The Numerical Solution for Quadratic Optimal Control Problems by Using Chebyshev and Legendre Polynomials

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Abstract

The purpose of this paper is to solve quadratic optimal control problems (QOCP) numerically with the assist of once Chebyshev and Legendre polynomials as basic functions to find the solution for optimal control (QOC) approximately. We will explain the algorithms of solution by examples and use the Mathcad's Program to reach the exact result.

Introduction

The optimal control problem is to find a control $u^*(t)$ which minimizes a given performance index while satisfying the system state equations and constraints. **[١]**

We use the approximation methods to solve the optimal control problem depending on the Chebyshev polynomials in the first time and Legendre polynomials, after that we will approximate these solutions of continuous time linear. To reach the approximate solutions we use the linear multi- term differential equations of $u(t)$ and $x(t)$ for both Chebyshev and Legendre polynomials and make the terms of these equations as square matrix to find these values by matrices system.

When we use these polynomials in approximate solutions, the results were evaluated by using index with $n = ٥$.

We will explain these algorithms by taking some examples for the quadratic control problems.

The linear quadratic problem is stated as follows;

Minimize the quadratic continuous time

Cost function $J = \int_{t_٠}^{t_f} (x^T Q x + u^T R u)\, dt$ …(١)

Subject to the linear system state equations;

$\dot{x}(t) = Dx(t) + Eu(t),$ …(٢)

where the initial condition $x(٠) = x_٠$ and the matrices $(D, E, Q$ and $R)$ are constants. **[٣]**

Chebyshev Polynomials of the First Kind of Degree n [٢]

The Chebyshev polynomials $T_n(x)$ can be obtained by means of Rodrigue's formula

$$T_n(x) = \frac{(-٢)^n n!}{(٢n)!}\sqrt{١ - x^٢}\,\frac{d^n}{dx^n}\left(١ - x^٢\right)^{n-١/٢} \qquad n = ٠, ١, ٢, \cdots$$

We find that
$$T_٠(x) = ١$$
$$T_١(x) = x$$
$$T_٢(x) = ٢x^٢ - ١$$
$$T_٣(x) = ٤x^٣ - ٣x$$
$$T_٤(x) = ٨x^٤ - ٨x^٢ + ١$$
$$\vdots$$

In this paper we use recurrence formula for $T_n(x)$. When the first two Chebyshev polynomials $T_٠(x)$, $T_١(x)$ are known, all other polynomials $T_n(x)$, $n \geq ٢$ can be obtained by means of the recurrence formula

$$T_{n+١}(x) = ٢xT_n(x) - T_{n-١}(x) \qquad \ldots(٣)$$

Therefore, the polynomials will be on the forms:

$$T_٢(x) = ٢xT_١(x) - T_٠(x)$$
$$T_٣(x) = ٢xT_٢(x) - T_١(x)$$
$$T_٤(x) = ٢xT_٣(x) - T_٢(x)$$
$$\vdots$$

Algorithm of Solution

To begin in solution we have to approximate both states variables x(t) and control variables u(t) by using Chebyshev polynomials as follows:

$$x(t) \approx \sum_{i=٠}^{n} a_i T_i(t) \qquad \ldots(٤)$$
$$u(t) \approx \sum_{i=٠}^{n} b_i T_i(t) \qquad \ldots(٥)$$

Where $٠ \leq t \leq ١$, and a_i, b_i are unknown parameters.

Then by expanding x(t) and u(t) into five order ($n = ٥$) in eqs. (٤) and (٥), we get:

$$x(t) = a_٠T_٠(t) + a_١T_١(t) + a_٢T_٢(t) + a_٣T_٣(t) + a_٤T_٤(t) + a_٥T_٥(t) \quad \ldots(٦)$$
$$u(t) = b_٠T_٠(t) + b_١T_١(t) + b_٢T_٢(t) + b_٣T_٣(t) + b_٤T_٤(t) + b_٥T_٥(t) \quad \ldots(٧)$$

Where $T_i(t)$ are Chebyshev polynomials, which can be found by (٣).

