighborhood of S is denoted by $\mathfrak{O}_\rho(S)$.
For each $v \in V$ with $v \neq x_*$ there can be found $m_-(v) \in \mathbb{Z}_+$, $m_+(v) \in \mathbb{Z}_-$ and a finite sequence
$u = u(v) = \{u(v)_{-m_-(v)}, u(v)_{-m_-(v)+1}, \dots, u(v)_{m_+(v)-1}\}$,
which are uniquely defined by
$u(v)_0 = v, \quad u(v)_i = \varphi(\mathbb{s}, u(v)_{i-1})$,
for $i = -m_-(v)+1, \dots, m_+(v)-1$
such that
$u(v)_{-m_-(v)}, \varphi(s, u(v)_{m_+(v)-1}) \in \mathfrak{O}_{\delta(\varepsilon)}(\{x_*\}), \quad u(v)_i \notin \mathfrak{O}_{\delta(\varepsilon)}(\{x_*\})$,
for $-m_-(v) < i < m_+(v)$.
Consider a given integer $k > k(\varepsilon)$ and a given sequence $\mathbf{b} \in \mathbb{Z}^m$.
Define a sequence $\mathbf{w} = \{w_n | n \in \mathbb{Z}\}$ by

$$\mathbf{w} = \begin{cases} w_{n+kj} = u(v_j)_n & \text{for } -m_-(v_j) \leq n < m_+(v_j)\ ;\ v_j \neq x_* \\ w_n = x_* & \text{for all other } n \end{cases}$$

Then $\mathbf{w}$ is a $\mathbb{G}_{\mathbb{S}}$-$\delta(\varepsilon)$-pseudo orbit of φ.
Hence, by the assumed $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing of φ, for any bounded action ψ with $\mathbb{d}_0(\varphi,\psi) < \delta(\varepsilon)$, the set $Z_\psi(\mathbf{b})$ of all $\mathbb{G}_{\mathbb{S}}$-orbits $\mathbf{z}$ satisfying $\mathbb{d}(z_{kn}, v_{\mathbf{b}_n}) < \varepsilon$ for all n is not empty.
Furthermore, by the assumed $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing of φ, this set $Z_\psi(\mathbf{b})$ contains a $\mathbb{G}_{\mathbb{S}}$-orbit of minimal period pk if $\mathbf{b}$ is $\mathbb{G}_{\mathbb{S}}$-periodic with minimal period p.
Standard constructions using Zorn's Lemma [14] allow a (single-valued) selector Z_ψ of the multi-valued action $\mathbf{b} \to Z_\psi(\mathbf{b})$ to be chosen which satisfies conditions $\boldsymbol{S1} - \boldsymbol{S3}$.

Indeed, let us denote by $\mathbf{Z}$ the totality of single-valued selectors Z_ψ which are defined on subsets of $\mathcal{D}(Z) \subset \mathbb{Z}^m$ and satisfy conditions $\boldsymbol{S1} - \boldsymbol{S3}$ and consider this set as being partially ordered by inclusion of the corresponding graphs $\mathbf{Gr}(Z_\psi) = \{(\mathbf{b}, Z_\psi(\mathbf{b})) : \mathbf{b} \in \mathcal{D}(Z)\}$.

By the construction every chain $\hat{\mathbf{Z}}$ (that is, linearly ordered subset) of $\mathbf{Z}$ has an upper bound, the graph of which is defined as the union $\cup_{Z_\psi \in \hat{\mathbf{Z}}} \mathbf{Gr}(Z_\psi)$.

Hence by Zorn's Lemma there exists a maximal element Z_* in the set $\mathbf{Z}$.

Suppose that the strict inclusion $\mathcal{D}(Z_*) \subset \mathbb{Z}^m$ holds. Then there exist an element

$\mathbf{b}_* \in \mathbb{Z}^m \setminus \mathcal{D}(Z_*)$. If for some positive integer i the sequence $\mathbf{b}_*$ is the i^{th}-shift of a sequence $\mathbf{b_0} \in \mathcal{D}(Z_*)$ then an action

$$Z_0(\mathbf{b}) = \begin{cases} Z_*(\mathbf{b}) & \text{if } \mathbf{b} \in \mathcal{D}(Z_*) \\ \mathrm{Sh}^{-ik}\, Z_*(\mathbf{b_0}) & \text{if } \mathbf{b} = \mathbf{b}_* \end{cases}$$

satisfies conditions $S1 - S3$ and strictly dominates Z_*, which contradicts the definition of Z_*.

