Stability of Picard $-S$ Iteration processes for $\xi-$ Uniformly Accretive Mapping.

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Abstract:

In this paper , we study the stability and the strong convergence to unique fixed point of $\xi-$ uniformly accretive mapping by using picard $-S$ Iteration process in Banach space.

Keywords: picard $-S$ Iteration , $\varphi-$ Uniformly Accretive , $B-$ Stabilty

1.Introduction:

Osilike [1], established the stability results of Mann and Ishikawa iterations for Lipschitz strongly pseudocontractive self mapping in uniformly smooth Banach space. Also Osilike [2], established the stability results of Mann and Ishikawa iterations for $\xi-$ strongly pseudocontractive self mapping in Banach space.

On the other hand , Zeqing , Lili and shin [3] , established the convergence result to the unique fixed point of strongly pseudocontractive self mapping in Banach space by Ishikawa iteration with errors , then he proved the stability results of this iteration . Liu , Xu and Kang [4] proved that Ishikawa iteration converge strongly to the unique fixed point of locally strongly pseudocontractive self mapping in uniformly smooth Banach space, then he proved the stability results of this iteration.

2.Preliminaires:

In this section , some basic definitions and lemmas which needed are presented

Definition(2.1), [5]

Let H is a normed space . A mapping $B: H \to H$ is said to be lipschitizian if $\exists\, L > 0$ such that $\|Bx - By\| \le L\,\|x - y\|$, $\forall x, y \in H$. ... (1)

Definition(2.2), [4]

Let H is a normed space with the dual space H^* and the mapping $J: H \to 2^{H^*}$ is defined by

$J(x)=\{f\in H^*:\langle x,f\rangle=\|f\|\|x\|,\|f\|=\|x\|\}$, $\forall x\in H$ is said to be normalized duality mapping , where f is a linear function .

Definition(2.3), [6]:

Let H is a normed space and D is a nonempty subset of H.

A mapping $B:D\rightarrow D$ is said to be $\xi-$ Uniformly Accretive . If $\exists$ strictly increasing $\boldsymbol{\xi}:[0,\infty[\rightarrow[0,\infty[$ *with* $\boldsymbol{\xi}(0)=0$ such that

$$\langle Bx-By,j(x-y)\rangle\geq\boldsymbol{\xi}\ \|x-y\|\quad\forall x,y\in B\qquad\text{... (2)}$$

Definition(2.4), [7]:

Let H is a normed space , D is a nonempty subset of H and $B:D\rightarrow D$ is a mapping, for $x_0\in D$. If the sequence $\{x_n\}$ define by

$$x_{n+1}=By_n$$
$$y_n=(1-\tau_n)\,z_n+\tau_nBz_n$$
$$z_n=(1-\omega_n)x_n+\omega_nBx_n,\forall n\geq 0\qquad\text{... (3)}$$

Then $\{x_n\}$ is said to be $\boldsymbol{S}$ – iteration process of B with two sequences $\{\tau_n\}$ and $\{\omega_n\}\subset[0,1[$.

Lemma(2.5), [8]and[9]

Let H is a normed space and $J:H\rightarrow 2^{H^*}$ the duality mapping .Then

$(i)\|x+y\|^2\leq\|x\|^2+2\langle y,j(x+y)\rangle\ \forall x,y\in H\ and\ \forall j(x+y)\in J(x+y)$.

$(ii)\langle y,j(x)\rangle\leq\|y\|\|x\|\quad\forall y\in H\ and\ \forall j(x)\in J(H)$

Definition(2.6), [4]:

Let H is real Banach space , D is a nonempty subset of H and $B:D\rightarrow D$ is a mapping, for $x_0\in D$ and $\{x_n\}\subset B$ defined by $x_{n+1}=f(B,x_n)$ ... (4)

Suppose that $F(B)=\{x\in H:Bx=x\}\neq\emptyset$ and $\{x_n\}\rightarrow\rho\in F(B)$. Let $\{y_n\}$ is a sequence in D and $\{\eta_n\}$ is a sequence in $[0,\infty[$ defined by $\eta_n=\|y_{n+1}-f(B,x_n)\|$, $\forall n\geq 0$. If $\eta_n\rightarrow 0$ implies that $\{y_n\}\rightarrow\rho$, then the sequence $\{x_n\}\subset B$ defined by (4) is said to be $B-$stable .