We can evaluate $a_i, b_i (i = ٠, \cdots, ٥)$ as follows:

We claim that $t = ٠, t = ٠.٥$ and $t = ١$ in eqs. (٦) and (٧) to find four equations $x(٠)$, $x(٠.٥), x(١), u(٠), u(٠.٥)$ and $u(١)$. Differentiating (٦) with respect to t, and put $t = ٠$ and substituting the result in (٢), seven equations have been obtained with twelve variables.

We need also to find variables, $\dot{x}(٠), \ddot{x}(٠), x^{(٣)}(٠), x^{(٤)}(٠), x^{(٥)}(٠)$.

After that, we use Gauss elimination procedure for solving the above system to find $a_i, b_i (i = ٠, \cdots, ٥)$.
Finally, substitute a_i, b_i in (٦) and (٧) and put them in (١) to find the approximating solution.

Example ١

Minimize $J = \int_{٠}^{١} \left(x^{٢} + u^{٢}\right) dt$ …(٨)

Subject to $\dot{x} = u, \; x(٠) = ١$ …(٩)

The exact value for x and u are $x(t) = \frac{\cosh(١-t)}{\cosh ١}, \quad u(t) = \frac{-\sinh(١-t)}{\cosh ١}$

The optimal value of the performance in this problem is $J =$ ٠.٧٦١٥٩٤١٥٦.[٤]

Solution

Now we use the previous algorithm to solve this function approximately with Chebyshev polynomials into $(n = ٥)$

$x(t) \approx \sum_{i=٠}^{٥} a_i T_i(t)$ …(١٠)

$u(t) \approx \sum_{i=٠}^{٥} b_i T_i(t)$ …(١١)

Where $٠ \le t \le ١$ and $T_i(t)$ are Chebyshev polynomials

Then (١٠) and (١١) will be

$$x(t) = a_{٠}T_{٠}(t) + a_{١}T_{١}(t) + a_{٢}T_{٢}(t) + a_{٣}T_{٣}(t) + a_{٤}T_{٤}(t) + a_{٥}T_{٥}(t)$$
$$= (a_{٠} - a_{٢} + a_{٤}) + (a_{١} - ٣a_{٣} + ٥a_{٥})t + (٢a_{٢} - ٨a_{٤})t^{٢} + (٤a_{٣} - ٢٠a_{٥})t^{٣} + ٨a_{٤}t^{٤} + ١٦a_{٥}t^{٥} \quad \ldots(١٢)$$

$$u(t) = b_{٠}T_{٠}(t) + b_{١}T_{١}(t) + b_{٢}T_{٢}(t) + b_{٣}T_{٣}(t) + b_{٤}T_{٤}(t) + b_{٥}T_{٥}(t)$$
$$= (b_{٠} - b_{٢} + b_{٤}) + (b_{١} - ٣b_{٣} + ٥b_{٥})t + (٢b_{٢} - ٨b_{٤})t^{٢} + (٤b_{٣} - ٢٠b_{٥})t^{٣} + ٨b_{٤}t^{٤} + ١٦b_{٥}t^{٥} \quad \ldots(١٣)$$

Now, to evaluate the control points $a_i, b_i (i = ٠, \cdots, ٥)$, we have to find the following values from (١٢) and (١٣):

$$x(٠) = a_{٠} - a_{٢} + a_{٤}$$
$$x(٠.٥) = a_{٠} + ٠.٥a_{١} - ٠.٥a_{٢} - a_{٣} - ٠.٥a_{٤} + ٠.٥a_{٥}$$
$$x(١) = a_{٠} + a_{١} + a_{٢} + a_{٣} + a_{٤} + a_{٥}$$
$$u(٠) = b_{٠} - b_{٢} + b_{٤}$$
…(١٤)
$$u(٠.٥) = b_{٠} + ٠.٥b_{١} - ٠.٥b_{٢} - b_{٣} - ٠.٥b_{٤} + ٠.٥b_{٥}$$
$$u(١) = b_{٠} + b_{١} + b_{٢} + b_{٣} + b_{٤} + b_{٥}$$

