Using Bernstein Polynomials Method for Solving High-order Nonlinear Volterra-Fredholm Integro Differential Equation

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Abstract

In this paper, the Bernstein Polynomial method is used to find an approximate solution initials values problem for high-order nonlinear Volterra Fredholm integro differential equation of the second kind. Some different examples considered and the solution discussed numerically and display graphically. By enhancing the degree of Bernstein Polynomial, we can improve the accuracy results. The simulation results were also compared with other researchers' work.

1. Introduction

Integral and integro-differential equations play an important role in characterizing many social, biological, physical and engineering problems. Nonlinear integral and inegro-differential equations are usually hard to solve analytically and exact solutions are rather difficult to be obtained. Many numerical methods have been presented to solve this problem such as Legendre wavelets method [1] presented by Razzaghi and Yousefi to solve a high-order nonlinear Volterra-Fredholm integro-differential equation. Cerdik-Yaslan and Akyuz-Dascioglu [2] established Chebyshev Polynomial method for the solution nonlinear Fredholm-Volterra integro-differential equations. In addition, Reihani and Abadi [3] presented a numerical method address rationalized Haar functions method for solving Fredholm and Volterra integral equations. In fact, Araghi and Behzadi, [4] have solved nonlinear Volterra-Fredholm integro-differential equations by utilizing the modified Adomian decomposition method. Finally, from a recent study Yüzbasi et al. [5] have used a collocation approach for solving high-order linear Fredholm-Volterra integro-differential equations.

Historically, the approximation functions plays an important role in real and applied field. In mathematics, they have been used to solve (Non-linear equations, ODE, PDE, and etc.). These functions, reduce the original problems to those of solving a system of linear, nonlinear algebraic equations. A typical example is Bernstein Polynomial [6], which possesses useful properties such as orthogonality and the capability to represent functions at different levels of resolution.

The nonlinear Volterra-Fredholm differential equation of the second kind can be expressed in general form as follows [7] [8] :

$$y(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)[y(t)]^r \, dt + \lambda_2 \int_a^b k_2(x,t)[y(t)]^s \, dt \qquad \ldots (1)$$

Where, $k_1(x,t)$, $k_2(x,t)$ and $f(x)$ are known functions, a,b are constant values, r,s are integer numbers and $y(x)$ is the unknown function to be determined. Thus, high-order nonlinear Volterra-Fredholm integro-differential equation of the second kind can be expressed as follows [8] [9]:

$$\sum_{i=0}^{m} \mu_i(x)\left[y^{(i)}(x)\right] = f(x) + \int_a^x k_1(x,t)[y(t)]^r \, dt + \int_a^b k_2(x,t)[y(t)]^s \, dt \qquad \ldots (2)$$

where $\mu_i(x)$, $i = 0,1,2,\ldots,m$, $\mu_m(x) \neq 0$ are known functions with the initial conditions $y^{(i)}(a) = y_i$, $i = 0,1,2,\ldots,m-1$.

In the present paper, we have introduced an approximation method to solve the high-order nonlinear Volterra-Fredholm integro-differential equation of the second kind by using the Bernstein polynomial method. The paper is organized as follows: The Bernstein Polynomial method is described in the second section. In third section we present the Bernstein Polynomial approximation which method required for our function. In the fourth section the numerical finding to demonstrate the accuracy and applicability of the propose method by considering two examples are reported.

2. Bernstein Polynomials Method

The Bernstein polynomial method of degree n is defined as [10] [11]

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad \text{for} \quad i = 0,1,2,\ldots,n \qquad \ldots (3)$$

where n is the degree of Bernstein polynomial, i is the index of polynomials, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, and $t \in [a,b]$ is a variable.