On the other hand, if the sequence $\mathbf{b}_*$ cannot be represented as a shift of a sequence $\mathbf{b} \in \mathcal{D}(Z_*)$ then define $Z_0(\mathbf{b})$ as an arbitrary element from the nonempty set $Z_\psi(\mathbf{b})$ of all $\mathbb{G}_\mathbb{S}$-orbits $\mathbf{z}$ satisfying $\mathbb{d}(z_{kn}, v_{\mathbf{b}_n}) < \varepsilon$ for all n, again the action Z_0 satisfies $\boldsymbol{S1} - \boldsymbol{S3}$ and strictly dominates Z_*, and we arrive again at a contradiction.

The Lemma 3.1 called that the selected action Z_ψ also satisfies condition $\boldsymbol{S4}$.

■

The next theorem provides a simple means of locating $\mathbb{G}$-homoclinic orbits.

Theorem 3.3. Let $\mathbf{w} = \{v_i \in \mathbb{X} | i = 0, \dots, p-1\}$ be a $\mathbb{G}_\mathbb{S}$-periodic δ-pseudo orbit of φ which is $\mathbb{G}_\mathbb{S}$-$(\mathbb{a}, \mathbb{b})$-bi-shadowing and $\mathbb{G}_\mathbb{S}$-$(\mathbb{a}, \mathbb{b})$-periodic bi-shadowing on $\{\mathbf{w}\}$ and $\mathbb{G}$-c-expansive in $\mathbb{X}$. Suppose that $\delta \le \mathbb{b}$, $\mathbb{d}(\varphi(\mathbb{s}, v_0), v_0) \le \mathbb{b}$, for $\mathbb{s} \in \mathbb{S}$ and

$$2\mathbb{a}\delta < \max_{i,j} \mathbb{d}(v_i, v_j), \quad \mathbb{a}(\delta + \mathbb{d}(\varphi(\mathbb{s}, v_0), v_0)) < c, \quad \text{for } \mathbb{s} \in \mathbb{S}. \qquad (4)$$

Then φ has a $\mathbb{G}_\mathbb{S}$-homoclinic orbit $\mathbf{x}$ in an open $\mathbb{a}\delta$-neighborhood of $\{\mathbf{w}\}$.

Proof: The point v_0 is clearly a $\mathbb{G}_\mathbb{S}$-$(\mathbb{d}(\varphi(\mathbb{s}, v_0), v_0))$-pseudo-equilibrium of φ. By the assumption that $\mathbb{d}(\varphi(\mathbb{s}, v_0), v_0) \le \mathbb{b}$ and the $\mathbb{G}_\mathbb{S}$-$(\mathbb{a}, \mathbb{b})$-periodic bi-shadowing there exists a proper $\mathbb{G}_\mathbb{S}$-equilibrium x_* of φ satisfying

$$\mathbb{d}(x_*, v_0) \le \mathbb{d}(\varphi(\mathbb{s}, v_0), v_0). \qquad (5)$$

Now consider the bi-infinite sequence $\mathbf{y} = \{y_n \mid n \in \mathbb{Z}\}$ defined by

$$\mathbf{y} = \begin{cases} y_n = v_0 & \text{for } n < 0 \text{ or } n \geq p, \\ y_n = v_n & \text{otherwise,} \end{cases}$$

which is obviously a $\mathbb{G}_{\mathbb{S}}$-δ-pseudo orbit of φ.

In view of the inequality $\delta \leq \mathbb{b}$, there thus exists a $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{x} = \{x_n \mid n \in \mathbb{Z}\}$ in the

$\mathbb{a}\delta$-neighborhood of a $\mathbb{G}_{\mathbb{S}}$-pseudo orbit $\mathbf{y}$. The elements of this $\mathbb{G}_{\mathbb{S}}$-orbit are not all identical because of the first inequality in (4).