Then we find $\dot{x}(٠)$:

$$\dot{x}(٠) = a_{١} - ٣a_{٣} + a_{٥} \quad \ldots(١٥)$$

After that, we substitute eqs. (١٥) and (١٣) into (٩):

$$\dot{x}(٠) = u(٠) = -a_{١} + ٣a_{٣} - ٥a_{٥} + b_{٠} - b_{٢} + b_{٤} = ٠$$
$$\ddot{x}(٠) = \dot{u}(٠) = -٤a_{٢} + ١٦a_{٤} + b_{١} - ٣b_{٣} + ٥b_{٥} = ٠$$

$x^{(٣)}(٠) = \grave{u}(٠) = -٢٤a_٣ + ١٢٠a_٥ + ٤b_٢ - ١٦b_٤ = ٠$ …(١٦)

$x^{(٤)}(٠) = u^{(٣)}(٠) = -١٩٢a_٤ + ٢٤b_٣ - ١٢٠b_٥ = ٠$

$x^{(٥)}(٠) = u^{(٤)}(٠) = -١٩٢٠a_٥ + ١٩٢b_٤ = ٠$

$x^{(٦)}(٠) = u^{(٥)}(٠) = -١٩٢٠b_٥ = ٠$

Transform eqs. (١٤-١٦) to the matrix form as follows:

$EG = H$ …(١٧)

Where,

$$E = \begin{bmatrix} ١ & ٠ & -١ & ٠ & ١ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ \\ ١ & ٠.٥ & -٠.٥ & -١ & -٠.٥ & ٠.٥ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ \\ ١ & ١ & ١ & ١ & ١ & ١ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ١ & ٠ & -١ & ٠ & ١ & ٠ \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ١ & ٠.٥ & -٠.٥ & -١ & -٠.٥ & ٠.٥ \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ١ & ١ & ١ & ١ & ١ & ١ \\ ٠ & -١ & ٠ & ٣ & ٠ & -٥ & ١ & ٠ & -١ & ٠ & ١ & ٠ \\ ٠ & ٠ & -٤ & ٠ & ١٦ & ٠ & ٠ & ١ & ٠ & -٣ & ٠ & ٥ \\ ٠ & ٠ & ٠ & -٢٤ & ٠ & ١٢٠ & ٠ & ٠ & ٤ & ٠ & -١٦ & ٠ \\ ٠ & ٠ & ٠ & ٠ & -١٩٢ & ٠ & ٠ & ٠ & ٠ & ٢٤ & ٠ & -١٢٠ \\ ٠ & ٠ & ٠ & ٠ & ٠ & -١٩٢٠ & ٠ & ٠ & ٠ & ٠ & ١٩٢ & ٠ \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ١٩٢٠ \end{bmatrix},$$

$$G = \begin{bmatrix} a_٠ \\ a_١ \\ a_٢ \\ a_٣ \\ a_٤ \\ a_٥ \\ b_٠ \\ b_١ \\ b_٢ \\ b_٣ \\ b_٤ \\ b_٥ \end{bmatrix}, H = \begin{bmatrix} ١ \\ ٠.٧٣٠٧٦٢٨٢٥٨ \\ ٠.٦٤٨٠٥٤٢٧ \\ -٠.٧٦١٥٩٤١٥ \\ -٠.٣٣٧٦٩٨٠٣٩٧ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \end{bmatrix}$$

Then we use the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = ٠, \cdots, ٥)$ over and above J^*

$a_٠ = ١.٢٦٤$ $b_٠ = -٠.٩٥٥$

$a_١ = -٠.٨٥٧$ $b_١ = ١.١١٤$

$a_٢ = ٠.٢٦٩$ $b_٢ = -٠.١٩٥$

$a_٣ = -٠.٠٣٢$ $b_٣ = ٠.٠٣٨$

$a_٤ = ٤.٧٦ \times ١٠^{-٣}$ $b_٤ = -١.٧٨ \times ١٠^{-٣}$

$a_{٠} = -١.٧٨ \times ١٠^{-٤}$ $b_{٠} = ٠$

And $J^{*} = ٠.٧٦١٤٠٨٦٣٨٤٥٧$

Example ٢

Minimize $J = \frac{١}{٢}\int_{٠}^{١}\left(٢x^{٢} + u^{٢}\right)dt$ …(١٨)