Lemma(2.7), [10]

Let $\{\xi_n\}$ is a sequence in $[1,\infty[$ and $\{\varpi_n\}$ is a sequence in $[0,1]$ and $\sum_{n=1}^{\infty}\varpi_n=\infty$. If $\exists$ strictly increasing $\xi:[0,\infty[\rightarrow[0,\infty[$ such that

$\varsigma_{n+1}^2\leq\varsigma_n^2-\varpi_n\varphi(\varsigma_{n+1})+\varepsilon_n,\forall n\geq n_0$, where $n_0\in N$ and $\varepsilon_n=O(\varpi_n)$, then

$\varsigma_n\rightarrow 0\ as\quad n\rightarrow\infty$.

3. Main Threorem

In this section , by using picard $-S$ Iteration process ,the stability and the strong convergence to unique fixed point of $\xi-$ uniformly accretive mapping in Banach space are presented .

Theorem(3.1):

Let D is a nonempty convex and bounded subset of a real Banach space H and $B: D \to D$ is lipschitzian and $\xi-$ uniformly accretive mapping . Let $\{x_n\}$ define as in(3) with the sequences $\{\tau_n\}$ and $\{\omega_n\} \in (0,1)$ satisfying the following

(i) $\lim_{n\to\infty} \tau_n = 0$ (ii) $\lim_{n\to\infty} \omega_n = 0$.

$(iii) \sum_{n=1}^{\infty} \tau_n = \infty, \forall n \in N$. If $F(B) \neq \emptyset$, then $\{x_n\}$ converges strongly to the unique fixed point of B.

Proof:

From conditions (3)and(1), then

$$\|x_{n+1}-\rho\|^2 = \|By_n - B\rho\|^2$$
$$\le L^2\|y_n-\rho\|^2$$
$$\le L^2\|(1-\tau_n)\, z_n + \tau_n Bz_n - \rho\|^2$$
$$= L^2\|(1-\tau_n)(1-\omega_n)x_n + (1-\tau_n)\,\omega_n Bx_n + \tau_n Bz_n - \rho\|^2$$
$$\le L^2(1-\tau_n)^2(1-\omega_n)^2\|x_n\|^2 + L^2\|(1-\tau_n)\,\omega_n Bx_n + \tau_n Bz_n - \rho\|^2$$
$$\le L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1 + L^2\|(1-\tau_n)Bx_n + \tau_n Bz_n - \rho\|^2$$

Where $Q_1 = sup\,\|x_n\|$,$\forall\, n \ge 0$

From codition (3), lemma $\{(2.5), (i)\}$, conditions (1)and(2) and lemma $\{(2.5), (ii)\}$, then

$$\|x_{n+1}-\rho\|^2$$
$$= L^2\|(1-\tau_n)Bx_n + \tau_n Bz_n - \rho\|^2 + L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1$$
$$= L^2\|(Bx_n-\rho) - \tau_n(Bx_n - Bz_n)\|^2 + L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1$$
$$\le L^2\|(1-\tau_n)(Bx_n-\rho) - \tau_n(Bx_n - Bz_n)\|^2 + L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1$$
$$\le (1-\tau_n)^2\|Bx_n-\rho\|^2 - 2\tau_n\,\langle Bx_n - Bz_n\,, j(x_{n+1}-\rho)\rangle + L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1$$
$$\le (1-\tau_n)^2\, L^2\|Bx_n-\rho\|^2 - 2\tau_n L^2\,\langle Bx_{n+1}-\rho\,, j(x_{n+1}-\rho)\rangle + 2\tau_n L^2\,\langle Bz_n-\rho\,, j(x_{n+1}-\rho)\rangle + 2\tau_n L^2\,\langle Bx_{n+1}-Bx_n\,, j(x_{n+1}-\rho)\rangle + L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1$$

$$\le (1-\tau_n)^2\, L^4\|x_n-\rho\|^2 - 2\tau_n\, L^2\,\xi\,\|x_{n+1}-\rho\| + 2\tau_n\, L^3\|z_n-\rho\|\|x_{n+1}-\rho\| + 2\tau_n\, L^3\,\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| + L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1$$
$$= (1-\tau_n)^2\, L^4\|x_n-\rho\|^2 - 2\tau_n L^2\;\xi\,\|x_{n+1}-\rho\|$$

