Block and Weddle Methods For Solving nth Order Linear Delay Fredholm Integro-Differential Equations

Atheer Jawad Kadhim
Department of Applied Sciences
University of Technology

Abstract

A proposed method is presented to solve *nth order linear delay Fredholm integro-differential equations* (DFIDE's) numerically using fourth-order six steps block and weddle methods. New algorithms with the aid of Matlab language are derived to treat numerically three types (retarded, neutral and mixed) of the first order linear delay Fredholm integro-differential equations as well as nth order linear delay Fredholm integro-differential equations using block and Weddle methods. Comparison between the numerical and exact results has been given for numerical examples for solving different types of linear DFIDE's for conciliated the accuracy of the results of the proposed method. Finally, the results are arranged in tabulated form and suitable graphing is given for every example.

1. Introduction

One of the most important and applicable subjects of applied mathematics, and in developing modern mathematics is the integral equations. The names of many modern mathematicians notably Fredholm, Volterra, Cauchy and others are associated with this topic [1].

The name integral equation was introduced by Bois-Reymond in 1888 and in 1959 Volterra's book "Theory of Functional and of Integral and Integro-Differential Equations" appeared [2].

The integral and integro-differential equations formulation of physical problems are more elegant and compact than the differential equation formulation, since the boundary conditions can be satisfied and embedded in the integral or integro-differential equation. Also the form of the solution to an integro-differential equation is often more stable for today's extremely fast machine computation [1, 3].

To facilitate the presentation of the material that followed, a brief review of some background on the linear *delay integro-differential equation* and their types are given in the following section.

2. Delay Integro-Differential Equation (DIDE):

The integro-differential equation is an equation involving one or more unknown function u(x) together with both differential and integral operations on x. It means that it is an equation containing derivative of the unknown function u(x) which appears outside the integral sign [4].

The general form of linear integro-differential equation is:

$$\frac{d^n u(x)}{dx^n}+\sum_{i=0}^{n-1} p_i(x)\frac{d^i u(x)}{dx^i}=g(x)+\lambda\int_a^{b(x)} k(x,t)u(t)dt \quad x\in[a,b(x)] \qquad \ldots (1)$$

with initial conditions: $\sum_{i=0}^{n-1}\frac{d^i u(x_0)}{dx^i}=u_i$, where g(x), $p_i(x)$, k(x,t) are known functions of *x*, $k(x,t)$ is called the kernel of the integral equation, u(x) is the unknown function, λ is a scalar parameter and a and b(x) are the limits of integral either are given constants or functions of *x*. The above equation is called ordinary integro-differential equation since the derivatives are taken with respect to single variable [1,5].

The form of first order linear integro-differential equation is:

$$\frac{du(x)}{dx}+p(x)u(x)=g(x)+\int_a^{b(x)} k(x,t)u(t)dt \quad x\in[a,b(x)]. \qquad \ldots (2)$$

with initial condition $u(x_0)=u_0$.

The delay integro-differential equation is a delay differential equation in which the unknown function u(x) can appear under an integral sign. The general form of nth order delay integro-differential equation is given by [6,7]:

$$\sum_{i=0}^{n} p_i(x)\frac{d^i u(x)}{dx^i}+\sum_{i=1}^{n} q_i(x)\frac{d^i u(x-\tau_i)}{dx^i}+\sum_{i=0}^{n} r_i(x)u(x-\tau_i)=$$
$$g(x)+\lambda\int_a^{b(x)} k(x,t)u(t-\tau)dt \qquad x\in[a,b(x)] \qquad \ldots (3)$$

with initial functions:
$$\begin{aligned} u(x)&=\phi(x)\\ u'(x)&=\phi'(x)\\ &\vdots\\ u^{(n-1)}(x)&=\phi^{(n-1)}(x) \end{aligned} ,$$

where g(x), $p_i(x)$, $q_i(x)$, k(x,t) are known functions of *x*, $k(x,t)$ is called the kernel of the integral equation, u(x) is the unknown function, λ is a scalar parameter (in this work λ=1), a and b(x) are the limits of the integral either are given constants or functions of *x* and $\tau, \tau_1, \tau_2, ..., \tau_n$ are fixed positive numbers. The integral equation in eq.(3) can be classified into different kinds according to the limits of integral and the kernel. If the limits a and b(x) in eq.(3) are constants (b(x) = b) then equation (3) is called a delay *Fredholm* integro-differential equation while if a in eq.(3) is a constant while (b(x) = *x*), eq.(3) is called a delay Volterra integro-differential equation [7,8].

Delay integro-differential equations are classified into three types [1,7,8] :-

- Equation (3) is called Retarded type if the derivatives of unknown function appear without difference argument (i.e. the delay comes in u only) and the delay appears in the integrand unknown function (i.e. $\tau \neq 0)$.
- Equation (3) is called a Neutral type if the highest-order derivative of unknown function appears with difference argument (i.e. the delay doesn't come in u) and the delay does not appear in the integrand function (i.e. $\tau = 0)$.
- All other DIDE's in eq.(3) are called mixed types, which are combination of the previous two types.

To present the delay in the linear integro-differential equation in eq.(2) three types can be introduced as follows [9,10]:

1. Retarded integro-differential equation when the delay comes in the unknown function u(x) involved in the integrand sign.

$$\frac{du(x)}{dx} + p(x)u(x) + r(x)u(x-\tau_1) = g(x) + \int_a^{b(x)} k(x,t)u(t-\tau)dt \quad x \in [a,b(x)]$$

With initial function $u(x) = \phi(x)$ … (4)

where τ, τ_1 are given positive constants called the "time delay" or "difference argument".

2. Neutral integro-differential equation when the delay comes in the derivative of u(x) outside the integral.

$$\frac{du(x-\tau)}{dx} + p(x)u(x) = g(x) + \int_a^{b(x)} k(x,t)u(t)dt \quad x \in [a,b(x)]$$

With initial function $u(x) = \phi(x)$ … (5)

3. Mixed integro-differential equation (sometimes called the advanced type), i.e. a combination of the previous two types.

$$q(x)u'(x-\tau_1) + r(x)u(x-\tau_2) + p(x)u(x) + w(x)u'(x) = g(x) + \int_a^{b(x)} k(x,t)u(t-\tau)dt$$

with initial function $u(x) = \phi(x)$ … (6)

where τ, τ_1, τ_2 are positive constants.

2.1 Delay Fredholm Integro-Differential Equation (DFIDE):

The following three types of linear delay Fredholm integro-differential equations are defined when b(x)=b in equations (4), (5) and (6) where b is a constant [1,11,12]:

1. Retarded Fredholm integro-differential equation:

$$\frac{du(x)}{dx} + p(x)u(x) + r(x)u(x-\tau_1) = g(x) + \int_a^b k(x,t)u(t-\tau)dt \qquad a \le x \le b$$

… (7)

2. Neutral Fredholm integro-differential equation :

$$\frac{du(x-\tau)}{dx} + p(x)u(x) = g(x) + \int_a^b k(x,t)u(t)dt \qquad a \le x \le b \qquad \text{… (8)}$$

3. Mixed Fredholm integro-differential equation :

$$q(x)u'(x-\tau_1) + r(x)u(x-\tau_2) + p(x)u(x) + w(x)u'(x) = g(x) + \int_a^b k(x,t)u(t-\tau)dt$$

$$a \le x \le b$$

… (9)

3. Weddle Method:

Weddle method is one of basic formula of quadrature approximation methods for integration. Quadrature rule is generic name given to any numerical method for the approximate calculation of definite integral I[u] of the function u(t) over finite integral [a,b] which is [1,3] : $I[u]=\int_a^b u(t)dt \qquad a<b$.

