

# Solving Partial Differential Equations by Homotopy Perturbation Method

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## ABSTRACT

In this paper, we submitted a good tool to solve linear and nonlinear partial differential equations which is called homotopy perturbation method. This method makes hard problems so easy to solve, in our paper we gave some various examples for linear and nonlinear partial differential equations by using this method.

## 1.INTRODUCTION

Recently, many mathematicians seek new techniques to find exact and approximate solutions for nonlinear partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. Some modern methods have been appeared like homotopy perturbation method which is analytic technique for solving linear and nonlinear problems. The first mathematician proposed this was Ji-Huan in 1999, [1] . This method gives an analytical exact and approximate solutions of nonlinear partial differential equations easily without transforming the equation or linearizing the problem with a very good results. In [2], Abdul-Sattar J. AL-Saif and Dhifaf A.Abood solved Korteweg-de Vries (Kdv) equation and convergence study of homotopy perturbation method [3], R. Taghipour used the the homotopy perturbation method to solve linear and nonlinear parabolic equations, [4] A.A. Hemeda presented the modification homotopy perturbation method of the fractional order in Caputo sense and [5] D.D. Ganji, H. Mirgolbabaei, Me. Mjansari and Mo. Miansari employed the homotopy perturbation method to find solutions of linear and nonlinear systems of ordinary differential equations and differential equations of the order three. In this work, we present homotopy perturbation method for solving inhomogeneous heat problem and vibrating beam problem of the fourth order as linear examples,

## Solving Partial Differential Equations by Homotopy Perturbation Method .....Aqeel Falih Jaddoa

inhomogeneous advection problem as nonlinear example and one system of nonlinear partial differential equations.

### 2. Analysis of the Homotopy Perturbation Method

In this section we submit the homotopy perturbation method by studying the following differential equation:-

$$A(u) - f(r) = 0, \quad r \in \Omega \quad \dots (1)$$

Considering the boundary conditions of:-

$$B\left(u, \frac{\partial u}{\partial r}\right) = 0, \quad r \in \Gamma \quad \dots (2)$$

Where A is a general differential operator, B a boundary operator, f(r) a known analytic function and  $\Gamma$  is the boundary of the domain  $\Omega$ .

The operator A can be generally divided into two parts of L and N where L is linear part, while N is the nonlinear part in the differential equation, so we can rewrite the equation (1) as:

$$L(u) + N(u) - f(r) = 0 \quad \dots (3)$$

By using Homotopy technique, we construct a homotopy as:

$v(r, p): \Omega \times [0, 1] \rightarrow R$ , which satisfies:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \quad \dots (4)$$

Or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad \dots (5)$$

Where  $p \in [0, 1]$  is an embedding parameter and  $u_0$  is an initial approximation of the differential equation which satisfies boundary conditions.

Obviously, considering equation (4) and (5), we will have:-

$$H(v, 0) = L(v) - L(u_0) = 0 \quad \dots (6)$$

$$H(v, 1) = L(v) + N(v) - f(r) = 0 \quad \dots (7)$$

By changing process of p from zero to unity is just of  $v(r, p)$  from  $u_0(r)$  to  $u(r)$ .

In topology, this is called deformation and  $L(v) - L(u_0)$ ,  $L(v) + N(v) - f(r)$  are homotopy.

The basic assumption is that the solution of the equation (4) and (5) can be expressed as a power series in p:

$$v(x, t) = v_0 + pv_1 + p^2v_2 + \dots \quad \dots (8)$$

Setting  $p=1$  results in the approximate solution of the differential equation will be:-

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad \dots (9)$$

The series (9) is convergent for most cases.

However, the convergence rate depends on nonlinear operator, so we will give some applications of method.

