Mathematical Study Of Weighted Generalized Exponential Distribution

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Abstract

This paper deals with studying a weighted Generalized Exponential distribution with the known shape parameter α=2 and scale parameter λ>0 and driving it's probability distribution function, cumulative distribution function , reliability function. Also driving the moment generating function and other moments.Also we driving the estimator for λ by MLE and MOM.

Keywords: Weighted Generalizedd Exponential ,Scale parameter ,Cumulative distribution function, Reliability function ,Hazard function , MLE ,MOM

1.Introducion

The weighted probability distribution ,is used in many fields of real life such as medicine ,ecology , reliability ,and so on .The concept of weighted distribution can be traced to the work of Fisher(1943)[2] in connection with his studies on how method of ascertainment can influence the from of distribution of recorded observations .

Later it was introduced and formulated in general terms by Rao (1965) [8] in connection of modeling statistical data.

The usefulness and application of the weighted distribution to biased samples in various area including medicine , ecology and branching process in Patil and Rao (1978)[6] ,Gupta and Keating(1985)[3] and Gupta and Kirmani(1990)[4].

Also El-Shaarawi ,Abdel H.(2002)[1] represent the size-biased sampling and say that the suitable weight function is w(x, β) then he drawing references about the parameter . Application examples for weighted distribution,was introduced by Jing(2010)[5] and Priyadarshani (2011)[7]

Here we introduce the one parameter Weighted Generalizedd Exponential distribution and some important concepts.

The mathematical definition of the weighted distribution is as follows ,let (Ω,r,p) be a probability space X:Ω→H be a random variable (r.v)where H=(a,b) be an interval on real line with (a>0) and (b>a) can be finite or infinite. When the distribution function F(x) of x is absolutely continuous with probability density function f(x) and w(x) be anon-negative weight function satisfying W_W=E[w(x)] < ∞, then the r.v. X_W having pdf:

$$f_w(x) = \frac{w(x)f(x)}{W_D} \qquad a<x<b \qquad \text{....................(1)}$$

Is said to have weighted distribution where:

$$W_D = \int_0^\infty w(x)f(x)dx \qquad \text{…………………. (2)}$$

2.The aim of this papers:

Derivation of one parameter Weighted Generalized Exponential (WGED) with some of statistical properties.

3. Generalized Exponential Distribution (GED):

The pdf of (GED) defined by:

$$f(x)=\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x} \qquad x>0 \ , \ \alpha,\lambda>0 \qquad \text{…………(3)}$$

where λ is scale parameter and α is the shape parameter ,when α=2 , equation (2) is:

$$f(x)=2\lambda\left(1-e^{-\lambda x}\right) e^{-\lambda x}\text{………………………(4)}$$

one parameter (GED).

4.Weighted Generalized Exponential Distribution (WGED):

Consider the weight function w(x)=x and the (GED) is given by (4) , then :

4.1 the pdf of (WGED):

$$W_D = \int_0^\infty w(x)f(x)dx$$

$$=\int_0^\infty x\, 2\lambda\left(1-e^{-\lambda x}\right) e^{-\lambda x}\, dx$$

$$=2\lambda \int_0^\infty x\, e^{-\lambda x}\, dx - 2\lambda \int_0^\infty x\, e^{-2\lambda x}\, dx$$

To solve: $2\lambda \int_0^\infty x\, e^{-\lambda x}\, dx$

Let y= $\lambda x \rightarrow x=\frac{y}{\lambda} \rightarrow dx=\frac{dy}{\lambda}$

Then $2\lambda \int_0^\infty x\, e^{-\lambda x}\, dx = 2 \int_0^\infty y\, e^{-y} \frac{dy}{\lambda}$

$=\frac{2}{\lambda}\, \Gamma(2) =\frac{2}{\lambda}$

And the same we solve :

$2\lambda \int_0^\infty x\, e^{-2\lambda x}\, dx =\frac{1}{2\lambda}$

$W_D=\frac{2}{\lambda} - \frac{1}{2\lambda}$

$W_D =\frac{3}{2\lambda}$

Then the pdf of(WGED) is:

$f_w(x)=\frac{4}{3}\, x\, \lambda^2 \left(1-e^{-\lambda x}\right) e^{-\lambda x}$ ……………….(5)

4.2 the cdf of (WGED):

$F_w(x)= \int_0^x f_w(x)dx$

$=\int_0^x \frac{4}{3}\, x\, \lambda^2 \left(1-e^{-\lambda x}\right) e^{-\lambda x} dx$

$F_W(x) =-\frac{4}{3}\, x\lambda\, e^{-\lambda x}-\frac{4}{3} e^{-\lambda x}+\frac{2}{3}\, x\lambda\, e^{-2\lambda x}+\frac{1}{3}\, e^{-2\lambda x}+1$……………(6)

5.The Statistical Properties of (WGED):

In this section , we present the statistical properties of (WGED) throughout computing the moment generating function ,mean , variance, the r^{th} moment , reliability function ,hazard function and the reverse hazard function as follow :