For instance, the Bernstein polynomials of degree 2 are

$$B_0^2(t) = (1-t)^2$$

$$B_1^2(t) = 2t(1-t)$$

$$B_2^2(t) = t^2$$

In addition, derivatives of the n^{th} degree Bernstein polynomials are polynomials of degree $n-1$. By using the definition of the Bernstein polynomial Eqn. (3). The derivative can be written as a linear combination of Bernstein polynomials [10] [11].

$$\frac{d}{dt}B_k^n(t) = \frac{d}{dt}\binom{n}{k}t^k(1-t)^{n-k} = n\left(B_{k-1}^{n-1}(t) - B_k^{n-1}\right), \quad 0 \le k \le n \qquad \ldots(4)$$

3. Bernstein Polynomials Approximation Method

In this section, we consider the Bernstein polynomial approximation solution. Any function $y(x) \in L^2([a,b])$ can be expanded into a Bernstein polynomial series of finite terms [10] [11]:

$$y(x) = c_0B_0^n + c_1B_1^n + c_3B_2^n + \cdots + c_nB_n^n, \qquad -\infty \le a \le x \le b \le \infty \qquad \ldots(5)$$

where $B_0^n(x), B_1^n(x), B_2^n(x), \ldots, B_n^n(x)$ are Bernstein polynomials terms which defined in Eqn. (3), $c_0, c_1, c_2, \ldots, c_n$ are unknown Bernstein polynomial coefficients, then Eqn. (5) can be decomposed as

$$y(x) = \sum_{i=0}^{n} c_iB_i^n(x) \qquad \ldots(6)$$

By utilizing Eqn. (6) and substituting into Eqn. (2), we obtain

$$\sum_{i=0}^{m}\mu_i(x)\left[\sum_{l=0}^{n}c_lB_l^n(x)\right]^i = f(x) + \int_a^x k_1(x,t)\left[\sum_{l=0}^{n}c_lB_l^n(t)\right]^r dt + \int_a^b k_2(x,t)\left[\sum_{l=0}^{n}c_lB_l^n(t)\right]^s dt \qquad \ldots(7)$$

The Eqn. (7) can be rewritten in a simplify form as

$$\begin{aligned}\sum_{i=0}^{m}\mu_i(x)\left[c_0B_0^n(x) + c_1B_1^n(x) + c_2B_2^n(x) + \ldots + c_nB_n^n(x)\right]^i &= f(x) \\ &+ \int_a^x k_1(x,t)\left[c_0B_0^n(t) + c_1B_1^n(t) + c_2B_2^n(t) + \ldots + c_nB_n^n(t)\right]^r dt \\ &+ \int_a^b k_2(x,t)\left[c_0B_0^n(t) + c_1B_1^n(t) + c_2B_2^n(t) + \ldots + c_nB_n^n(t)\right]^s dt\end{aligned} \qquad \ldots(8)$$

Finally, extension the Bernstein polynomial terms into Eqn. (8) by using Eqn. (3), we obtained

$$\begin{aligned}\sum_{i=0}^{m}\mu_i(x)\left[c_0(1-x)^n + c_1x(1-x)^{n-1} + c_2x^2(1-x)^{n-2} + \ldots + c_nx^n\right]^i &= f(x) \\ &+ \int_a^x k_1(x,t)\left[c_0(1-t)^n + c_1t(1-t)^{n-1} + c_2t^2(1-t)^{n-2} + \ldots + c_nt^n\right]^r dt \\ &+ \int_a^b k_2(x,t)\left[c_0(1-t)^n + c_1t(1-t)^{n-1} + c_2t^2(1-t)^{n-2} + \ldots + c_nt^n\right]^s dt\end{aligned} \qquad \ldots(9)$$

Now by integrating the terms in the right hand side of Eqn. (9) after simplifying and derivatives the terms in the left hand side using Eqn. (4).

Once this equation is simplified and represent as a nonlinear equation include x as a variable. Then substitute the collocation points x_i in the interval $[a,b]$, which can be calculated as follows

$$x_i = a + i\,\frac{b-a}{n}, \quad i = 0,1,2,\ldots,n \qquad \ldots(10)$$

the result obtained from Eqn. (9). At the end, we establish a system of nonlinear equations involve $(n+1)$ the unknown coefficients c_i, $i = 0,1,2,\ldots,n$ which can be determined by solving the nonlinear system using the Newton - Raphson method, which can be solved using MATLAB solver [12].

4. Numerical Examples and Results

In this section, we consider the following examples of nonlinear higher-order Volterra-Fredholm integro-differential equation. These examples are studied in [9] [13] by using differential transform method.