Weddle formula approximates the function on the interval $[t_0,t_6]$ by a curve that possess through seven points as:

$$\int_{t_0}^{t_6} f(t)dt=\frac{3H}{10}\left[f_0+5f_1+f_2+6f_3+f_4+5f_5+f_6\right] \qquad \ldots(10)$$

where $t_0,\ldots,t_6$ are called integration nodes, $H=\frac{(t_6-t_0)}{N}$, N is the number of intervals $[t_0,t_1],[t_1,t_2],\ldots,[t_5,t_6]$ and $f_i=f(t_i)$ where i = 0,1,…,6.

When it is applied over the interval [a,b], the composite Weddle rule is obtained as [1,5] :

$$\int_a^b f(t)dx=\frac{3H}{10}\begin{bmatrix} f_0+5f_1+f_2+6f_3+f_4+5f_5+2f_6+5f_7+f_8+6f_9+f_{10}+ \\ 5f_{11}+\cdots+2f_{N-6}+5f_{N-5}+f_{N-4}+6f_{N-3}+f_{N-2}+5f_{N-1}+f_N \end{bmatrix} \quad \ldots(11)$$

where a,b are the limit of the integral, $H=\frac{(b-a)}{N}$, N is the number of intervals which is the multiple of (6), $f_i=f(t_i)$ t_0 = a, t_N = b and $t_i=a+iH$ are called the integration nodes which are lying in the interval [a,b] where i = 0,1,…,N.

In order to solve three types (retarded, neutral and mixed) of linear delay Fredholm integro-differential equations numerically, Weddle method is valid for the integral of DFIDE's.

4. Block Method:

Block method provides easy and efficient mean for the solution of the many problems. The concept of block method is essentially an extrapolation procedure and has the advantage of being self-starting. Block method was described for differential equation by Milne and Young [13,14].

In this research block method was employed for finding the numerical solution for three types of linear delay Fredholm integro-differential equations.

Consider the following first order differential equation :

$y'=f(t,y(t))$ with initial condition $y(t_0)=y_0$ …(12)

A block method up to the fourth-order for eq.(12) is computed by:

$$y_{n+1} = y_n + 2hy'_n \qquad order\,2$$
$$y_{n+2} = y_n + 2hy'_{n+1} \qquad order\,2$$
$$y_{n+1} = y_n + (h/2)[y'_n + y'_{n+1}] \qquad order\,3$$
$$y_{n+2} = y_n + (h/2)[y'_n + y'_{n+2}] \qquad order\,3$$
$$y_{n+1} = y_n + (h/12)[5y'_n + 8y'_{n+1} - y'_{n+2}] \qquad order\,4$$
$$y_{n+2} = y_n + (h/3)[y'_n + 4y'_{n+1} + y'_{n+2}] \qquad order\,4$$

Block method of second and third order have been little used for ordinary differential equations, in general, and delay differential equations in particular because they required more evaluation of the function f.

However, the following fourth order block method, which is most popular and more efficient for dealing with differential equations.

Let

$$\left.\begin{aligned}
B_1 &= f(t_n, y(t_n)) \\
B_2 &= f(t_n + h, y(t_n) + hB_1) \\
B_3 &= f\left(t_n + h, y(t_n) + \frac{h}{2}B_1 + \frac{h}{2}B_2\right) \\
B_4 &= f(t_n + 2h, y(t_n) + 2hB_3) \\
B_5 &= f\left(t_n + h, y(t_n) + \frac{h}{12}(5B_1 + 8B_3 - B_4)\right) \\
B_6 &= f\left(t_n + 2h, y(t_n) + \frac{h}{3}(B_1 + B_4 + 4B_5)\right)
\end{aligned}\right\} \qquad \ldots (13)$$

Then the fourth order-six steps block method may be written in the form:

$$y_{n+1} = y_n + \frac{h}{12}(5B_1 + 8B_3 - B_4) \qquad \ldots (14)$$

$$y_{n+2} = y_n + \frac{h}{3}(B_1 + 4B_5 + B_6) \qquad \ldots (15)$$

5. The Solution of Linear Delay Fredholm Integro-Differential Equation (DFIDE) Using Block and Weddle Methods:

In this section three types (retarded, neutral and mixed) of linear delay Fredholm integro-differential equations have been solved using block and Weddle methods.

5.1 The Solution of First Order Linear Delay Fredholm Integro-Differential Equation (DFIDE) Using Block and Weddle Methods:

In this subsection the block including fourth order and Weddle method are candidates to find the numerical solutions for the first order DFIDE's as follows:

Recall three types (retarded, neutral and mixed) of linear DFIDE's in section (2.1). The numerical solution of these equations can be found using fourth order block and Weddle methods as follows:

The DFIDE's in (7), (8) and (9) can be written as:

1. Retarded Fredholm integro-differential equation :

$$\frac{du(x)}{dx} = f\big(x,u(x),u(x-\tau),g(x),I[Q(x,t)]\big) \qquad \dots (1٦)$$

where $I[Q(x,t)]$ is the finite integral on [a,b] in eq.(7) and $Q(x,t)=k(x,t)\,u(t-\tau)$.

2. Neutral Fredholm integro-differential equation :

$$\frac{du(x-\tau)}{dx} = f\big(x,u(x),g(x),I[Q(x,t)]\big) \qquad \dots (1٧)$$

where $I[Q(x,t)]$ is the finite integral on [a,b] in eq.(8) and $Q(x,t)=k(x,t)\,u(t)$.

3. Mixed Fredholm integro-differential equation:

$$\left.\begin{aligned} &\frac{du(x-\tau_1)}{dx} = f\big(x,u(x),u(x-\tau),g(x),I[Q(x,t)]\big) \\ \text{or} \quad &\frac{du(x)}{dx} = f\big(x,u(x),u(x-\tau),u'(x-\tau),g(x),I[Q(x,t)]\big) \end{aligned}\right\} \qquad \dots (18)$$

where $I[Q(x,t)]$ is the finite integral on [a,b] in eq.(9) and $Q(x,t)=k(x,t)\,u(t-\tau)$.