We must show that a $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{x}$ is $\mathbb{G}_{\mathbb{S}}$-homoclinic. To this end it sufficies to establish the limit relationships $\lim_{n\to\infty} x_{-n} = \lim_{n\to\infty} x_n = x_*$. Suppose that

$$\lim_{n\to\infty} x_n = x_* \qquad (6)$$

does not hold. Then there exists a sequence of indices $i_m \to \infty$ and an $\varepsilon_1 > 0$ such that

$$\mathbb{d}(x_{i_m}, x_*) > \varepsilon_1, \qquad m = 1,2,\ldots. \qquad (7)$$

Consider a coordinate-wise limit point $\mathbf{x}^* = \{x_n^* \mid n \in \mathbb{Z}\}$ of the sequence of shifted $\mathbb{G}_{\mathbb{S}}$-orbits

$$\mathbf{x}^m = \{x^m_{-i_m}, x^m_{-i_m+1}, \ldots\}$$

defined by $x^m_{-i_m} = x_n$ for $n = 0,1,2,\ldots$. Then (7) implies

$$\mathbb{d}(x_0^*, x_*) > \varepsilon_1. \qquad (8)$$

Now every sequence $\mathbf{x}^m$ is an $\mathbb{G}_{\mathbb{S}}$-orbit of φ, so $\mathbf{x}^*$ is also a $\mathbb{G}_{\mathbb{S}}$-orbit of φ because φ is an action. Furthermore, $\mathbf{x}^*$ satisfies the inequalities

$$\mathbb{d}(x_n^*, x_*) < c \qquad (9)$$

for all n because of (5) and the second inequality in (4). The inequalities (9) and (8) contradict the $\mathbb{G}$-c-expansivity, so the limit (6) must be exist.

The proof of $\lim_{n\to\infty} x_{-n} = x_*$ is similarly. ■

Corollary 3.4. Let φ be $\mathbb{G}$-c-expansive on $\mathbb{Y}$ with $c = \delta/_k$ and both $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing and $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing on $\mathbb{Y}$ with respect to bounded actions $\psi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ and $\mathbf{x}$ be a $\mathbb{G}_{\mathbb{S}}$-homoclinic orbit of φ contained in $\mathbb{Y}$ and define $k(\varepsilon)$ by (3) and $\delta(\varepsilon)$ by

$$\delta(\varepsilon) = \frac{1}{3}\min\{\mathbb{b}, \varepsilon/\mathbb{a}\}.$$

Then every action ψ satisfying $\mathbb{d}_0(\varphi,\psi) < \delta(\varepsilon)$ is $\mathbb{G}_{\mathbb{S}}$-(ε,k)-chaotic on a neighborhood of $\{\mathbf{x}\}$ for any positive integer $k \geq k(\varepsilon)$.

4- $\mathbb{G}$-Chaotic Behavior with $\mathbb{G}$-Chain Components:

During the study of the chaotic behavior with Chain Components of maps, Park and Lee in [15] presented a number of concepts in chain recurrent in the metric space, In what follows we generalize it to a $\mathbb{G}$-space.

Definition 4.1. A point $x \in \mathbb{X}$ is called **$\mathbb{G}_{\mathbb{S}}$-δ-chain recurrent** for φ if there exists a finite