Example 1

Consider the nonlinear Volterra-Fredholm integro-differential equation as follows:

$$x^4 y^{(4)}(x) - y''(x) + y'(x) = -\frac{x^6}{30} - \frac{x^4}{6} - \frac{x^2}{2} + \frac{14}{3}x - \frac{1}{2} + \int_0^x (x-t)y^2(t)dt$$
$$-2\int_0^1 (x+t)y(t)dt, \qquad 0 \le x \le 1 \qquad \ldots(11)$$

with initial conditions $y(0)=1,\ y'(0)=0,\ y''(0)=2$ and $y'''(0)=0$ the exact solution is $y(x) = 1 + x^2$.

The numerical results of example1 are shown in Table1 and Figure (1). Table 1 has simulation results obtained from three orders of Bernstein polynomials as well as the exact solution with differential transforms method reslutes [13]. Figure (1) Shows the comparison between the exact solution and the approximate solutions for various n orders.

Table (1): Simulation results of Example 1 for *n*=1, 2, 3

X	*Exact solution*	*Differential transforms method*	*Bernstein polynomial method*		
			n=1	*n=2*	*n=3*
0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0100	1.0100	1.0000	1.0100	1.0100
0.2	1.0400	1.0400	1.0000	1.0400	1.0400
0.3	1.0900	1.0900	1.0000	1.0900	1.0900
0.4	1.1600	1.1600	1.0000	1.1600	1.1600
0.5	1.2500	1.2500	1.0000	1.2500	1.2500
0.6	1.3600	1.3600	1.0000	1.3600	1.3600
0.7	1.4900	1.4900	1.0000	1.4900	1.4900
0.8	1.6400	1.6400	1.0000	1.6400	1.6400
0.9	1.8100	1.8100	1.0000	1.8100	1.8100
1	2.0000	2.0000	1.0000	2.0000	2.0000

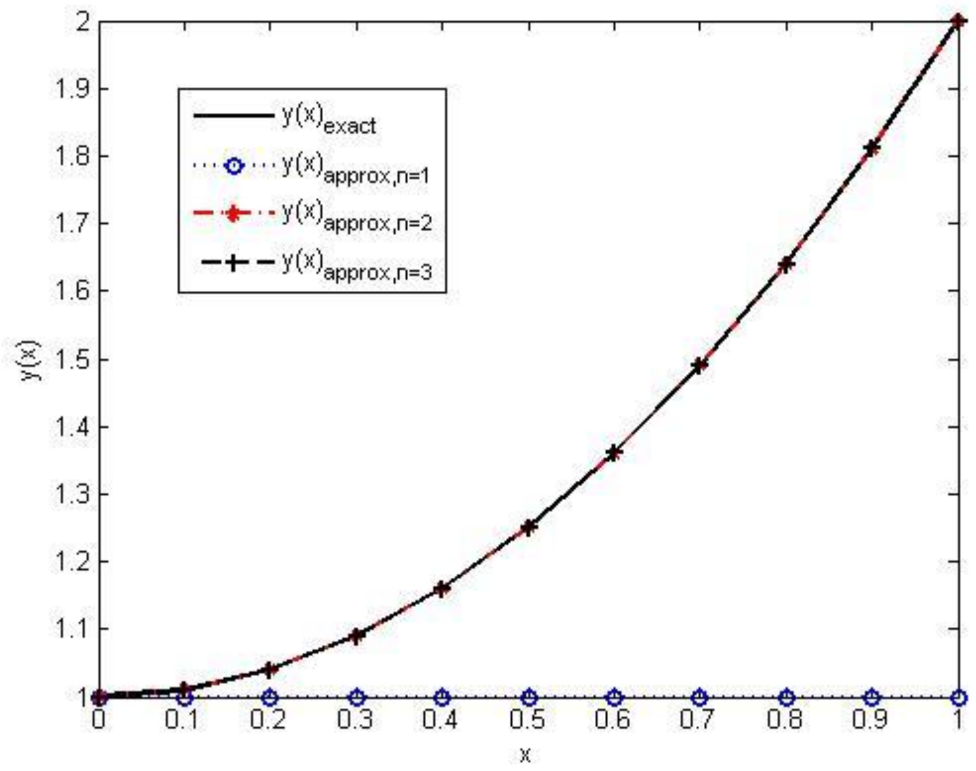

Figure (1): Approximation and exact solution of Example 1

Example 2

We Consider the nonlinear Volterra-Fredholm integro-differential equation as [13]:

$$y^{(8)}(x)-\pi^8 y(x)=\frac{x}{2}-\int_0^x y^2(t)\,dt+\frac{\sin(2\pi\, x)}{2\pi}\int_0^1[\cos(\pi\, t)-y(t)]y(t)\,dt \quad , \quad 0\le x\le 1 \ldots(12)$$

with initial conditions

$y(0)=0$, $y'(0)=\pi$, $y''(0)=0$, $y'''(0)=-\pi^3$, $y^{(4)}(0)=0$, $y^{(5)}(0)=\pi^5$, $y^{(6)}(0)=0$ and $y^{(7)}(0)=-\pi^7$.