Hence, applying Weddle method in eq.(11) for computing the results for the integration of equations (16), (17) and (18) yields:

$$I[Q(x,t)]=\int_a^b Q(x,t)dt = Weddle\big(Q(x,t),a,b,N\big)$$

$$=\frac{3H}{10}\begin{bmatrix} Q(x,t_0)+5Q(x,t_1)+Q(x,t_2)+6Q(x,t_3)+Q(x,t_4)+ \\ 5Q(x,t_5)+\cdots+2Q(x,t_{N-6})+5Q(x,t_{N-5})+Q(x,t_{N-4})+ \\ 6Q(x,t_{N-3})+Q(x,t_{N-2})+5Q(x,t_{N-1})+Q(x,t_N) \end{bmatrix} \qquad \dots (19)$$

where $t_0 = a \ \ and \ \ t_N = b$ are the limit of the integral in DFIDE's (7),(8) and (9) , $H = \frac{(b-a)}{N}$, N is the number of intervals $([t_0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N])$ which is multiple of (6), and $t_i = a + iH$ where i = 0,1,…,N.

By using the result of the integral $I[Q(x,t)]$ in eq.(19), the numerical solution of DFIDE using fourth order block method is computed as follows:

Consider DFIDE's (17) and (18). To find the numerical solution of these equations using fourth order block method the following transformation is done:

Let $y = (x - \tau) \Rightarrow x = y + \tau$ and equations (17) and (18) become:

- $\frac{du(x-\tau)}{dx} = \frac{du(y)}{dy} = f\left((y+\tau), u(y+\tau), g(y+\tau), I[Q(y+\tau, t)]\right)$ …(20)
- $\frac{du(x-\tau_1)}{dx} = \frac{du(y)}{dy} = f\left((y+\tau_1), u(y+\tau_1), u((y+\tau_1)-\tau), g(y+\tau_1), I[Q(y+\tau_1, t)]\right)$ …(21)

So the general form of first order linear DFIDE in equations (16), (20) and (21) can be written as:

$$\frac{du(x)}{dx} = f\left(x, u(x), u(x-\tau), u'(x-\tau), g(x), I[Q(x,t)]\right) \qquad \ldots (22)$$

with initial function $u(x) = \phi(x)$.

By applying block method for DFIDE in eq.(22) using equations (13), (14) and (15), one gets the following formula:

$$u(x_{j+1}) = u(x_j) + \frac{h}{12}(5B_1 + 8B_3 - B_4) \qquad \ldots (23)$$

$$u(x_{j+2}) = u(x_j) + \frac{h}{3}(B_1 + 4B_5 + B_6) \qquad \ldots (24)$$

where

$$\left.\begin{array}{l}
B_1 = f\left(x_j, u(x_j), \phi(x_j-\tau), \phi'(x_j-\tau), g(x_j), Weddle\left(Q(x_j,t),a,b,N\right)\right) \\
B_2 = f\begin{pmatrix} x_j+h, u(x_j)+hB_1, \phi(x_j+h-\tau), \phi'(x_j+h-\tau), g(x_j+h), \\ Weddle\left(Q(x_j+h,t),a,b,N\right) \end{pmatrix} \\
B_3 = f\begin{pmatrix} x_j+h, u(x_j)+\frac{h}{2}B_1+\frac{h}{2}B_2, \phi(x_j+h-\tau), \phi'(x_j+h-\tau), g(x_j+h), \\ Weddle\left(Q(x_j+h,t),a,b,N\right) \end{pmatrix} \\
B_4 = f\begin{pmatrix} x_j+2h, u(x_j)+2hB_3, \phi(x_j+2h-\tau), \phi'(x_j+2h-\tau), g(x_j+2h), \\ Weddle\left(Q(x_j+2h,t),a,b,N\right) \end{pmatrix} \\
B_5 = f\begin{pmatrix} x_j+h, u(x_j)+\frac{h}{12}(5B_1+8B_3-B_4), \phi(x_j+h-\tau), \phi'(x_j+h-\tau), g(x_j+h), \\ Weddle\left(Q(x_j+h,t),a,b,N\right) \end{pmatrix} \\
B_6 = f\begin{pmatrix} x_j+2h, u(x_j)+\frac{h}{3}(B_1+B_4+4B_5), \phi(x_j+2h-\tau), \phi'(x_j+2h-\tau), g(x_j+2h), \\ Weddle\left(Q(x_j+2h,t),a,b,N\right) \end{pmatrix}
\end{array}\right\}$$

… (25)

for each j=0,1,…,m. where $(m + 1)$ is the number of points ($x_0, x_1, ..., x_m$).

The numerical solution using fourth order ***block and Weddle methods*** of three types (retarded, neutral and mixed) linear ***DFIDE's*** in (16), (20) and (21) can be summarized by the following algorithm :

BWM-DFIDE Algorithm :

Step 1: Input a, b, x_0 , m ,N where $a\ and\ b$ are the limit of the integral in DFIDE, x_0 is the initial value, N is the number of intervals in eq.(19) and $(m + 1)$ is the number of points ($x_0, x_1, ..., x_m$).

Step 2: Define $Q(x,t)$ in DFIDE as eq.(16) or eq.(17) or eq.(18).

Step 3: Define the function $g(x)$ in the DFIDE.

Step 4: Set $h = \frac{(x_m - x_0)}{m}$.

Step 5: Set j=0

Step 6: Compute:

$$B_1 = f\left(x_j, u(x_j), \phi(x_j-\tau), \phi'(x_j-\tau), g(x_j), Weddle\left(Q(x_j,t),a,b,N\right)\right)$$

Step 7: Compute:

$$B_2 = f\begin{pmatrix} x_j + h, u(x_j) + h\,B_1, \phi(x_j + h - \tau), \phi'(x_j + h - \tau), g(x_j + h), \\ Weddle\big(Q(x_j + h, t), a, b, N\big) \end{pmatrix}$$

Step 8: Compute:

$$B_3 = f\begin{pmatrix} x_j + h, u(x_j) + \frac{h}{2}B_1 + \frac{h}{2}B_2, \phi(x_j + h - \tau), \phi'(x_j + h - \tau), g(x_j + h), \\ Weddle\big(Q(x_j + h, t), a, b, N\big) \end{pmatrix}$$

Step 9: Compute:

$$B_4 = f\begin{pmatrix} x_j + 2h, u(x_j) + 2hB_3, \phi(x_j + 2h - \tau), \phi'(x_j + 2h - \tau), g(x_j + 2h), \\ Weddle\big(Q(x_j + 2h, t), a, b, N\big) \end{pmatrix}$$

Step 10: Compute:

$$B_5 = f\begin{pmatrix} x_j + h, u(x_j) + \frac{h}{12}(5B_1 + 8B_3 - B_4), \phi(x_j + h - \tau), \phi'(x_j + h - \tau), g(x_j + h), \\ Weddle\big(Q(x_j + h, t), a, b, N\big) \end{pmatrix}$$

Step 11: Compute:

$$B_6 = f\begin{pmatrix} x_j + 2h, u(x_j) + \frac{h}{3}(B_1 + B_4 + 4B_5), \phi(x_j + 2h - \tau), \phi'(x_j + 2h - \tau), g(x_j + 2h), \\ Weddle\big(Q(x_j + 2h, t), a, b, N\big) \end{pmatrix}$$

where $Weddle(Q(x,t),a,b,N)$ in steps (6,7,…,11) is Weddle method in eq.(19) for computing the result of the integral of DFIDE as:

$$\int_a^b Q(x,t)dt = Weddle\big(Q(x,t), a, b, N\big)$$

$$= \frac{3H}{10}\begin{bmatrix} Q(x,a) + 5Q(x,t_1) + Q(x,t_2) + 6Q(x,t_3) + Q(x,t_4) + \\ 5Q(x,t_5) + \cdots + 2Q(x,t_{N-6}) + 5Q(x,t_{N-5}) + Q(x,t_{N-4}) + \\ 6Q(x,t_{N-3}) + Q(x,t_{N-2}) + 5Q(x,t_{N-1}) + Q(x,b) \end{bmatrix}$$

where $t_i = a + iH$, $H = \frac{(b-a)}{N}$ and i = 0,1,…,N.