5-1.The moment generating function of this distribution is:

$M_x(t)=E(e^{tx}) =\int_0^\infty e^{tx} fw(x)dx$

$=\int_0^\infty \frac{4}{3}\, e^{tx}\, x\, \lambda^2 \left(1-e^{-\lambda x}\right) e^{-\lambda x} dx$

$=\frac{4}{3}\, \lambda^2 \int_0^\infty x\, e^{tx-\lambda x}\, dx-\frac{4}{3}\, \lambda^2 \int_0^\infty x\, e^{tx-2\lambda x}\, dx$

To solve: $\frac{4}{3}\, \lambda^2 \int_0^\infty x\, e^{tx-\lambda x}\, dx$

$=\frac{4}{3}\, \lambda^2 \int_0^\infty x\, e^{-x(-t+\lambda)}\, dx$

Let $x(-t+\lambda)=y \rightarrow x=\frac{y}{-t+\lambda}$

$dx = \frac{dy}{-t+\lambda}$

(5)

then

$\frac{4}{3}\, \lambda^2 \int_0^\infty x\, e^{-x(-t+\lambda)}\, dx=\frac{4}{3}\, \lambda^2 \int_0^\infty \frac{y}{-t+\lambda}\, e^{-y}\, \frac{dy}{-t+\lambda}$

$=\frac{4}{3(-t+\lambda)^2}\, \lambda^2 \int_0^\infty y\, e^{-y}\, dy \rightarrow= \frac{4}{3(-t+\lambda)^2}\, \lambda^2\, \Gamma(2)$

$=\frac{4}{3(-t+\lambda)^2}\, \lambda^2$

Now to solve **:** $\boldsymbol{\frac{4}{3}\, \lambda^2 \int_0^\infty x\, e^{tx-2\lambda x}\, dx}$

$=\frac{4}{3}\, \lambda^2 \int_0^\infty x\, e^{-x(-t+2\lambda)}\, dx$

Let $x(-t+2\lambda)=y \rightarrow x=\frac{y}{-t+2\lambda} \rightarrow dx = \frac{dy}{-t+2\lambda}$

Then:

$\frac{4}{3}\lambda^2\int_0^\infty x\,e^{-x(-t+2\lambda)}\,dx=\frac{4}{3}\lambda^2\int_0^\infty \frac{y}{-t+2\lambda}\,e^{-y}\,\frac{dy}{-t+2\lambda}$

$=\frac{4}{3(-t+2\lambda)^2}\lambda^2\int_0^\infty y\,e^{-y}\,dy$

$=\frac{4}{3(-t+2\lambda)^2}\lambda^2\,\Gamma(2)$

$=\frac{4}{3(-t+2\lambda)^2}\lambda^2$

Then:

$M_x(t)=\frac{4}{3(-t+\lambda)^2}\lambda^2-\frac{4}{3(-t+2\lambda)^2}\lambda^2$ ……………………………..(7)

5-2.The mean:

$E(x)=\mu=\int_0^\infty x f w(x)dx$

$=\int_0^\infty \frac{4}{3}\,x\,x\,\lambda^2\left(1-e^{-\lambda x}\right)e^{-\lambda x}dx$

$=\int_0^\infty \frac{4}{3}\,x^2\,\lambda^2\left(1-e^{-\lambda x}\right)e^{-\lambda x}dx$

$\therefore E(x)=\frac{7}{3\lambda}$ ……………………………..(8)

5-3.The variance:

$\sigma^2=E(x^2)-E^2(x)$

$\mathbf{E(x^2)}=\frac{4}{3}\lambda^2\int_0^\infty x^3\left(1-e^{-\lambda x}\right)e^{-\lambda x}dx$

$=\frac{15}{2\lambda^2}$

Then:

$\sigma^2=\frac{37}{18\lambda^2}$ ………………………………….(9)

5-4.The the rth moment:

$\mathbf{E(x^r)}=\frac{4}{3}\lambda^2\int_0^\infty x^{r+1}\left(1-e^{-\lambda x}\right)e^{-\lambda x}dx$

Then

$\mathbf{E(x^r)}=\frac{(2^{r+2}-1)}{3(2\lambda)^2}\Gamma(r+2)$ ………………………….....(10)

For the case r=1,2,3,4 we have:

r=1 $\mathbf{E(x)}=\frac{7}{3\lambda}$

r=2 $\mathbf{E(x^2)}=\frac{15}{2\lambda^2}$

r=3 $\mathbf{E(x^3)}=\frac{31}{\lambda^3}$

r=4 $\mathbf{E(x^4)}=\frac{315}{2\lambda^4}$

5-5.The reliability function:

This function can be derived using the cumulative distribution function and given by:

$R(X)=1-F_W(x)$

$R(X)=\frac{4}{3}x\lambda\, e^{-\lambda x}+\frac{4}{3}e^{-\lambda x}-\frac{2}{3}x\lambda\, e^{-2\lambda x}-\frac{1}{3}\, e^{-2\lambda x}$ ………………(11)