Subject to $\dot{x} = \frac{x}{٢} + u,\ x(٠) = ١$ …(١٩)

The exact value for x and u are $x(t) = \frac{٢e^{٣t}+e^{٣}}{e^{٣t/٢}(٢+e^{٣})},\ u(t) = \frac{٢(e^{٣t}-e^{٣})}{e^{٣t/٢}(٢+e^{٣})}$…(٢٠)

The optimal value of the performance in this problem is $J = ٠.٨٦٤١٦٤٤٩٧$. **[٤]**

Solution

We will solve this example in the same method of the previous example that depends on Chebyshev polynomials with $(n = ٥)$.

The change will be in eqs. (١٦) because we will substitute eqs. (١٥) and (١٣) into (١٩):

$$\left.\begin{aligned}
&\dot{x}(٠) = \frac{x(٠)}{٢} + u(٠) = a_{٠} - ٢a_{١} - a_{٢} + ٦a_{٣} + a_{٤} - ١٠a_{٥} + ٢b_{٠} - ٢b_{٢} + ٢b_{٤} = ٠\\
&\ddot{x}(٠) = \frac{\dot{x}(٠)}{٢} + \dot{u}(٠) = a_{١} - ٨a_{٢} - ٣a_{٣} + ٣٢a_{٤} + ٥a_{٥} + ٢b_{١} - ٦b_{٣} + ١٠b_{٥} = ٠\\
&x^{(٣)}(٠) = \frac{\ddot{x}(٠)}{٢} + \ddot{u}(٠) = ٤a_{٢} - ٤٨a_{٣} - ١٦a_{٤} + ٢٤٠a_{٥} + ٨b_{٢} - ٣٢b_{٤} = ٠\\
&x^{(٤)}(٠) = \frac{x^{(٣)}(٠)}{٢} + u^{(٣)}(٠) = ٢٤a_{٣} - ٣٨٤a_{٤} - ١٢٠a_{٥} + ٤٨b_{٣} - ٢٤٠b_{٥} = ٠\\
&x^{(٥)}(٠) = \frac{x^{(٤)}(٠)}{٢} + u^{(٤)}(٠) = ١٩٢a_{٤} - ٣٨٤٠a_{٥} + ٣٨٤b_{٤} = ٠\\
&x^{(٦)}(٠) = \frac{x^{(٥)}(٠)}{٢} + u^{(٥)}(٠) = ١٩٢٠a_{٥} + ٣٨٤٠b_{٥} = ٠
\end{aligned}\right\} \quad …(٢١)$$

Therefore the terms of the matrices E, H will change according to eqs. (٢١) and eq. (٢٠) respectively.

Then we use also the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = ٠, \cdots, ٥)$ over and above J^*

$a_{٠} = ١.٦١٩$ $b_{٠} = -٢.٧٠٥$

$a_{١} = -١.٥٥٧$ $b_{١} = ٣.٥$

$a_{٢} = ٠.٦٣٩$ $b_{٢} = -٠.٩٩٦$

$a_{٣} = -٠.١١١$ $b_{٣} = ٠.٢٢$

$a_{٤} = ٠.٠٢١$ $b_{٤} = -٠.٠١٩$

$a_{٥} = -٨.٧٥ \times ١٠^{-٤}$ $b_{٥} = ٤.٣٧٥ \times ١٠^{-٤}$

And $J^* = ٠.٨٦٤٨٥٣٦١١٦٧٩$

Legendre Polynomials of the Degree n

The Legendre polynomials are the everywhere regular solutions of Legendre's equation, **[٥]**

$$(١ - x^٢)\acute{\acute{u}} - ٢x\acute{u} + mu = [(١ - x^٢)\acute{u}]' + mu = ٠ \qquad \ldots(٢٢)$$

which are possible only if

$$m = n(n + ١), \qquad n = ٠, ١, ٢, \cdots$$

The solution for a particular value of n is $P_n(x)$. It is a polynomial of degree n. If n is even/ odd then the polynomial is even/ odd. They are normalized such that $P_n(x) = ١$.