Step 12: Compute :

$$x_{j+1} = x_j + h$$

$$u(x_{j+1}) = u(x_j) + \frac{h}{12}(5B_1 + 8B_3 - B_4)$$

$$u(x_{j+2}) = u(x_j) + \frac{h}{3}(B_1 + 4B_5 + B_6)$$

Step 13: Put $j = j+1$

Step 14: If $j = m$ then stop.

Else go to (step 6)

5.2 The Solution of n^th Order Linear Delay Fredholm Integro-Differential Equation (DFIDE) Using Block and Weddle Methods:

The general form of nth-order linear DFIDE in eq.(3) can be written as:

$$f\begin{pmatrix} x, p_0(x)u(x), p_1(x)u'(x),\dots, p_{n-1}(x)u^{(n-1)}(x), p_n(x)u^{(n)}(x), q_1(x)u'(x-\tau_1), \\ \dots, q_n(x)u^{(n)}(x-\tau_n),\dots, r_0(x)u(x-\tau_0),\dots, r_n(x)u(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix} = 0 \quad \dots (26)$$

where $I[Q(x,t)]$ is the finite integral on [a,b] , $x \in [a,b]$ and $Q(x,t) = k(x,t)\, u(t-\tau)$.

with initial functions:

$$u(x) = \phi(x)$$
$$u'(x) = \phi'(x)$$
$$\vdots$$
$$u^{(n-1)}(x) = \phi^{(n-1)}(x)$$

To solve neutral and mixed types of DFIDE in eq.(26) using fourth order block and Weddle methods the same transformation in eq.(20) and eq,(21) is done respectively: Hence, eq.(26) can be written as:

$$\frac{d^n u(x)}{dx^n} = f\begin{pmatrix} x, p_0(x)u(x), p_1(x)u'(x),\dots, p_{n-1}(x)u^{(n-1)}(x), q_1(x)u'(x-\tau_1),\dots, \\ q_{n-1}(x)u^{(n-1)}(x-\tau_{n-1}),\dots, r_0(x)u(x-\tau_0),\dots, r_n(x)u(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix} \quad \dots (27)$$

Obviously, the nth order equation (27) with difference argument may be replaced by a system of nth-equation of first order DFIDE's as follows:
Let

$$v_1(x) = u(x)$$
$$v_2(x) = u'(x)$$
$$\vdots$$
$$v_{n-1}(x) = u^{(n-2)}(x)$$
$$v_n(x) = u^{(n-1)}(x)$$

Then, one gets the following system of the first order equations:

$$v_1'(x) = v_2(x)$$
$$v_2'(x) = v_3(x)$$
$$\vdots$$
$$v_{n-1}'(x) = v_n(x)$$
$$v_n'(x) = f\begin{pmatrix} x, p_0(x)v_1(x), p_1(x)v_2(x),\dots, p_{n-1}(x)v_n(x), q_1(x)v_2(x-\tau_1),\dots, q_{n-1}(x) \\ v_n(x-\tau_{n-1}),\dots, r_0(x)v_1(x-\tau_0),\dots, r_n(x)v_1(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix}$$

… (28)

The above system of the first order linear DFIDE's can be treated numerically by using fourth order block and Weddle methods as follows:

$$u_i(x_{j+1}) = u_i(x_j) + \frac{h}{12}(5B_{1i} + 8B_{3i} - B_{4i}) \quad \dots (29)$$

$$u_i(x_{j+2}) = u_i(x_j) + \frac{h}{3}(B_{1i} + 4B_{5i} + B_{6i}) \quad \dots (30)$$

where

$$B_{1i}=f_i\begin{pmatrix} x_j,\, p_0(x_j)v_1(x_j),\dots,p_{n-1}(x_j)v_n(x_j),q_1(x_j)v_2(x_j-\tau_1),\dots,q_{n-1}(x_j)v_n(x_j-\tau_{n-1}),\\ \dots,r_0(x_j)\phi(x_j-\tau_0),\dots,r_n(x_j)\phi(x_j-\tau_n),g(x_j),Weddle\big(Q(x_j,t),a,b,N\big)\end{pmatrix}$$

$$B_{2i}=f_i\begin{pmatrix} x_j+h,\, p_0(x_j+h)v_1(x_j)+hB_{11},\dots,p_{n-1}(x_j+h)v_n(x_j)+hB_{1n},q_1(x_j+h)\\ v_2(x_j-\tau_1)+hB_{11},\dots,q_{n-1}(x_j+h)v_n(x_j-\tau_{n-1})+hB_{1n},\dots,r_0(x_j+h)\phi(x_j+h-\tau_0)\\ ,\dots,r_n(x_j+h)\phi(x_j+h-\tau_n),g(x_j+h),Weddle\big(Q(x_j+h,t),a,b,N\big)\end{pmatrix}$$

$$B_{3i}=f_i\begin{pmatrix} x_j+h,\, p_0(x_j+h)v_1(x_j)+\frac{h}{2}B_{11}+\frac{h}{2}B_{21},\dots,p_{n-1}(x_j+h)v_n(x_j)+\frac{h}{2}B_{1n}+\frac{h}{2}B_{2n},\\ q_1(x_j+h)v_2(x_j+h-\tau_1),\dots,q_{n-1}(x_j+h)v_n(x_j+h-\tau_{n-1}),\dots,r_0(x_j+h)\phi(x_j+h-\tau_0)\\ ,\dots,r_n(x_j+h)\phi(x_j+h-\tau_n),g(x_j+h),Weddle\big(Q(x_j+h,t),a,b,N\big)\end{pmatrix}$$

$$B_{4i}=f_i\begin{pmatrix} x_j+2h,\, p_0(x_j+2h)v_1(x_j)+2hB_{31},\dots,p_{n-1}(x_j+2h)v_n(x_j)+2hB_{3n},q_1(x_j+2h)\\ v_2(x_j+2h-\tau_1),\dots,q_{n-1}(x_j+2h)v_n(x_j+2h-\tau_{n-1}),\dots,r_0(x_j+2h)\phi(x_j+2h-\tau_0)\\ ,\dots,r_n(x_j+2h)\phi(x_j+2h-\tau_n),g(x_j+2h),Weddle\big(Q(x_j+2h,t),a,b,N\big)\end{pmatrix}$$