The proposed method was applied to obtain the numerical solution for Example 2. In Table 2, the approximation solution for $n=1,3,4,6,7$ using the Bernstein polynomial method with differential transforms method [13] and the exact solution are listed. The computational results using the present method for $n=1,3,4,6,7$ together with exact solution are shown in Figure (2).

Table (2): Simulation results of Example 2 for *n*=1, 3, 4, 6, 7

x	*Exact solution*	*Differential transforms method*	*Bernstein polynomial method*				
			n=1	*n=3*	*n=4*	*n=6*	*n=7*
0	0	0	0	0	-0.0000	0	0.0000
0.1	0.3090	0.3090	0.3142	0.3090	0.3100	0.3090	0.3090
0.2	0.5878	0.5878	0.6283	0.5870	0.5954	0.5878	0.5878
0.3	0.8090	0.8090	0.9425	0.8029	0.8312	0.8091	0.8090
0.4	0.9511	0.9511	1.2566	0.9259	0.9929	0.9520	0.9510
0.5	1.0000	1.0000	1.5708	0.9248	1.0557	1.0045	0.9998
0.6	0.9511	0.9511	1.8850	0.7687	0.9949	0.9670	0.9503
0.7	0.8090	0.8090	2.1991	0.4266	0.7858	0.8552	0.8058
0.8	0.5878	0.5878	2.5133	-0.1326	0.4036	0.7030	0.5774
0.9	0.3090	0.3090	2.8274	-0.9398	-0.1764	0.5660	0.2794
1	0.0000	0.0000	3.1416	-2.0261	-0.9789	0.5240	-0.0752