$\mathbb{G}_s$-δ-pseudo orbit $\{x_n \mid n = 0, \dots, n\}$ of φ with $x_0 = x_n = x$, that is connecting x with itself.
A point $x \in \mathbb{X}$ is called $\mathbb{G}_s$**-chain recurrent** for φ if for any $\delta > 0$ there exists a finite
$\mathbb{G}_s$-δ-pseudo orbit connecting x with itself.
Let $\mathbb{G}_s$-$CR(\varphi)$ denote the set of all $\mathbb{G}_s$-chain recurrent points of φ.
Note that the $\mathbb{G}_s$-chain recurrent set $\mathbb{G}_s$-$CR(\varphi)$ is compact and φ-invariant.
We define a relation $\sim$ on $\mathbb{G}_s$-$CR(\varphi)$ by $x \sim y$ if for any $\delta > 0$ there exist two finite
$\mathbb{G}_s$-δ-pseudo orbits $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}$ is connecting x with y and $\mathbf{y}$ is connecting y with x. Two such points are called $\mathbb{G}_s$**-chain equivalent**. It is an equivalence relation. The equivalence classes are called the $\mathbb{G}_s$**-chain components** of φ.
Now we show that if an action φ is $\mathbb{G}_s$-$(\mathbb{a}, \mathbb{b})$-bi-shadowing and $\mathbb{G}_s$-$(\mathbb{a}, \mathbb{b})$-periodic
bi-shadowing on the $\mathbb{G}_s$-chain recurrent set $\mathbb{G}_s$-$CR(\varphi)$ then all nearby perturbations of an action φ behave chaotically on a neighborhood of each $\mathbb{G}_s$-chain component of φ whenever it has a fixed point.
Let $\boldsymbol{C}$ be a $\mathbb{G}_s$-chain component of φ, and x and y be any two points in $\boldsymbol{C}$. For any $\delta > 0$, we denote $\mathbb{O}(x, y, \delta)$ the set of all finite $\mathbb{G}_s$-δ-pseudo orbits in $\mathbb{G}$-$CR(\varphi)$ from x to y.
Note that we can choose $\mathbb{G}_s$-δ-pseudo orbit from x to y which belongs to $\mathbb{G}_s$-$CR(\varphi)$.
For any $\mathbf{x} \in \mathbb{O}(x, y, \delta)$, we let $\mathrm{card}(\mathbf{x})$ be the cardinal number of the set $\mathbf{x}$, and
$\mathrm{card}_\delta(x, y) = \inf\{\mathrm{card}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{O}(x, y, \delta)\}$.
Lemma 4.2. Let $\boldsymbol{C} \subset \mathbb{X}$ be a $\mathbb{G}_s$-chain component of φ. For any $\delta > 0$, we can choose a positive integer $N = N(\delta)$ such that
$\sup\{\mathrm{card}_\delta(x, y) \mid x, y \in \boldsymbol{C}\} \leq N$.
Proof. Suppose not, then there exists $\delta_0 > 0$ such that for any $N > 0$,
$\sup\{\mathrm{card}_\delta(x, y) \mid x, y \in \boldsymbol{C}\} > N$.
Hence, we can select $(x_n, y_n) \in \boldsymbol{C} \times \boldsymbol{C}$ satisfying $\mathrm{card}_{\delta_0}(x_n, y_n) \geq n$ and $\mathrm{card}_{\delta_0}(x_n, y_n) \leq \mathrm{card}_{\delta_0}(x_{n+1}, y_{n+1})$ for all $= 1, 2, \dots$. Since $\boldsymbol{C}$ is compact, we may choose subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, which are convergent; say
$\lim_{k\to\infty} x_{n_k} = x$,and $\lim_{k\to\infty} y_{n_k} = y$.
Since $\boldsymbol{C}$ is compact, then $x, y \in \boldsymbol{C}$.
To complete the proof, it is enough to show that $\mathrm{card}_{\delta_0}(x, y) = \infty$.

Suppose that $\mathrm{card}_{\delta_0}(x,y)$ is finite; say $\mathrm{card}_{\delta_0}(x,y) = L$. Then there exists a $\mathbb{G}_{\mathbb{S}}$-δ_0-pseudo orbit $\mathbf{x} = \{v_1, \dots, v_L\}$ in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$ satisfying $v_1 = x$ and $v_L = y$. Then we can choose a sufficiently large integer $\alpha > L$ such that
$\tilde{\mathbf{x}} = \{x_\alpha, v_2, \dots, v_{L-1}, y_\alpha\}$
becomes a $\mathbb{G}_{\mathbb{S}}$-δ_0-pseudo orbit in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$. Hence we have
$\tilde{\mathbf{x}} \in \mathbb{O}(x_\alpha, y_\alpha, \delta_0)$, $\mathrm{card}(\tilde{\mathbf{x}}) = L$, and $\mathrm{card}_{\delta_0}(x_\alpha, y_\alpha) \le L$.
However this contradicts to the fact that
$\mathrm{card}_{\delta_0}(x_\alpha, y_\alpha) \ge \alpha \ge L$. ■