$$B_{5i}=f_i\begin{pmatrix} x_j+h,\, p_0(x_j+h)v_1(x_j)+\frac{h}{12}(5B_{11}+8B_{31}-B_{41}),\dots,p_{n-1}(x_j+h)v_n(x_j)+\\ \frac{h}{12}(5B_{1n}+8B_{3n}-B_{4n}),q_1(x_j+h)v_2(x_j+h-\tau_1),\dots,q_{n-1}(x_j+h)v_n(x_j+h-\tau_{n-1}),\dots,\\ r_0(x_j+h)\phi(x_j+h-\tau_0),\dots,r_n(x_j+h)\phi(x_j+h-\tau_n),g(x_j+h),Weddle\big(Q(x_j+h,t),a,b,N\big)\end{pmatrix}$$

$$B_{6i}=f_i\begin{pmatrix} x_j+2h,\, p_0(x_j+2h)v_1(x_j)+\frac{h}{3}(B_{11}+B_{41}+4B_{51}),\dots,p_{n-1}(x_j+2h)v_n(x_j)+\\ \frac{h}{3}(B_{1n}+B_{4n}+4B_{5n}),q_1(x_j+2h)v_2(x_j+2h-\tau_1),\dots,q_{n-1}(x_j+2h)v_n(x_j+2h-\tau_{n-1})\\ ,\dots,r_0(x_j+2h)\phi(x_j+2h-\tau_0),\dots,r_n(x_j+2h)\phi(x_j+2h-\tau_n),g(x_j+2h),\\ Weddle\big(Q(x_j+2h,t),a,b,N\big)\end{pmatrix}$$

… (31)

for each i=1,2,…,n. and j=0,1,…,*m* where (*m* + 1) is the number of points ($x_0,x_1,\dots,x_m$) and *Weddle*(*Q*(*x*,*t*),*a*,*b*,*N*) is Weddle method in eq.(19).

The numerical solution using fourth order ***b**lock and **W**eddle **m**ethods* of three types (retarded, neutral and mixed) **nth** order linear ***DFIDE'****s* can be summarized by the following algorithm :

BWM-nDFIDE Algorithm :

Step 1: Input a, b, x_0, m, N, n where $a\ and\ b$ are the limit of the integral in DFIDE, x_0 is the initial value, N is the number of intervals in eq.(19), (m + 1) is the number of points ($x_0, x_1, ..., x_m$) and n is the order of DFIDE.

Step 2: Define $Q(x,t)$ as in eq.(26) and the function $g(x)$ in the nth order DFIDE.

Step 3: Set $h = \frac{(x_m - x_0)}{m}$ and j = 0.

Step 4: For each i=1,2,…,n compute:

$$B_{1i} = f_i\begin{pmatrix} x_j, p_0(x_j)v_1(x_j), \ldots, p_{n-1}(x_j)v_n(x_j), q_1(x_j)v_2(x_j - \tau_1), \ldots, q_{n-1}(x_j)v_n(x_j - \tau_{n-1}), \\ \ldots, r_0(x_j)\phi(x_j - \tau_0), \ldots, r_n(x_j)\phi(x_j - \tau_n), g(x_j), Weddle\big(Q(x_j, t), a, b, N\big) \end{pmatrix}$$

$$B_{2i} = f_i\begin{pmatrix} x_j + h, p_0(x_j + h)v_1(x_j) + hB_{11}, \ldots, p_{n-1}(x_j + h)v_n(x_j) + hB_{1n}, q_1(x_j + h) \\ v_2(x_j - \tau_1) + hB_{11}, \ldots, q_{n-1}(x_j + h)v_n(x_j - \tau_{n-1}) + hB_{1n}, \ldots, r_0(x_j + h)\phi(x_j + h - \tau_0) \\ , \ldots, r_n(x_j + h)\phi(x_j + h - \tau_n), g(x_j + h), Weddle\big(Q(x_j + h, t), a, b, N\big) \end{pmatrix}$$

$$B_{3i} = f_i\begin{pmatrix} x_j + h, p_0(x_j + h)v_1(x_j) + \frac{h}{2}B_{11} + \frac{h}{2}B_{21}, \ldots, p_{n-1}(x_j + h)v_n(x_j) + \frac{h}{2}B_{1n} + \frac{h}{2}B_{2n}, \\ q_1(x_j + h)v_2(x_j + h - \tau_1), \ldots, q_{n-1}(x_j + h)v_n(x_j + h - \tau_{n-1}), \ldots, r_0(x_j + h)\phi(x_j + h - \tau_0) \\ , \ldots, r_n(x_j + h)\phi(x_j + h - \tau_n), g(x_j + h), Weddle\big(Q(x_j + h, t), a, b, N\big) \end{pmatrix}$$

$$B_{4i}=f_i\begin{pmatrix} x_j+2h,\, p_0(x_j+2h)v_1(x_j)+2hB_{31},\ldots,\, p_{n-1}(x_j+2h)v_n(x_j)+2hB_{3n},\, q_1(x_j+2h) \\ v_2(x_j+2h-\tau_1),\ldots,q_{n-1}(x_j+2h)v_n(x_j+2h-\tau_{n-1}),\ldots,r_0(x_j+2h)\phi(x_j+2h-\tau_0) \\ ,\ldots,r_n(x_j+2h)\phi(x_j+2h-\tau_n),\, g(x_j+2h), Weddle\left(Q(x_j+2h,t),a,b,N\right) \end{pmatrix}$$

$$B_{5i}=f_i\begin{pmatrix} x_j+h,\, p_0(x_j+h)v_1(x_j)+\frac{h}{12}(5B_{11}+8B_{31}-B_{41}),\ldots,\, p_{n-1}(x_j+h)v_n(x_j)+ \\ \frac{h}{12}(5B_{1n}+8B_{3n}-B_{4n}),q_1(x_j+h)v_2(x_j+h-\tau_1),\ldots,q_{n-1}(x_j+h)v_n(x_j+h-\tau_{n-1}),\ldots, \\ r_0(x_j+h)\phi(x_j+h-\tau_0),\ldots,r_n(x_j+h)\phi(x_j+h-\tau_n),\, g(x_j+h), Weddle\left(Q(x_j+h,t),a,b,N\right) \end{pmatrix}$$

$$B_{6i}=f_i\begin{pmatrix} x_j+2h,\, p_0(x_j+2h)v_1(x_j)+\frac{h}{3}(B_{11}+B_{41}+4B_{51}),\ldots,\, p_{n-1}(x_j+2h)v_n(x_j)+ \\ \frac{h}{3}(B_{1n}+B_{4n}+4B_{5n}),q_1(x_j+2h)v_2(x_j+2h-\tau_1),\ldots,q_{n-1}(x_j+2h)v_n(x_j+2h-\tau_{n-1}) \\ ,\ldots,r_0(x_j+2h)\phi(x_j+2h-\tau_0),\ldots,r_n(x_j+2h)\phi(x_j+2h-\tau_n),\, g(x_j+2h), \\ Weddle\left(Q(x_j+2h,t),a,b,N\right) \end{pmatrix}$$

$$u_i(x_{j+1})=u_i(x_j)+\frac{h}{12}(5B_{1i}+8B_{3i}-B_{4i})$$

$$u_i(x_{j+2})=u_i(x_j)+\frac{h}{3}(B_{1i}+4B_{5i}+B_{6i}) \quad \text{and} \quad x_{j+1}=x_j+h$$

where $Weddle(Q(x,t),a,b,N)$ in step (4) is Weddle method in eq.(19) for computing the result of the integral of DFIDE as:

$$\int_a^b Q(x,t)dt = Weddle\left(Q(x,t),a,b,N\right)$$

$$=\frac{3H}{10}\begin{bmatrix} Q(x,a)+5Q(x,t_1)+Q(x,t_2)+6Q(x,t_3)+Q(x,t_4)+ \\ 5Q(x,t_5)+\cdots+2Q(x,t_{N-6})+5Q(x,t_{N-5})+Q(x,t_{N-4})+ \\ 6Q(x,t_{N-3})+Q(x,t_{N-2})+5Q(x,t_{N-1})+Q(x,b) \end{bmatrix}$$

where $t_k=a+kH$, $H=\frac{(b-a)}{N}$ and $k=0,1,\ldots,N$.