$+\ \tau_n\,L^3\{\|z_n-\rho\|^2+\|x_{n+1}-\rho\|^2\}+\ 2\tau_n\,L^3\ \|x_{n+1}-x_n\|\|x_{n+1}-\rho\|$

$$(1-\tau_n\,L^3)\|x_{n+1}-\rho\|^2\leq(1-\tau_n)^2\,L^4\|x_n-\rho\|^2-2\tau_nL^2\ \ \xi\ \|x_{n+1}-\rho\|$$
$$+\ \tau_n\,L^3\|z_n-\rho\|^2+\ \tau_n\,L^3\ \|x_{n+1}-x_n\|\ Q_2$$
$$+L^2(1-\tau_n)^2(1-\ \omega_n)^2Q_1$$

Where $Q_2=sup\,\|x_{n+1}-\rho\|\ \ ,\forall\,n\geq 0$

Since D is bounded set in H and $\|z_n\|,\|Bz_n\|,\|\,z_{n-1}\|$ and$\|Bz_{n-1}\|$in D , then $\{\|z_n\|,\|Bz_n\|\,,\|\,z_{n-1}\|and\|Bz_{n-1}\|\,\}$ are bounded sequences . It follows from condition(3), and $(\tau_n and\ \omega_n\to 0\ as\ \ n\to\infty\ )$. *We get*

$$\|x_{n+1}-x_n\|=\|By_n-By_{n-1}\|$$
$$=L\|y_n-y_{n-1}\|$$
$$=L\|y_n\|+L\|y_{n-1}\|$$
$$=L\|(1-\tau_n)\,z_n+\tau_nBz_n\|+L\|(1-\tau_{n-1})\,z_{n-1}+\tau_{n-1}Bz_{n-1}\|$$
$$=L(1-\tau_n)\,\|\,z_n\|+\tau_n\|Bz_n\|+L(1-\tau_{n-1})\|\,z_{n-1}\|+\tau_{n-1}\|Bz_{n-1}\|$$
$$=L(1-\tau_n)\,\|\,z_n\|+\tau_n\|Bz_n\|+L(1-\tau_{n-1})\|\,z_{n-1}\|+\ \tau_{n-1}\|Bz_{n-1}\|$$
$$\to 0\ \ as\ \ n\to\infty\,.$$

From conditions (3) and (1), we get

$$\|z_n-\rho\|=\|\ \ (1-\omega_n\,)(x_n-\rho)+\omega_n(Bx_n-\rho)\|$$
$$\leq(1-\omega_n\,)\|x_n-\rho\,\|+\omega_n\|Bx_n-\rho\|$$
$$\leq(1-\omega_n\,)\|x_n-\rho\,\|+\omega_nL\,\|x_n-\rho\|$$
$$=\{(1-\omega_n\,)+\omega_nL\}\,\|x_n-\rho\|$$
$$\|z_n-\rho\|^2=\{(1-\omega_n\,)^2+2\,\omega_n\,L(1-\omega_n\,)+\omega_n^2\,L^2\}\,\|x_n-\rho\|^2$$
$$\tau_nL^3\|z_n-\rho\|^2+(1-\tau_n)^2\,L^4\|x_n-\rho\|^2=K_n\,\|x_n-\rho\|^2$$

Where ,

$$K_n=\ L^4-\tau_nL^3-2\,\tau_n\,\omega_n\,L^3+\tau_n\omega_n^2L^3\qquad+2\,\tau_n\,\omega_n\ \,L^4-2\tau_n\omega_n^2\,L^4+\tau_n\omega_n^2L^5+\ \tau_n^2\,L^4$$

$$\|x_{n+1}-\rho\|^2$$
$$\leq\frac{K_n}{V_n}\|x_n-\rho\|^2-\frac{2\tau_n\,L^2\,\xi\,\|x_{n+1}-\rho\|}{V_n}$$
$$+\frac{\tau_nL^3}{V_n}\ \|x_{n+1}-x_n\|\ Q_2$$
$$+\frac{L^2(1-\tau_n)^2(1-\ \omega_n)^2Q_1}{V_n}\qquad\dots.(5)$$