5-6.The hazard function:

$H(X)=\frac{f_w(x)}{R(x)}$

$H(x)=\frac{2x\lambda^2(1-e^{-\lambda x})\,e^{-\lambda x}}{2x\lambda\, e^{-\lambda x}+2e^{-\lambda x}-x\lambda\, e^{-2\lambda x}-\frac{1}{2}\, e^{-2\lambda x}}$ ……………………..(12)

5-7.The reverse hazard function:

$\varphi(x)=\frac{f_w(x)}{F_w(x)}$

$\varphi(x)=\frac{2x\lambda^2(1-e^{-\lambda x})\,e^{-\lambda x}}{-2x\lambda\, e^{-\lambda x}-2e^{-\lambda x}+x\lambda\, e^{-2\lambda x}+\frac{1}{2}\, e^{-2\lambda x}+\frac{3}{2}}$ …………….……..(13)

6-Estimation of the scale parameter

In this section , estimates of the scale parameter λ of the (WEGD) by method of moment (MOM) and maximum likelihood estimators (MLE).

6.1- Method of moment estimators (MOM)

To find the moment estimate's λ we solve the equation:

$m_r=E(x^r)$

$m_r=\frac{\sum_{i=1}^{n}x_i^r}{n}$

$E(x)=m_1=\frac{7}{3\lambda}$

$\mathbf{E(x^2)}=m_2=\frac{\sum_{i=1}^{n}x_i^2}{n}$

$\rightarrow\frac{\sum_{i=1}^{n}x_i^2}{n}=\frac{45}{6\lambda^2}\rightarrow 6\lambda^2.\frac{\sum_{i=1}^{n}x_i^2}{n}=45\rightarrow 6\lambda\,.\frac{7}{3\bar{x}}.\frac{\sum_{i=1}^{n}x_i^2}{n}=45$

$14\lambda\,.\frac{\sum_{i=1}^{n}x_i^2}{n}=45\,\bar{x}$

$\hat{\lambda}_{MOM}=\frac{45n\,\bar{x}}{14\sum_{i=1}^{n}x_i^2}$ …………………………..(14)

6.2- Maximum likelihood estimators (MLE)

$L(\lambda,x)=\prod_{i=1}^{n} f(xi,\lambda)$

$=\prod_{i=1}^{n}[\frac{4}{3}\ x\ \lambda^2\left(1-e^{-\lambda x}\right)e^{-\lambda x}]$

$=(\frac{4^n}{3})\ \lambda^{2n}.\sum_{i=1}^{n} x_i\ .\prod_{i=1}^{n}\left(1-e^{-\lambda x}\right).\ e^{-\lambda\sum_{i=1}^{n} x_i}$.

$\log L=n\ \log\frac{4}{3}+2n\log\lambda+\log\sum_{i=1}^{n} x_i$

$+\lambda\sum_{i=1}^{n} x_i+\sum_{i=1}^{n}\log\left(1-e^{-\lambda x}\right)-\lambda\sum_{i=1}^{n} x_i$

$$\frac{d\log L}{d\lambda}=\frac{2n}{\lambda}-\sum_{i=1}^{n} x_i+\sum_{i=1}^{n}\frac{x_i\ e^{-\lambda x_i}}{1-e^{-\lambda x_i}}=0$$

$$\frac{2n}{\hat{\lambda}}=\sum_{i=1}^{n} x_i-\sum_{i=1}^{n}\frac{x_i\ e^{-\hat{\lambda} x_i}}{1-e^{-\hat{\lambda} x_i}}$$

$$\hat{\lambda}_{MLM}=\frac{2n}{\sum_{i=1}^{n} x_i-\sum_{i=1}^{n}\frac{x_i\ e^{-\hat{\lambda} x_i}}{1-e^{-\hat{\lambda} x_i}}}\qquad \ldots\ldots\ldots\ldots(15)$$

7-Conclusion

The weighted probability distributions are used in many fields of real life such as medicine ,ecology , reliability and other fields ,here present one of the weighted derived from Generalized exponential distribution using weight function .Estimated need special procedure like Newton Raphson method or other methods to solve the estimated and obtain the solution of situation. For the researchers they can use the simulation procedure to find the estimator λ.

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الملخص:

يهتم هذا البحث بدراسة رياضية للتوزيع الاسي العام الموزون بمعلمة الشكل المعلومة $\alpha=2$ ومعلمة القياس المجهولة λ ,وذلك بدراسة واشتقاق بعض الخــواص الإحصائية المهمة الخاصة بهذا التوزيع لدالة الكثافة الاحتمالية ,الدالة التجميعية,دالة المعولية وغيرها من الدوال. كذلك تم اشتقاق الدالة المولدة للعزوم و العزوم الأخرى .ومن ثم اشتقاق تقدير للمعلمة λ بطريقتي الإمكان الأعظم و العزوم.