Step 5: Put $j=j+1$

Step 6: If $j=m$ then stop.

Else go to (step 4)

6. Numerical Examples :

Example (1):

Consider the following *mixed* Fredholm integro-differential equation of the first order:

$$\frac{du(x-1)}{dx} = 1 - \frac{7x}{12} + \int_0^1 x\,t\,u(t-\tfrac{1}{2})dt \qquad x \geq 0$$

with initial function : $u(x) = x + 1$.

The exact solution of the above linear DFIDE is: $u(x) = x + 1 \qquad x \geq 0$.

When the algorithm (BWM-DFIDE) is applied and the transformation in eq.(21) is done, table (1) presents the comparison between the exact and numerical solutions of the above mixed Fredholm integro-differential equation using block and Weddle methods for *m*=10, h=0.1 , $x_j = jh,\ j = 0,1,\ldots,m$ and *m*=100, h=0.01 , depending on least square error (L.S.E.).

Table (1) The solution of DFIDE for Ex.(1).

x	*Exact*	*Block and Weddle Method (BWM-DFIDE) u(x)*	
		h=0.1	*h=0.01*
0	1.0000	1.0000	1.0000
٠.1	1.1000	1.1000	1.1000
٠.2	1.2000	1.2000	1.2000
٠.3	1.3000	1.3000	1.3000
٠.4	1.4000	1.4000	1.4000
٠.5	1.5000	1.5000	1.5000
٠.6	1.6000	1.6000	1.6000
٠.7	1.7000	1.7000	1.7000
٠.8	1.8000	1.8000	1.8000
٠.9	1.9000	1.9000	1.9000
1	2.0000	2.0000	2.0000

L.S.E.	0.35e-30	0.44e-38

Figure (1) shows the solution of linear mixed *Fredholm integro-differential* equation, which was given in example (1) by using block and Weddle methods (BWM-DFIDE algorithm) with the exact solutions.

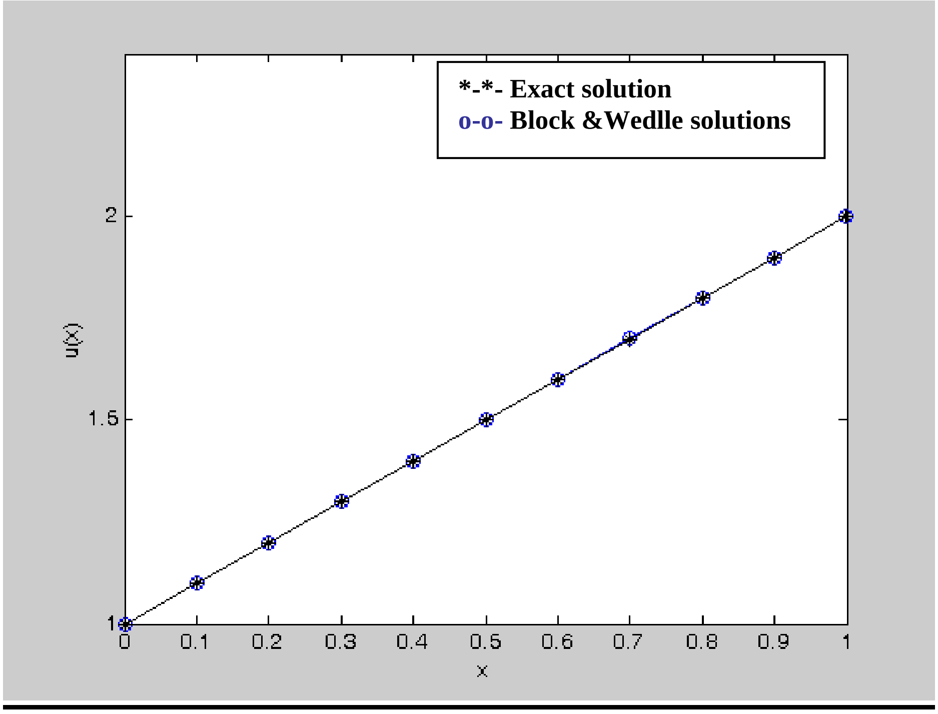

Fig.(1) The comparison between the exact and block & Weddle solutions for mixed Fredholm integro-differential equation in Ex.(1).

The numerical solution by using block and Weddle methods (BWM-DFIDE algorithm) is good when taking h small. Different values of h and the corresponding L.S.E. coming from our choices are listed in table (2) where $0 \le x \le 1$.

Table(2) The L.S.E. of Ex.(1) when h=0.05 ,0.02 and 0.001

The block and Weddle solutions (BWM-DFIDE Algorithm)		
h	***m***	***L.S.E.***
0.05	20	0.463e-32
0.02	50	0.569e-35
0.001	1000	0.114e-47

Example (2):

Consider the following *retarded* Fredholm integro-differential equation of the second order:

$$\frac{d^2u(x)}{dx^2}+x\frac{du(x)}{dx}+u(x)+xu(x-\tfrac{\pi}{2})=x\big(\cos 1+\sin 1-1\big)+\int_0^1 xtu(t-\tfrac{\pi}{2})dt \qquad 0\le x\le 1$$

with initial functions : $u(x)=\cos(x-\tfrac{\pi}{2})$
$u'(x)=-\sin(x-\tfrac{\pi}{2})$

The exact solution of the above linear DFIDE is: $u(x)=\sin x \qquad x\ge 0$.

The above DFIDE can be replaced by a system of two first order DFIDE's as:

$$v_1'(x)=v_2(x), \qquad 0\le x\le 1$$

$$v_2'(x)=-xv_2(x)-v_1(x)-xv_1(x-\tfrac{\pi}{2})+x\big(\cos 1+\sin 1-1\big)+\int_0^1 xtv_1(t-\tfrac{\pi}{2})dt \quad 0\le x\le 1$$

with initial functions: $v_1(x)=\cos(x-\tfrac{\pi}{2})$
$v_2(x)=-\sin(x-\tfrac{\pi}{2})$

and exact solutions: $exact_1=v_1(x)=\sin x \qquad x\ge 0$
$exact_2=v_2(x)=\cos x \qquad x\ge 0$

When the algorithm (BWM-nDFIDE) is applied, table (3) presents the comparison between the exact and numerical solution of the above retarded Fredholm integro-differential equation using block and Weddle methods for m=10, h=0.1, $x_j=jh$, $j=0,1,...,m$ depending on least square error (L.S.E.).