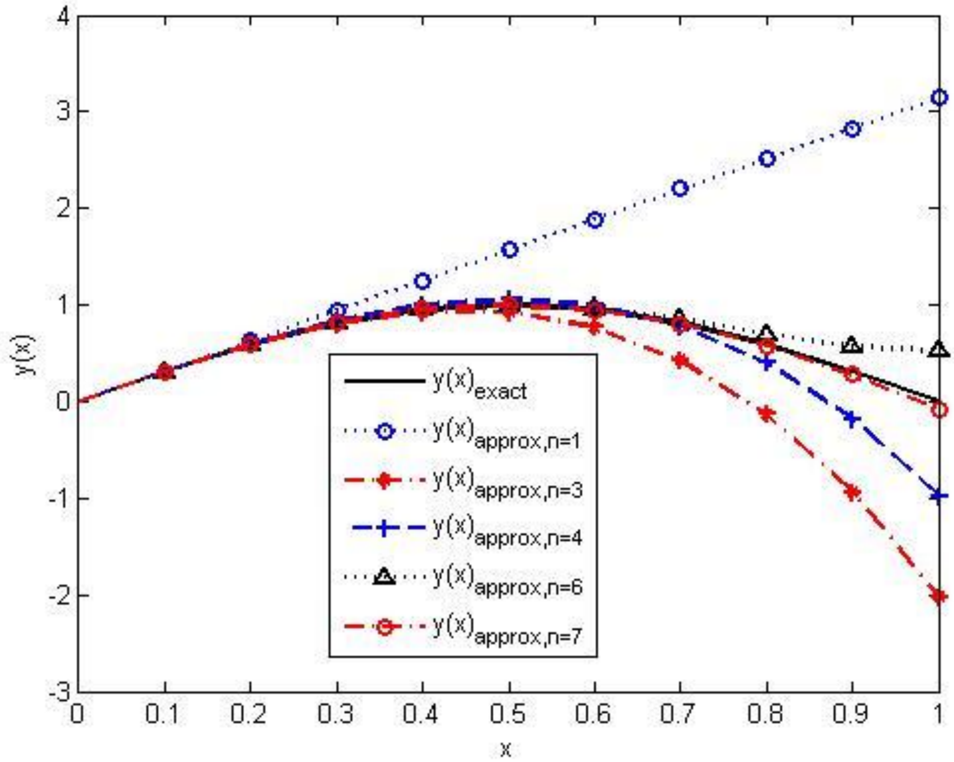

Figure (2): Approximation and exact solution of Example 2

5. Conclusion

Most integro differential equations are difficult to solve analytically, in many cases it requires to obtain the approximate solution. In this work, the Bernstein polynomial method for the solution of high-order nonlinear Volterra-Fredholm integro-differential equations is successfully implemented. The proposed approach is simple and has been tested on two examples to illustrate the efficiency of the present method. The simulation results are also compared with the other researcher's work. It is clear that from the figures and tables, more accurate results can be obtained by increasing the n^{th} degree of the Bernstein polynomial.

6. References

[1] Razzaghi, M. and Yousefi, S. "Legendre wavelets method for nonlinear Volterra-Fredholm integral equations". *Mathematics and Computers in Simulation*, Vol. 70, pp. 1-8. doi:10.1016/j.matcom.2005.02.035. (2005).

[2] Cerdik-Yaslan, H., & Akyuz-Dascioglu, A. "Chebyshev polynomial solution of nonlinear Fredholm-Volterra integro-differential equations". *J. Arts. Sci*, Vol. 5, pp. 89-101. (2006).

[3] Reihani, M.H. and Abadi, Z., "Rationalized Haar functions method for solving Fredholm and Volterra integral equations". *Journal of Computational and Applied Mathematics*, Vol. 200, pp. 12-20. doi:10.1016/j.cam.2005.12.026. (2007).

[4] Araghi, M.A. and Behzadi, Sh.S. "Solving nonlinear Volterra-Fredholm integro-differential equations using the modified Adomian decomposition method". *Computational Methods in Applied Mathematics*, Vol. 9, pp. 321-331. (2009).

[5] Yüzbasi, S., Sahin, N. and Yildirim, A., "A collocation approach for solving high-order linear Fredholm- Volterra integro-differential equations". *Mathematical and Computer Modelling*, Vol. 55, pp. 547-563. doi:10.1016/j.mcm.2011.08.032. (2012).

[6]. Kenneth I. Joy "Bernstein polynomials", University of California, Davis. , 2000.

[7] Jerri, A., "*Introduction to integral equations with applications*:" John Wiley & Sons. (1999).

[8] Wazwaz, A.-M., "*Linear and nonlinear integral equations: methods and applications*:" Springer. (2011).

[9] Wang W., "An algorithm for solving higher-order nonlinear Volterra-Fredholm integro-differential equation with mechanization". *Appl. Math. Comput.*; Vol. 172, pp. 1-23. (2006).

[10] Joy, K. I., "Bernstein polynomials". *On-line Geometric Modelling Notes, http://en. wikipedia. org/wiki/Bernstein*, pp. 1-13. (2000).

[11] Gzyl, H., & Palacios, J. L., "On the approximation properties of Bernstein polynomials via probabilistic tools". *Boletın de la Asociación Matemática Venezolana*, Vol. X, No. 1, pp. 5-13. (2003).

[12] Xue, D., & Chen, Y., "*Solving applied mathematical problems with MATLAB*:" CRC Press. pp. 172-173, (2011).

[13] Behiry, S. H., & Mohamed, S. I. "Solving high-order nonlinear Volterra-Fredholm integro-differential equations by differential transform method". *Natural Science*, Vol. 4, pp. 581-587. (2012).

استخدام متعدد حدود برنشتن لحل معادلة فولتيرا فريدهولم التفاضلية التكاملية غير الخطية ذات الرتب العالية

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الخلاصة:

في هذا البحث استعملت طريقة متعددة حدود برنشتن لإيجاد الحل التقريبي لمعادلة فولتيرا فريدهولوم التفاضلية التكاملية غير الخطية ذات الرتب العالية من النوع الثاني. بعض الامثلة اخُذت بالاعتبار والحل لها نُوقش عددياً وبُين بيانياً. وبزيادة درجة متعددة حدود برنشتن فان النتائج تكون أكثر دقة. كما ان النتائج تم مقارنتها مع نتائج سابقة .