Theorem 4.3. Let $\boldsymbol{C}$ be a $\mathbb{G}_{\mathbb{S}}$-chain component of an action φ on $\mathbb{X}$. Suppose φ is both
$\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-bi-shadowing and $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a},\mathbb{b})$-periodic bi-shadowing on the $\mathbb{G}_{\mathbb{S}}$-chain recurrent set $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$, and let $\varepsilon > 0$ be arbitrary with $\varepsilon < \mathrm{diam}\,\boldsymbol{C}$. Let $r(\varepsilon) = \min\left\{\varepsilon, \frac{\varepsilon}{2\mathbb{a}}, \frac{\mathbb{b}}{2}\right\}$. Then every action $\psi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ with $\mathbb{d}_0(\varphi,\psi) < r(\varepsilon)$ is $\mathbb{G}_{\mathbb{S}}$-(ε, k)-chaotic on a neighborhood of $\boldsymbol{C}$ for any $k \ge 2N(r(\varepsilon))$ if $\boldsymbol{C}$ bas a fixed point, where $N(r(\varepsilon))$ is an integer corresponding to the number $r(\varepsilon)$ as in Lemma 4.2.

Proof. Let $\varepsilon > 0$ be arbitrary with $\varepsilon < \mathrm{diam}\,\boldsymbol{C}$, and let an action $\psi: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be satisfying

$$\mathbb{d}_0(\varphi,\psi) < r(\varepsilon), \qquad \text{where } r(\varepsilon) = \min\left\{\varepsilon, \frac{\varepsilon}{2\mathbb{a}}, \frac{\mathbb{b}}{2}\right\}.$$

Let $N(r(\varepsilon))$ be an integer corresponding to the number $r(\varepsilon)$ as in Lemma 4.2, and let $k > 0$ be a fixed integer with $k \ge 2N(r(\varepsilon))$. Let $\{v_1, \dots, v_m\}$ be a finite subset of $\boldsymbol{C}$ with
$\min_{i \ne j} \mathbb{d}(v_i, v_j) \ge 2\varepsilon.$
Now we are going to construct an action $Z_\psi: \mathbb{Z}^m \to \mathbb{O}(\psi)$ which have the properties $\boldsymbol{S1} - \boldsymbol{S4}$. For each sequence $\mathbf{b} = \{b_n | n \in \mathbb{Z}\} \in \mathbb{Z}^m$, we can associate a sequence
$V(\mathbf{b}) = \{\dots, v_{b_{-1}}, v_{b_0}, v_{b_1}, \dots\}$
in the product space.
For each integer $i \in \mathbb{Z}$ we may find a $\mathbb{G}_{\mathbb{S}}$-$r(\varepsilon)$-pseudo orbit $\mathbf{x}_i$ in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$ from v_{b_i} to $v_{b_{i+1}}$ with $\mathrm{card}(\mathbf{x}_i) = k$. To show this, we let $p \in \boldsymbol{C}$ be a fixed point of φ. Since
$\sup\{\mathrm{card}_{r(\varepsilon)}(x,y) \mid x, y \in \boldsymbol{C}\} \le N(r(\varepsilon)),$
we can choose a $\mathbb{G}_{\mathbb{S}}$-$r(\varepsilon)$-pseudo orbit $\mathbf{y}_0$ in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$ from v_{b_i} to p with $\mathrm{card}(\mathbf{y}_0) \le N(r(\varepsilon))$, and a $\mathbb{G}_{\mathbb{S}}$-$r(\varepsilon)$-pseudo orbit $\mathbf{y}_1$ in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$ from p to $v_{b_{i+1}}$ with $\mathrm{card}(\mathbf{y}_1) \le N(r(\varepsilon))$. By connecting two $\mathbb{G}_{\mathbb{S}}$-pseudo orbits $\mathbf{y}_0$ and $\mathbf{y}_1$, we