Table (3) The solution of DFIDE for Ex.(2).

x	*Exact₁*	*Block and Weddle Method (BWM-nDFIDE)* $v_1(x)$	*Exact₂*	*Block and Weddle Method (BWM-nDFIDE)* $v_2(x)$
0	**0.0000**	**0.0000**	**1.0000**	**1.0000**
٠.1	**0.0998**	**0.0998**	**0.9950**	**0.9950**
٠.2	**0.1987**	**0.1987**	**0.9801**	**0.9801**
٠.3	**0.2955**	**0.2955**	**0.9553**	**0.9553**
٠.4	**0.3894**	**0.3894**	**0.9211**	**0.9211**
٠.5	**0.4794**	**0.4794**	**0.8776**	**0.8776**
٠.6	**0.5646**	**0.5646**	**0.8253**	**0.8253**
٠.7	**0.6441**	**0.6442**	**0.7648**	**0.7648**
٠.8	**0.7172**	**0.7174**	**0.6967**	**0.6967**
٠.9	**0.7833**	**0.7833**	**0.6216**	**0.6216**
1	**0.8415**	**0.8415**	**0.5403**	**0.5403**
L.S.E		**0.539e-8**	**L.S.E**	**0.747e-9**

Figure (2) shows the solution of linear retarded *Fredholm integro-differential* equation, which was given in example (2) by using block and Weddle methods (BWM-nDFIDE algorithm) with the exact solutions.

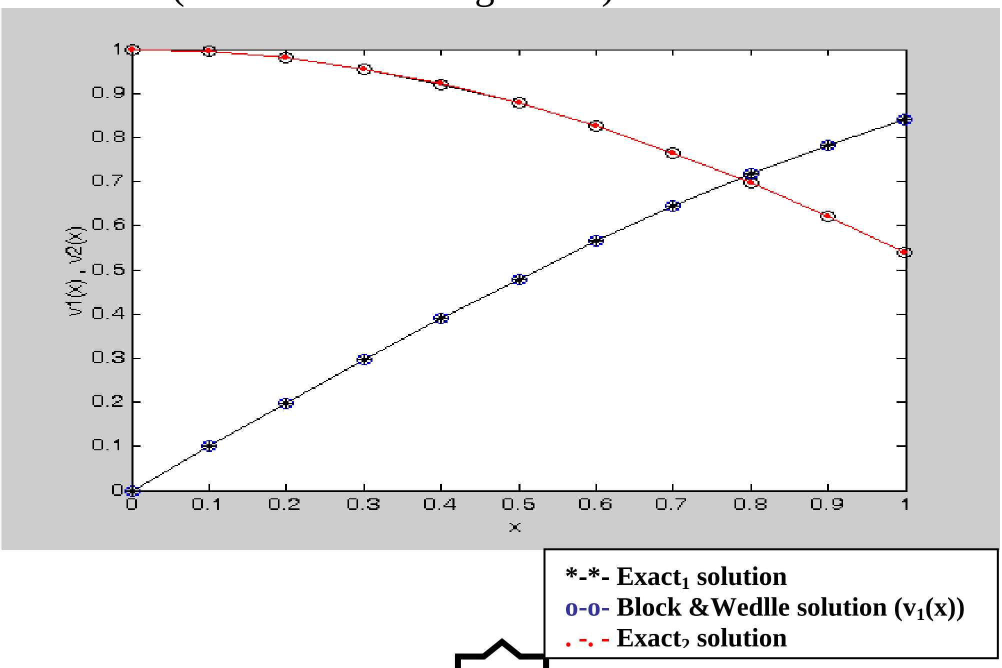

Fig.(2) The comparison between the exact and block & Weddle solutions for retarded Fredholm integro-differential equation in Ex.(2)

Example (3):

Consider the following *Neutral* Fredholm integro-differential equation of third order:

$$\frac{d^3u(x-1)}{dx^3}+x\frac{d^2u(x)}{dx^2}+x^2\frac{du(x-1)}{dx}+u(x)=\left(x^2-\frac{3}{2}x-\frac{2}{3}\right)+\int_0^1(x+2t)u(t)dt \quad 0\le x\le 1$$

with initial functions : $u(x)=x+2$, $u'(x)=1$, $u''(x)=0$

The exact solution of the above linear DFIDE is:

$$u(x)=x+2 \qquad x\ge 0$$

When the transformation in eq.(20) is done and the above DFIDE is replaced by a system of three first order DFIDE's, the algorithm (BWM-nDFIDE) is applied to solve this equation. Table (4) presents the comparison between the exact and numerical solution of the above Neutral Fredholm integro-differential equation using block and Weddle methods for *m*=10, h=0.1, $x_j=jh$, $j=0,1,\dots,m$ and *m*=100, h=0.01 , depending on least square error (L.S.E.).

Table (4) The solution of DFIDE for Ex.(3).

x	$Exact_1$	Block and Weddle Method (BWM-nDFIDE) $v_1(x)$		$Exact_2$	Block and Weddle Method (BWM-nDFIDE) $v_2(x)$		$Exact_3$	Block and Weddle Method (BWM-nDFIDE) $v_3(x)$	
		h=0.1	h=0.01		h=0.1	h=0.01		h=0.1	h=0.01
0	2.0000	2.0000	2.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.1	2.1000	2.1000	2.1000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.2	2.2000	2.2000	2.2000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.3	2.3000	2.3000	2.3000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.4	2.4000	2.4000	2.4000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.5	2.5000	2.5000	2.5000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.6	2.6000	2.6000	2.6000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.7	2.7000	2.7000	2.7000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.8	2.8000	2.8000	2.8000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000

0.9	2.9000	2.9000	2.9000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
1	3.0000	3.0000	3.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
L.S.E.		0.0000	0.0000	**L.S.E.**	0.0000	0.0000	**L.S.E.**	0.0000	0.0000

Figure (3) shows the solution of linear neutral *Fredholm integro-differential* equation, which was given in example (3) by using block and Weddle methods (BWM-nDFIDE algorithm) with the exact solutions.