$$V_n=(1-\tau_nL^3)$$

Since $\tau_n\ \to 0\ \ as\qquad n\to\infty\,,\ \ \exists\,n_0\,\in N$ such that $V_n\leq 1,\forall\,n\geq n_0$. therefor , it follows from (5) that

$$\|x_{n+1}-\rho\|^2=\|x_n-\rho\|^2-2\tau_n\ \ L^2\xi\,\|x_{n+1}-\rho\|$$
$$+\,\tau_n\,\{L^3\,\|x_{n+1}-x_n\|\,Q_2+\ W_n\ \ Q_3\}+L^2(1-\tau_n)^2(1-\omega_n)^2Q_1$$

$,Q_3=sup\|x_n-\rho\|^2$ Where

$W_n = \frac{\tau_n L^4}{{\tau_n}^2} - 2\,\omega_n L^3 + \omega_n^2 L^3 + 2\,\omega_n\, L^4 - 2\omega_n^2 L^4 + \omega_n^2 L^5 + \tau_n L^4, \forall\, n \geq n_0$

Since $\tau_n \to 0$ and $\omega_n \to 0$ as $n \to \infty$, we get $W_n \to 0$ as $n \to \infty$ and $L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1 \to 0$ as $n \to \infty$.

Let us denote $\varsigma_n = \|x_n - \rho\|^2$

$\varpi_n = 2\,\tau_n$

And using lemma (2.7), we get $\varsigma_n \to 0\ as\ n \to \infty$.

Assume that $\rho_1\ and\ \rho_2 \in F(B)$. Since B is $\xi-$uniformly accretive mapping, there exists

$j(\rho_1 - \rho_2) \in J(\rho_1 - \rho_2)$ such that

$\|\rho_1 - \rho_2\|^2 = \langle (B\rho_1 - B\rho_2), j(\rho_1 - \rho_2)\rangle \geq \xi\|\rho_1 - \rho_2\|$.

Wet get $\|\rho_1 - \rho_2\|^2 \geq \xi\|\rho_1 - \rho_2\|$. This implies that $\rho_1 = \rho_2$.

Now , we prove the main result

Theorem(2) : Assume that H, D and B as in the Theorem (1) and $F(B) \neq \emptyset$ .Let $\{x_n\}$ is a sequence define by , for $x_0 \in D$,

$$x_{n+1} = Bk_n$$
$$k_n = (1-\tau_n)\,z_n + \tau_n B z_n$$
$$z_n = (1-\omega_n)x_n + \omega_n Bx_n, \forall n \geq 0 \qquad \dots(6)$$

with the sequences $\{\tau_n\}$ and $\{\omega_n\} \in (0,1)$ satisfying the following

(i) $lim_{n\to\infty}\,\tau_n = 0$ (ii) $lim_{n\to\infty}\,\omega_n = 0$ $, \forall n \in N$.

Let $\{y_n\}$ is a sequence in D Define $\{\eta_n\} \subset [0,\infty[$ by

$$\vartheta_n = (1-\omega_n)y_n + \omega_n B y_n$$
$$\eta_n = \|y_{n+1} - (1-\tau_n)\vartheta_n - \tau_n B\vartheta_n\|^2, \ \forall n \geq 0 \qquad \dots(7)$$

Then

(1) $\lim_{n\to\infty}\|x_n - \rho\| = 0, \rho \in F(B)$.

(2)$\|y_{n+1} - \rho\|^2 \leq \|y_n - \rho\|^2 - 2\tau_n L^2\,\xi\,\|y_{n+1} - \rho\| + \eta_n$
$+ \tau_n\{L\,\|y_{n+1} - y_n\|\,Q_2 + W_n\,\|y_n - \rho\|^2\} + L^2(1-\tau_n)^2(1-\omega_n)^2 Q_1$

Where

$W_n = \frac{\tau_n L^4}{{\tau_n}^2} - 2\,\omega_n L^3 + \omega_n^2 L^3 + 2\,\omega_n\, L^4 - 2\omega_n^2 L^4 + \omega_n^2 L^5 + \tau_n L^4, \forall\, n \geq n_0$

(3) $\lim_{n\to\infty} y_n = \rho \leftrightarrow \lim_{n\to\infty}\eta_n = 0$.