### 3. Application of the HPM.

#### Example 3.1

The inhomogeneous heat problem : -

$$\left. \begin{aligned} u_t &= u_{xx} + \sin x \\ u(x, 0) &= \cos x \end{aligned} \right\} \dots (10)$$

And a given solution  $u(x,t) = \cos x e^{-t} + \sin x (1 - e^{-t})$ , [6,p. 81]

A corresponding to the homotopy perturbation method we will have:-

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p \left[ \frac{\partial^2 u}{\partial x^2} + \sin x - \frac{\partial u_0}{\partial t} \right] \dots (11)$$

We suppose that the solution of the problem (10) is in the form:-

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \dots (12)$$

We substitute (12) into (11) and we equate the coefficients of like power p, we will have the set of differential equations:-

$$\left. \begin{aligned} p^0: \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} &= 0 \\ p^1: \frac{\partial u_1}{\partial t} &= \left[ \frac{\partial^2 u_0}{\partial x^2} + \sin x - \frac{\partial u_0}{\partial t} \right] \\ p^2: \frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} \\ p^3: \frac{\partial u_3}{\partial t} &= \frac{\partial^2 u_2}{\partial x^2} \\ p^4: \frac{\partial u_4}{\partial t} &= \frac{\partial^2 u_3}{\partial x^2} \\ &\vdots \end{aligned} \right\} \dots (13)$$

Solve equations (13) to get the solutions:-

$$\left. \begin{aligned} u_1 &= (\sin x - \cos x)t \\ u_2 &= (\sin x - \cos x) \frac{t^2}{2!} \\ u_3 &= (\sin x - \cos x) \frac{t^3}{3!} \\ u_4 &= (\sin x - \cos x) \frac{t^4}{4!} \\ &\vdots \end{aligned} \right\} \dots (14)$$

So the solution will be

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + u_3 + u_4 \dots \quad (15) \\ &= \cos x + (\sin x - \cos x)t + (\sin x - \cos x) \frac{t^2}{2!} + (\sin x - \cos x) \frac{t^3}{3!} + (\sin x - \cos x) \frac{t^4}{4!} + \dots \\ &= \cos x \left( 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) + \sin x \left( t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) \end{aligned}$$

## Solving Partial Differential Equations by Homotopy Perturbation Method .....Aqeel Falih Jaddoa

$$u(x,t) = \cos x e^{-t} + \sin x (1 - e^{-t}).$$

### Example 3.2

The inhomogeneous advection problem :-

$$\left. \begin{aligned} u_t + uu_x &= x \\ u(x,0) &= 2 \end{aligned} \right\} \dots(16)$$

And a given solution  $u(x,t) = 2 \operatorname{sech} t + x \tanh t$ , [6,p. 358]

A corresponding to the homotopy perturbation method we will have:-

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p \left[ -u \frac{\partial u}{\partial x} + x - \frac{\partial u_0}{\partial t} \right] \dots(17)$$

We suppose that the solution of the problem (16) takes the form:-

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \dots(18)$$

We substitute (18) into (17) and we equate the coefficients of like power p, we will have the set of differential equations:-

$$\left. \begin{aligned} p^0: \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} &= 0 \\ p^1: \frac{\partial u_1}{\partial t} &= -u_0 \frac{\partial u_0}{\partial x} + x - \frac{\partial u_0}{\partial t} \\ p^2: \frac{\partial u_2}{\partial t} &= -\left( u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} \right) \\ p^3: \frac{\partial u_3}{\partial t} &= -\left( u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} \right) \\ p^4: \frac{\partial u_4}{\partial t} &= -\left( u_0 \frac{\partial u_3}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_3 \frac{\partial u_0}{\partial x} \right) \\ &\vdots \end{aligned} \right\} \dots(19)$$

Solve equations (19) to get the solutions:-

$$\left. \begin{aligned} u_1 &= xt \\ u_2 &= -t^2 \\ u_3 &= -x \frac{t^3}{3} \\ u_4 &= \frac{5}{12} t^4 \\ &\vdots \end{aligned} \right\} \dots(20)$$

So the solution will be:-

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + u_3 + u_4 + \dots \dots(21) \\ &= 2 + xt - t^2 - x \frac{t^3}{3} + \frac{5}{12} t^4 + \dots \\ &= 2 - t^2 + \frac{5}{12} t^4 + x \left( t - \frac{1}{3} t^3 \right) + \dots \\ &= 2 \left( 1 - \frac{t^2}{2!} + \frac{5}{4!} t^4 - \dots \right) + x \left( t - \frac{1}{3} t^3 + \dots \right) \end{aligned}$$

$$= 2 \operatorname{secht} + x \operatorname{tanht} .$$

**Example 3.3**

The vibrating beam problem of the fourth order:-

$$\left. \begin{aligned} u_{tt} &= -u_{xxxx} \\ u(x, 0) &= \sin \pi x + 0.5 \sin 3\pi x \end{aligned} \right\} \dots(22)$$