can construct a
$\mathbb{G}_{\mathbb{S}}$-$r(\varepsilon)$-pseudo orbit $\mathbf{x}_i$ in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$ from v_{b_i} to $v_{b_{i+1}}$ with $\mathrm{card}(\mathbf{x}_i) = k$.
Let $\mathbf{x}(\mathbf{b}) = \{\dots, \mathbf{x}_{-1}, \mathbf{x}_0, \mathbf{x}_1, \dots\} \equiv \{x_n | n \in \mathbb{Z}\}$. Then $\mathbf{x}(\mathbf{b})$ is a $\mathbb{G}_{\mathbb{S}}$-$r(\varepsilon)$-pseudo orbit of φ in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$. Since φ is $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a}, \mathbb{b})$-bi-shadowing on $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$ and
$r(\varepsilon) + \mathbb{d}_0(\varphi, \psi) \le 2r(\varepsilon) \le \mathbb{b}$,
there exists a $\mathbb{G}_{\mathbb{S}}$-orbit $\mathbf{z}(\mathbf{b}) = \{z_n \equiv z_g | n \in \mathbb{Z}, g \in \mathbb{G}\}$ of ψ such that
$\mathbb{d}(x_n, z_n) < \mathbb{a}(r(\varepsilon) + \mathbb{d}_0(\varphi, \psi)) \le 2\, \mathbb{a}\, r(\varepsilon) \le \varepsilon$
for all $n \in \mathbb{Z}$.
Let $\mathbf{b} = \{b_n\}$ be a $\mathbb{G}_{\mathbb{S}}$-periodic sequence in $\mathbb{Z}^m$ with period p. Since φ is $\mathbb{G}_{\mathbb{S}}$-$(\mathbb{a}, \mathbb{b})$-periodic bi-shadowing on $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$, by the same techniques as above, we can select a $\mathbb{G}_{\mathbb{S}}$- $r(\varepsilon)$-pseudo orbit $\mathbf{x}(\mathbf{b})$ of φ in $\mathbb{G}_{\mathbb{S}}$-$CR(\varphi)$ with period pk and a proper $\mathbb{G}_{\mathbb{S}}$-periodic orbit
$\mathbf{z}(\mathbf{b}) = \{z_n \equiv z_g | n = 0, \dots, pk-1\,;\ g \in \mathbb{G}\}$ of ψ with period pk and $\mathbb{d}(x_n, z_n) < \varepsilon$ for all $n = 0, \dots, pk-1$.
For each $\mathbf{b} \in \mathbb{Z}^m$ the set $Z_\psi(\mathbf{b})$ of all $\mathbb{G}_{\mathbb{S}}$-orbits $\mathbf{z}(\mathbf{b}) = \{z_n \equiv z_g | n \in \mathbb{Z}, g \in \mathbb{G}\}$ of ψ satisfying $\mathbb{d}(x_{nk}, v_{b_n}) < \varepsilon,\ n \in \mathbb{Z}$
is not empty. Moreover the set $Z_\psi(\mathbf{b})$ contains a $\mathbb{G}_{\mathbb{S}}$-orbit of ψ with period pk if $\mathbf{b}$ is a
$\mathbb{G}_{\mathbb{S}}$-periodic sequence in $\mathbb{Z}^m$ with period p.
By our construction, we can consider the totality $\mathbf{Z}$ of the single valued actions
$Z_\psi : \mathcal{D}(Z) \to \mathbb{O}(\psi)$ satisfying the conditions $\mathcal{S}1 - \mathcal{S}3$, where $\mathcal{D}(Z)$ is a subset of $\mathbb{Z}^m$ such that Z is defined on it. Consider the set $\mathbf{Z}$ as being partially ordered by inclusion of the set corresponding graphs:
$\mathbf{Gr}(Z_\psi) = \{(\mathbf{b}, Z_\psi(\mathbf{b})) | \mathbf{b} \in \mathcal{D}(Z)\}$.
Then every chain $\widehat{\mathcal{S}}$ of $\mathbf{Z}$ has an upper bound, and the graph of which is defined as the union $\cup_{Z_\psi \in \widehat{\mathcal{S}}} \mathbf{Gr}(Z_\psi)$. By the Zorn's Lemma, there exists a maximal element Z_* in $\mathbf{Z}$. Then we can see that $\mathcal{D}(Z_*) = \mathbb{Z}^m$.
Suppose not. Then there exists $\mathbf{a} \in \mathbb{Z}^m \setminus \mathcal{D}(Z_*)$. If $\mathbf{a} = \mathrm{Sh}^i(\mathbf{c})$ for some positive integer i and some $\mathbf{c} \in \mathcal{D}(Z_*)$ then the action $Z_0 : \mathcal{D}(Z_*) \cup \{\mathbf{a}\} \to \mathbb{O}(\psi)$ defined by