$$P_٠(x) = ١$$
$$P_١(x) = x$$
$$P_٢(x) = (٣x^٢ - ١)/٢$$
$$P_٣(x) = (٥x^٣ - ٣x)/٢$$
$$P_٤(x) = (٣٥x^٤ - ٣٠x^٢ + ٣)/٨$$
$$\vdots$$

In this paper, we use the recurrence formula

$$P_{n+١}(x) = \frac{٢n+١}{n+١} xP_n(x) - \frac{n}{n+١} P_{n-١}(x) \qquad \ldots(٢٣)$$

Therefore, the polynomials will be on the form

$$P_٢(x) = \frac{٣}{٢} xP_١(x) - \frac{١}{٢} P_٠(x)$$
$$T_٣(x) = \frac{٥}{٣} xP_٢(x) - \frac{٢}{٣} P_١(x)$$
$$T_٤(x) = \frac{٧}{٤} xP_٣(x) - \frac{٣}{٤} P_٢(x)$$
$$\vdots$$ **[٦]**

Algorithm of Solution

This algorithm is similar to the previous algorithm that depends on Chebyshev polynomials except that we put $P_n(x)$ instead of $T_n(x)$ as follows:

$x(t) \approx \sum_{i=٠}^{n} a_i P_i(t) \quad \ldots(٢٤)$

$u(t) \approx \sum_{i=٠}^{n} b_i P_i(t) \quad \ldots(٢٥)$

Where $٠ \leq t \leq ١$, and a_i, b_i are unknown parameters and $P_n(x)$ are Legendre polynomials.

Then by expanding $x(t)$ and $u(t)$ into five order $(n = ٥)$ in eqs. (٢٤) and (٢٥), we get:

$x(t) = a_٠P_٠(t) + a_١P_١(t) + a_٢P_٢(t) + a_٣P_٣(t) + a_٤P_٤(t) + a_٥P_٥(t)$ …(٢٦)

$u(t) = b_٠P_٠(t) + b_١P_١(t) + b_٢P_٢(t) + b_٣P_٣(t) + b_٤P_٤(t) + b_٥P_٥(t)$ …(٢٧)

Where $P_i(t)$ are Legendre polynomials, which can be found by (٢٣).

We can evaluate $a_i, b_i (i = ٠, \cdots, ٥)$ as follows:
We also claim that $t = ٠, t = ٠.٥$ and $t = ١$ in eqs. (٢٦) and (٢٧) to find four equations $x(٠)$, $x(٠.٥), x(١), u(٠), u(٠.٥)$ and $u(١)$. Differentiating (٢٦) with respect to t, and put $t = ٠$ and substituting the result in (٢), seven equations have been obtained with twelve variables.
We need also to find the variables, $\dot{x}(٠), \ddot{x}(٠), x^{(٣)}(٠), x^{(٤)}(٠), x^{(٥)}(٠)$.
After that, we use Gauss elimination procedure for solving the above system to find $a_i, b_i (i = ٠, \cdots, ٥)$.
Finally, substitute a_i, b_i in (٢٦) and (٢٧) and put them in (١) to find the approximating solution.
Now, we will repeat the solution of the previous examples by using the Legendre polynomials.