And a given solution  $u(x, t) = \sin \pi x \cos \pi^2 t + (0.5) (\sin 3\pi) (\cos 9\pi^2 t)$ , [7, p. 165]

A corresponding to the homotopy perturbation method we will have:-

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u_0}{\partial t^2} = p \left[ -\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u_0}{\partial x^2} \right] \dots(23)$$

We suppose that the solution of the problem (22) in the form:-

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \dots(24)$$

We substitute (24) into (23) and we equate the coefficients of like power p, we will have the set of differential equations:-

$$\left. \begin{aligned} p^0: \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} &= 0 \\ p^1: \frac{\partial^2 u_1}{\partial t^2} &= -\frac{\partial^4 u_0}{\partial x^4} \\ p^2: \frac{\partial^2 u_2}{\partial t^2} &= -\frac{\partial^4 u_1}{\partial x^4} \\ p^3: \frac{\partial^2 u_3}{\partial t^2} &= -\frac{\partial^4 u_2}{\partial x^4} \\ p^4: \frac{\partial^2 u_4}{\partial t^2} &= -\frac{\partial^4 u_3}{\partial x^4} \\ &\vdots \end{aligned} \right\} \dots(25)$$

Solve equations (25) to get the solutions:-

$$\left. \begin{aligned} u_1 &= -\pi^4 (\sin \pi x + (3^4)(0.5) \sin 3\pi x) \frac{t^2}{2} \\ u_2 &= \pi^8 (\sin \pi x + (3^8)(0.5) \sin 3\pi x) \frac{t^4}{4!} \\ u_3 &= -\pi^{12} (\sin \pi x + (3^{12})(0.5) \sin 3\pi x) \frac{t^6}{6!} \\ u_4 &= \pi^{16} (\sin \pi x + (3^{16})(0.5) \sin 3\pi x) \frac{t^8}{8!} \\ &\vdots \end{aligned} \right\} \dots(26)$$

So the solution will be:-

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots \dots(27)$$

$$\begin{aligned}
 &= (\sin \pi x + 0.5 \sin 3\pi x) - \pi^4 (\sin \pi x + (3^4)(0.5) \sin 3\pi x) \frac{t^2}{2} \\
 &\quad + \pi^8 (\sin \pi x + (3^8)(0.5) \sin 3\pi x) \frac{t^4}{4!} - \pi^{12} (\sin \pi x + (3^{12})(0.5) \sin 3\pi x) \frac{t^6}{6!} \\
 &\quad + \pi^{16} (\sin \pi x + (3^{16})(0.5) \sin 3\pi x) \frac{t^8}{8!} - \dots \\
 &= \sin \pi x \left( 1 - \frac{(\pi^2 t)^2}{2!} + \frac{(\pi^2 t)^4}{4!} - \frac{(\pi^2 t)^6}{6!} + \frac{(\pi^2 t)^8}{8!} - \dots \right) \\
 &\quad + (0.5) \sin 3\pi x \left( 1 - \frac{(3^2 \pi^2 t)^2}{2!} + \frac{(3^2 \pi^2 t)^4}{4!} - \frac{(3^2 \pi^2 t)^6}{6!} + \frac{(3^2 \pi^2 t)^8}{8!} - \dots \right) \\
 &= \sin \pi x \cos \pi^2 t + (0.5) (\sin 3\pi) (\cos 9\pi^2 t) .
 \end{aligned}$$

**Example 3.4**

Consider the nonlinear system of equations with initial conditions:-

$$\left. \begin{aligned}
 u_t &= uu_x + vu_y \\
 v_t &= uv_x + vv_y
 \end{aligned} \right\} \dots (28)$$

$$u(x, y, 0) = v(x, y, 0) = x + y$$

The solution is given by  $u(x, y, 0) = v(x, y, 0) = \frac{x+y}{1-2t}$ , [8]