$$Z_0(\mathbf{b}) = \begin{cases} Z_*(\mathbf{b}) & \text{if } \ \mathbf{b} \in \mathcal{D}(Z_*) \\ \mathrm{Sh}^{-ik}\, Z_*(\mathbf{c}) & \text{if } \ \mathbf{b} = \mathbf{a} \end{cases}$$

satisfies conditions $\mathcal{S}1 - \mathcal{S}3$ and $\mathcal{D}(Z_*) \subsetneq \mathcal{D}(Z_0)$, which contradicts the definition of Z_*.

Similarly if the sequence a cannot be represented as a shift of a sequence $\mathbf{b} \in \mathcal{D}(Z_*)$ then we can construct an action $Z_0: \mathcal{D}(Z_*) \cup \{\mathbf{a}\} \to \mathbb{O}(\psi)$ satisfying conditions $\boldsymbol{S1} - \boldsymbol{S3}$. This means that $\mathcal{D}(Z_*) = \mathbb{Z}^m$.

The fact that the action $Z_\psi: \mathbb{Z}^m \to \mathbb{O}(\psi)$ satisfies the condition $\boldsymbol{S4}$ follows from Lemma 3.1. ■

References

1. Walters, Peter. "On the pseudo orbit tracing property and its relationship to stability." *The structure of attractors in dynamical systems*. Springer, Berlin, Heidelberg, 231-244, (1978).
2. Pilyugin, Sergei Yu. *Shadowing in dynamical systems*. Springer, (2006).
3. AL-Sharaa, Iftichar MT, and Raad Safah AL-Joboury. "Asymptotic Fitting Shadowing Property." *Albahir journal* 7.13-14 (2018).
4. Ajam, M. H.O., Some Algebraic Results of Shadowing Property in Dynamical Systems. *Journal of Engineering and Applied Sciences*, 13(8 SI): 6395-6397, (2018).
5. Diamond P., Kloeden P. E., Kozyakin V. S., and Pokrovskii A. V., "Robustness of the Observable Behavior of Semihyperbolic Dynamical Systems", Avtomat. i Telemekh. (11), 148-159 (1995).
6. Ajam, M. H., On Bi-shadowing in Dynamical Systems. *MSc. thesis*, University of Babylon, (2017).
7. Ajam, M. H. O., and Talb, I. M. Some General Properties of Bi-shadowing Property. *Journal of University of Babylon for Pure and Applied Sciences*, *26*(5), 226-232, (2018).
8. Ajam, Mohammed H.O., and Iftichar M.T. Al-Shara'a. "Types of Expansivity on Bi-Shadowing Property." *IOP Conference Series: Materials Science and Engineering*. Vol. 928. No. 4. IOP Publishing, (2020).
9. Li, T. Y., and Yorke, J. A. (1975). Period Three Implies Chaos. *The American Mathematical Monthly*, *82*(10), 985-992.
10. Osipov, A. V., and Tikhomirov, S. B. (2014). Shadowing for Actions of Some Finitely Generated Groups. *Dynamical Systems*, *29*(3), 337-351.
11. Gilbert, L. (2014). *Elements of Modern Algebra*. Cengage Learning.
12. Ombach, J. (1987). Consequences of The Pseudo Orbits Tracing Property and Expansiveness. *Journal of the Australian Mathematical Society*, *43*(3), 301-313.
13. Diamond, P., Kloeden, P. E., Krasnosel'skii, M. A., and Pokrovskii, A. V. (1996). *Chaotic Dynamics in Nonsmooth Perturbations of Bishadowing*

Systems. Deakin University. Centre for Applied Dynamical Systems and Environmental Modelling.
14. Kelley, J. L. (1975). *General Topology*, Springer.
15. Park, T. Y., and Lee, K. H. (2001). Chaotic Behaviour of Chain Components in Bishadowing Systems. *Journal of the Korean Mathematical Society*, *38*(3), 613-621.