Solution ٠f Example ١

We use the previous algorithm to solve this function approximately with Legendre polynomials into $(n = ٥)$

$$x(t) \approx \sum_{i=٠}^{٥} a_i P_i(t) \qquad \ldots(٢٨)$$

$$u(t) \approx \sum_{i=٠}^{٥} b_i P_i(t) \qquad \ldots(٢٩)$$

Where $٠ \leq t \leq ١$ and $P_i(t)$ are Legendre polynomials
Then (٢٨) and (٢٩) will be

$$\begin{aligned} x(t) &= a_٠P_٠(t) + a_١P_١(t) + a_٢P_٢(t) + a_٣P_٣(t) + a_٤P_٤(t) + a_٥P_٥(t) \\ &= \left(a_٠ - \frac{١}{٢}a_٢ + \frac{٣}{٨}a_٤\right) + \left(a_١ - \frac{٣}{٢}a_٣ + \frac{١٥}{٨}a_٥\right)t + \left(\frac{٣}{٢}a_٢ - \frac{١٥}{٤}a_٤\right)t^٢ + \\ &\quad \left(\frac{٥}{٢}a_٣ - \frac{٣٥}{٤}a_٥\right)t^٣ + \frac{٣٥}{٨}a_٤t^٤ + \frac{٦٣}{٨}a_٥t^٥ \qquad \ldots(٣٠) \end{aligned}$$

$$\begin{aligned} u(t) &= b_٠P_٠(t) + b_١P_١(t) + b_٢P_٢(t) + b_٣P_٣(t) + b_٤P_٤(t) + b_٥P_٥(t) \\ &= \left(b_٠ - \frac{١}{٢}b_٢ + \frac{٣}{٨}b_٤\right) + \left(b_١ - \frac{٣}{٢}b_٣ + \frac{١٥}{٨}b_٥\right)t + \left(\frac{٣}{٢}b_٢ - \frac{١٥}{٤}b_٤\right)t^٢ + \\ &\quad \left(\frac{٥}{٢}b_٣ - \frac{٣٥}{٤}b_٥\right)t^٣ + \frac{٣٥}{٨}b_٤t^٤ + \frac{٦٣}{٨}b_٥t^٥ \qquad \ldots(٣١) \end{aligned}$$

Now, to evaluate the control points $a_i, b_i (i = ٠, \cdots, ٥)$, we have to find the following values from (٣٠) and (٣١):

$$\left.\begin{aligned} x(٠) &= a_٠ - \frac{١}{٢}a_٢ + \frac{٣}{٨}a_٤ \\ x(٠.٥) &= a_٠ + \frac{١}{٢}a_١ - \frac{١}{٨}a_٢ - \frac{٧}{١٦}a_٣ - \frac{٣٧}{١٢٨}a_٤ + \frac{٢٣}{٢٥٦}a_٥ \\ x(١) &= a_٠ + a_١ + a_٢ + a_٣ + a_٤ + a_٥ \\ u(٠) &= b_٠ - \frac{١}{٢}b_٢ + \frac{٣}{٨}b_٤ \\ u(٠.٥) &= b_٠ + \frac{١}{٢}b_١ - \frac{١}{٨}b_٢ - \frac{٧}{١٦}b_٣ - \frac{٣٧}{١٢٨}b_٤ + \frac{٢٣}{٢٥٦}b_٥ \\ u(١) &= b_٠ + b_١ + b_٢ + b_٣ + b_٤ + b_٥ \end{aligned}\right\} \ldots(٣٢)$$

Then we find $\dot{x}(٠)$:

$$\dot{x}(٠) = a_{١} - \frac{٣}{٢} a_{٣} + \frac{١٥}{٨} a_{٥} \qquad \ldots(٣٣)$$

After that, we substitute eqs. (١٥) and (١٣) into (٩):