Proof *:*

From Theorem (1), we get $lim_{n\to\infty}\|x_n - \rho\| = 0, \rho \in F(B)$. Then the proof of (1) is completed

By use condition (7),then

$$\|y_{n+1} - \rho\|^2 \leq \eta_n + \|(1-\tau_n)\vartheta_n + \tau_n B\vartheta_n - \rho\|^2 \qquad \dots(8)$$

Let $\gamma_n = (1-\tau_n)\vartheta_n + \tau_n B\vartheta_n$, then $(1-\tau_n)\vartheta_n = \gamma_n - \tau_n B\vartheta_n$

As the proof in Theorem (1) ,then

$\|\gamma_n - \rho\|^2 \leq \|y_n - \rho\|^2 - 2\tau_n L^2\,\xi\,\|y_{n+1} - \rho\| + \eta_n$

$$+\tau_n\{L^3\|y_{n+1}-y_n\|Q_2 + W_n\|y_n-\rho\|^2\} + L^2(1-\tau_n)^2(1-\omega_n)^2Q_1 \quad \dots(9)$$

Where $Q_1 = sup\|y_n\|$ and $Q_2 = sup\|y_{n+1}-y_n\|, \forall\, n\geq 0$

$W_n = \frac{\tau_n L^4}{\tau_n{}^2} - 2\,\omega_n L^3 + \omega_n^2 L^3 \quad +2\,\omega_n L^4 - 2\omega_n^2 L^4 + \omega_n^2 L^5 + \tau_n L^4, \forall\, n\geq n_0$

Hence $\|y_{n+1}-\rho\|^2 = \|y_n-\rho\|^2 - 2\tau_n L^2\xi\|y_{n+1}-\rho\| + \eta_n$
$+\tau_n\{L^3\|y_{n+1}-y_n\|Q_2 + W_n\, Q_3\} + L^2(1-\tau_n)^2(1-\omega_n)^2Q_1$

Where $Q_3 = sup\|y_n-\rho\|^2, \forall\, n\geq 0$. Then the proof of (2) is completed

Now suppose that $lim_{n\to\infty} y_n = \rho$. Then

$$\eta_n = \|y_{n+1}-(1-\tau_n)\vartheta_n - \tau_n B\vartheta_n\|^2$$
$$\leq \|y_{n+1}-\rho\|^2 + \|(1-\tau_n)\vartheta_n - \tau_n B\vartheta_n - \rho\|^2$$
$$\leq \|y_{n+1}-\rho\|^2 + \|y_n-\rho\|^2 - 2\tau_n L^2\,\xi\|y_{n+1}-\rho\|$$
$$+\tau_n\{L^3\|y_{n+1}-y_n\|Q_2 + W_n\, Q_3\}$$

It is clear that $\eta_n \to 0$ as $n\to\infty$.

Next , suppose that $lim_{n\to\infty}\eta_n = 0$. From (8)and(9), then

$\|y_{n+1}-\rho\|^2 = \|y_n-\rho\|^2 - 2\tau_n L^2\xi\|y_{n+1}-\rho\| + \eta_n$
$+\tau_n\{L^3\|y_{n+1}-y_n\|Q_2 + W_n\, Q_3\} + L^2(1-\tau_n)^2(1-\omega_n)^2Q_1$

Which mean that $y_n \to 0$ as $n\to\infty$ according to lemma (2.7). Then the proof of (2) is completed .

4. conclusions

the stability of picard $-\boldsymbol{S}$ Iteration for a fixed point of $\boldsymbol{\xi}-$ uniformly accretive mapping has been established in Banach space . Also picard $-\boldsymbol{S}$ Iteration converge strongly to unique fixed point of $\boldsymbol{\xi}-$ uniformly accretive mapping.

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الاستقرار للعمليات التكرارية من نمط بيكارد – أس للتطبيق المتسق الموحد.

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المستخلص:

في هذا البحث ندرس الاستقرار والتقارب للنقطة الصامدة الوحيدة للتطبيق الانكماشي المتسق بشكل موحد باستخدام المتتابعة التكرارية من نمط بيكارد – أس في فضاء بناخ .