A corresponding to the homotopy perturbation method we will have:-

$$\left. \begin{aligned}
 \frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} &= p \left[ u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} - \frac{\partial u_0}{\partial t} \right] \\
 \frac{\partial v}{\partial t} - \frac{\partial v_0}{\partial t} &= p \left[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \frac{\partial v_0}{\partial t} \right]
 \end{aligned} \right\} \dots (29)$$

Suppose that the solution for the system (28) of the form:-

$$\left. \begin{aligned}
 u &= u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \\
 v &= v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots
 \end{aligned} \right\} \dots (30)$$

We substitute (30) into (29) and we equate the coefficients of like power p, we will have the set of differential equations:-

$$\begin{aligned}
 p^0: \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} &= 0, \\
 \frac{\partial v_0}{\partial t} - \frac{\partial v_0}{\partial t} &= 0 \\
 p^1: \frac{\partial u_1}{\partial t} &= u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} \\
 \frac{\partial v_1}{\partial t} &= u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} \\
 p^2: \frac{\partial u_2}{\partial t} &= u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} \\
 \frac{\partial v_2}{\partial t} &= u_0 \frac{\partial v_1}{\partial x} + u_1 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_1}{\partial y} + v_1 \frac{\partial v_0}{\partial y} \\
 p^3: \frac{\partial u_3}{\partial t} &= u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_2}{\partial y} + v_1 \frac{\partial u_1}{\partial y} + v_2 \frac{\partial u_0}{\partial y} \\
 \frac{\partial v_3}{\partial t} &= u_0 \frac{\partial v_2}{\partial x} + u_1 \frac{\partial v_1}{\partial x} + u_2 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_2}{\partial y} + v_1 \frac{\partial v_1}{\partial y} + v_2 \frac{\partial v_0}{\partial y} \\
 &\vdots \\
 &\vdots
 \end{aligned}
 \tag{31}$$

Solve equations (31) to get the solutions:-

$$\begin{aligned}
 u_1 &= (x+y)(2t), & v_1 &= (x+y)(2t) \\
 u_2 &= (x+y)(2t)^2, & v_2 &= (x+y)(2t)^2 \\
 u_3 &= (x+y)(2t)^3, & v_3 &= (x+y)(2t)^3 \\
 u_4 &= (x+y)(2t)^4, & v_4 &= (x+y)(2t)^4 \\
 &\vdots & & \\
 &\vdots & & \\
 u_n &= (x+y)(2t)^n, & v_n &= (x+y)(2t)^n
 \end{aligned}
 \tag{32}$$

Solve the solution will be:-

$$\begin{aligned}
 u(x,t) &= u_0 + u_1 + u_2 + u_3 + \dots \tag{33} \\
 &= (x+y) + (x+y)(2t) + (x+y)(2t)^2 + (x+y)(2t)^3 + \dots \\
 &= (x+y)(1 + 2t + (2t)^2 + (2t)^3 + \dots) \\
 &= (x+y) \frac{1}{1-2t} = \frac{x+y}{1-2t}
 \end{aligned}$$

$$\begin{aligned}
 v(x,t) &= v_0 + v_1 + v_2 + v_3 + \dots \tag{34} \\
 &= (x+y) + (x+y)(2t) + (x+y)(2t)^2 + (x+y)(2t)^3 + \dots \\
 &= (x+y)(1 + 2t + (2t)^2 + (2t)^3 + \dots) \\
 &= (x+y) \frac{1}{1-2t} = \frac{x+y}{1-2t}
 \end{aligned}$$

## **CONCLUSION**

The results of this method give us evidence that the homotopy perturbation method is very effective and have a high accuracy to find solutions for the partial differential equations. We proved our claim when we obtained solutions of problems that equal even to the exact solution and the same results for modern methods like Adomian's decomposition method.

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## **المستخلص**

في هذا البحث قدمنا اداة جيدة لحل معادلات تفاضلية جزئية خطية وغير خطية والتي تسمى طريقة الهوموتوبي الترجافية. هذه الطريقة تجعل المسائل الصعبة أسهل حلها, في بحثنا أعطينا أمثلة متنوعة لمعادلات تفاضلية خطية وغير خطية باستخدام هذه الطريقة.