$$\left.\begin{array}{l}
\dot{x}(٠) = u(٠) = -a_{١} + \frac{٣}{٢}a_{٣} - \frac{١٥}{٨}a_{٥} + b_{٠} - \frac{١}{٢}b_{٢} + \frac{٣}{٨}b_{٤} = ٠ \\
\ddot{x}(٠) = \dot{u}(٠) = -٣a_{٢} + \frac{١٥}{٢}a_{٤} + b_{١} - \frac{٣}{٢}b_{٣} + \frac{١٥}{٨}b_{٥} = ٠ \\
x^{(٣)}(٠) = \ddot{u}(٠) = -١٥a_{٣} + \frac{١٠٥}{٢}a_{٥} + ٣b_{٢} - \frac{١٥}{٢}b_{٤} = ٠ \\
x^{(٤)}(٠) = u^{(٣)}(٠) = -١٠٥a_{٤} + ١٥b_{٣} - \frac{١٠٥}{٢}b_{٥} = ٠ \\
x^{(٥)}(٠) = u^{(٤)}(٠) = -٩٤٥a_{٥} + ١٠٥b_{٤} = ٠ \\
x^{(٦)}(٠) = u^{(٥)}(٠) = ٩٤٥b_{٥} = ٠
\end{array}\right\} \quad \ldots(٣٤)$$

Transform eqs. (٣٢-٣٤) to the matrix on the form of eq. (١٧)

Where,

$$E = \begin{bmatrix} ١ & ٠ & -\frac{١}{٢} & ٠ & \frac{٣}{٨} & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ \\ ١ & \frac{١}{٢} & \frac{-١}{٨} & \frac{-٧}{١٦} & -\frac{٣٧}{١٢٨} & \frac{٢٣}{٢٥٦} & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ \\ ١ & ١ & ١ & ١ & ١ & ١ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ١ & ٠ & -\frac{١}{٢} & ٠ & \frac{٣}{٨} & ٠ \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ١ & \frac{١}{٢} & -\frac{١}{٨} & -\frac{٧}{١٦} & -\frac{٣٧}{١٢٨} & \frac{٢٣}{٢٥٦} \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ١ & ١ & ١ & ١ & ١ & ١ \\ ٠ & -١ & ٠ & \frac{٣}{٢} & ٠ & -\frac{١٥}{٨} & ١ & ٠ & -\frac{١}{٢} & ٠ & \frac{٣}{٨} & ٠ \\ ٠ & ٠ & -٣ & ٠ & \frac{١٥}{٢} & ٠ & ٠ & ١ & ٠ & -\frac{٣}{٢} & ٠ & \frac{١٥}{٨} \\ ٠ & ٠ & ٠ & -١٥ & ٠ & \frac{١٠٥}{٢} & ٠ & ٠ & ٣ & ٠ & -\frac{١٥}{٢} & ٠ \\ ٠ & ٠ & ٠ & ٠ & -١٠٥ & ٠ & ٠ & ٠ & ٠ & ١٥ & ٠ & -\frac{١٠٥}{٢} \\ ٠ & ٠ & ٠ & ٠ & ٠ & -٩٤٥ & ٠ & ٠ & ٠ & ٠ & ١٠٥ & ٠ \\ ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٠ & ٩٤٥ \end{bmatrix},$$

$$G = \begin{bmatrix} a_٠ \\ a_١ \\ a_٢ \\ a_٣ \\ a_٤ \\ a_٥ \\ b_٠ \\ b_١ \\ b_٢ \\ b_٣ \\ b_٤ \\ b_٥ \end{bmatrix}, H = \begin{bmatrix} ١ \\ ٠.٧٣٠٧٦٢٨٢٥٨ \\ ٠.٦٤٨٠٥٤٢٧ \\ -٠.٧٦١٥٩٤١٥ \\ -٠.٣٣٧٦٩٨٠٣٩٧ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \\ ٠ \end{bmatrix}$$

Then we use the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = ٠, \cdots, ٥)$ over and above J^*

$a_٠ = ١.١٧٤$ $\qquad b_٠ = -٠.٨٩$

$a_١ = -٠.٨٣٨$ $\qquad b_١ = ١.٠٩١$

$a_٢ = ٠.٣٥٥$ $\qquad b_٢ = -٠.٢٥٩$

$a_٣ = -٠.٠٥١$ $\qquad b_٣ = ٠.٠٦١$

$a_٤ = ٨.٧٠٤ \times ١٠^{-٣}$ $\qquad b_٤ = -٣.٢٥٥ \times ١٠^{-٣}$

$a_٥ = -٣.٦١٦ \times ١٠^{-٤}$ $\qquad b_٥ = ٠$

And $J^* = ٠.٧٦١٣١٧٩٢٣٤٦٧$

Solution of Example٢

We use the same method of the previous algorithm to solve example (٢) approximately with Legendre polynomials into ($n = ٥$).