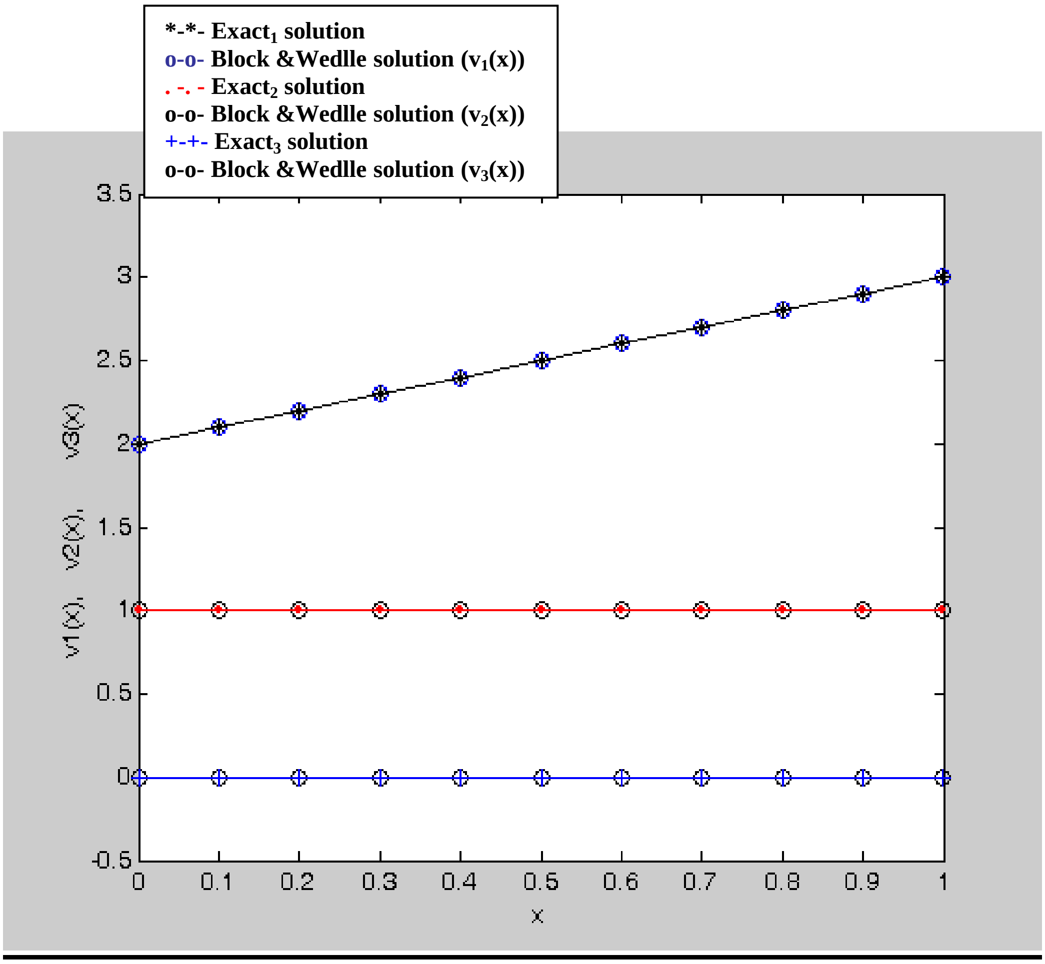

Fig.(3) The comparison between the exact and block & Weddle solution for neutral Fredholm integro-differential equation in Ex.(3).

7. Conclusion:

Block and Weddle methods have been presented to find the numerical solutions for three types (retarded, neutral and mixed) of nth-order linear delay Fredholm integro-differential equations. The results show a marked improvement in the least square errors (L.S.E.). From solving some numerical examples, the following points are included:

1. Block and Weddle methods solve three types (retarded, neutral and mixed) of nth-order linear DFIDE's and good results are obtained.
2. Block and Weddle methods proved their effectiveness in solving nth order linear DFIDE's where they give a better accuracy and consistent to the solution of three types DFIDE's.
3. Block and Weddle methods give qualified way for solving first order linear DFIDE's as well as nth-order of linear DFIDE
4. The good approximation depends on the size of h, if h is decreased then the number of points (knots) increases and the L.S.E. approaches to zero where this gives the advantage in numerical computation.
5. Block and Weddle methods solved linear DFIDE of any order by reducing the equation to a system of first order equations.

8. References:

1. Burgestaller R., "Integral and Integro-Differential Equation Theory Methods and Applications"; Edit by Agarwal R.P. Oregun D. Gordon and Breach Science Publisher, 2000.
2. Jerri, A.J., “Introduction to Integral Equations with Applications”; Marcel Dekker; 1985.
3. Burton, T.A. "Integral Equation with Delay"; Aeta Math hung 72, No 3, PP.233-242, 1998.
4. Delves, L. M. and Mohamed, J.L. "Computational Methods for Integral Equations" CombridgeUniversity Press, 1985.
5. Fatma T. "Numerical Solutions of Fredholm Integro-Differential Equations Using Spline Functions"; M.Sc. Thesis, University of Technology, 2002.
6. Marry R., “Laplace Transform”, Schaum’s Outline Series, New York, Prentice Hall, Inc., 1973.
7. Hayat A. "Approximated Solutions of Linear Delay Integro-Differential Equations "; M.Sc. Thesis, University of Technology, 2006.
8. Driver, R.D., "Ordinary and Delay Differential Equations", Springer verlag, New York, Inc. 1997.
9. Werbowki J. "On Some Asymptotic Behavior of Solution of Volterra-Integral Equation with Delay", Damostr. Math B.PP.579-584, 19٩9.

10. Al-Shather, A.H. :Some Application of Fractional Order Differential Operator in Differential and Delay Integro-Differential Equation", Ph.D. Thesis, College of Science, Al-Nahrain University, 2003.
11. Hartung F. and Gyori I., "On Numerical Solution for a Class of Linear Integro-Differential-Difference Equation with Time and State-Dependent Delays", http:/www.ma.man.ac.uk/nare parts, 2005.
12. Baker, C.H., "Retarded Integro-Differential Equations", http:/ www.ma.man.ac.uk/nare ports, 200٦.
13. Lambert, J.D., "Computational Methods in Ordinary Differential Equations", John Wiley & Sons Ltd., New York, 1979.
14. Shampin L.and Watts H.," The Stability of The Block Methods", J. Number Anal,Vol.13, pp. 101-114, 2003.

طريقتي البلوك و ويدل لحل معادلات فريدهولم التكاملية- التفاضلية التباطؤية الخطية من الرتب العليا

أثير جواد كاظم

قسم العلوم التطبيقية – الجامعة التكنولوجية

الخلاصة :

يقدم البحث طريقة مطورة مع اشتقاق خوارزميات جديدة لإيجاد الحل العددي لمعادلات فريدهولم التكاملية التفاضلية التباطؤية الخطية من الرتب العليا باستخدام طريقة البلوك من الرتبة الرابعة و طريقة ويدل. حيث تمت معالجة ثلاثة أنواع من معادلات فريدهولم التكاملية– التفاضلية التباطؤية الخطية من الرتبة الأولى عددياً مثلما لمعادلات فريدهولم التكاملية– التفاضلية التباطؤية الخطية من الرتب العليا باستخدام طريقتي البلوك من الرتبة الرابعة و ويدل حيث من الممكن ملاحظة كفاءة الطريقتين و سهولة الحسابات فيهما. استخدمت لغة (Matlab) لبرمجة خوارزميات هذا البحث. كما تمت مقارنة النتائج العددية و الحقيقية لثلاثة أنواع من الرتب المختلفة لمعادلات فريدهولم التكاملية التفاضلية التباطؤية الخطية من خلال بعض الأمثلة و الرسوم وقد تم الحصول على نتائج جيدة.