The change will be in eqs. (٣٤) because we will substitute eqs. (٣٣) and (٣١) into (١٩):

$$\begin{rcases}
\dot{x}(٠) = \frac{x(٠)}{٢} + u(٠) = a_٠ - ٢a_١ - \frac{١}{٢}a_٢ + ٣a_٣ + \frac{٣}{٨}a_٤ - \frac{١٥}{٤}a_٥ + ٢b_٠ - b_٢ + \frac{٣}{٤}b_٤ = ٠ \\
\ddot{x}(٠) = \frac{\dot{x}(٠)}{٢} + \dot{u}(٠) = a_١ - ٦a_٢ - \frac{٣}{٢}a_٣ + ١٥a_٤ + \frac{١٥}{٨}a_٥ + ٢b_١ - ٣b_٣ + \frac{١٥}{٤}b_٥ = ٠ \\
x^{(٣)}(٠) = \frac{\ddot{x}(٠)}{٢} + \ddot{u}(٠) = ٣a_٢ - ٣٠a_٣ - \frac{١٥}{٢}a_٤ + ١٠٥a_٥ + ٦b_٢ - ١٥b_٤ = ٠ \\
x^{(٤)}(٠) = \frac{x^{(٣)}(٠)}{٢} + u^{(٣)}(٠) = ١٥a_٣ - ٢١٠a_٤ - \frac{١٠٥}{٢}a_٥ + ٣٠b_٣ - ١٠٥b_٥ = ٠ \\
x^{(٥)}(٠) = \frac{x^{(٤)}(٠)}{٢} + u^{(٤)}(٠) = ١٠٥a_٤ - ١٨٩٠a_٥ + ٢١٠b_٤ = ٠
\end{rcases} \quad \ldots(٣٥)$$

$$x^{(٦)}(٠) = \frac{x^{(٥)}(٠)}{٢} + u^{(٥)}(٠) = ٩٤٥a_٥ + ١٨٩٠b_٥ = ٠$$

Therefore the terms of the matrices E, H will change according to eqs. (٢١) and eq. (٢٠) respectively.

Then we use also the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = ٠, \cdots, ٥)$ over and above J^*

$a_٠ = ١.٤٠٤$ $\quad$ $b_٠ = -٢.٣٧٢$

$a_١ = -١.٤٩$ $\quad$ $b_١ = ٣.٣٦٨$

$a_٢ = ٠.٨٣٧$ $\quad$ $b_٢ = -١.٣١٣$

$a_٣ = -٠.١٧٧$ $\quad$ $b_٣ = ٠.٣٥١$

$a_٤ = ٠.٠٣٧$ $\quad$ $b_٤ = -٠.٠٣٥$

$a_٥ = -١.٧٧٨ \times ١٠^{-٣}$ $\quad$ $b_٥ = ٨.٨٨٩ \times ١٠^{-٤}$

And $J^* = ٠.٨٦٤٢٨٣٤٨٠٥٧٨$

Conclusion

The Chebyshev and Legendre polynomials are considered in this paper, in order to use them to approximate the variables by using these polynomials.

we use the Chebyshev polynomials in the first solution and after that we use Legendre polynomials, to reach to the approximate solution J^*.

If we compare the results with respect to the exact solution that given, we can use these polynomials to solve quadratic optimal control problem numerically, with accurate results.

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المستخلص

الغرض الاساسي من هذا البحث هو حل مسائل السيطرة المثلى التربيعية عدديا باستخدام متعددة الحدود جيبيشيف وليجيندر واستخدامها كدوال اساسية لايجاد حلول تقريبية لمثل هذه المسائل، مع بيان خوارزميات هذه الحلول على بعض الامثلة وبأستخدام برنامج الماث كاد للوصول الى نتائج مضبوطة.