Prey-predator food web model with stage structure for predator

Alla tareq balasim

Abstract :

A three species food web model with stage structure for predator is proposed and analyzed. The boundness and local stability analysis was investigated. The global dynamic of the subsystem has been investigated numerically. Also the global stability of the axial equilibrium point was investigated using lyapunov function.

Introduction:

The idea about prey-predator model based on Lotka-Voltera prey-predator model which assumed that all predators of any species attack and depends for its growth on any type of prey regardless to its maturity. A similar models of prey-predator has been studied by many authors see [1-2], however, recently most research classify prey-predator model depending on life cycle of predators as mature and immature, the immature depending on its feeding on a mature one (parents), because of their weakness. So their ability for attack will be negligible, this pattern used by some authors like Wang and Chen [3] Xiao and Chan [4] and Beretta and Kuang [5] for stage model with time delays due to the gestation of the predator and the crowding of the prey. For asymptotic behavior pf predator-prey system, it is known from Poincare-Bendixson theorem that two dimensional continues time model can approach either an equilibrium point or a limit cycle, while three and higher dimensional models can exhibit complex behavior.

The mathematical model:

The three species food web model consisting of two preys x_1, x_2 and predators y_1, y_2 with stage structure for predator can be represented mathematically by the following system of differential equations:

$$\frac{dx_1}{dt} = x_1(r_1 - a_1x_1) - x_1y_2$$
$$\frac{dx_2}{dt} = x_2(r_2 - a_2x_2) - x_2y_2$$
$$\frac{dy_1}{dt} = (e_1x_1 + e_2x_2)y_2 - (m + d_1)y_1$$
$$\frac{dy_2}{dt} = m\,y_1 - d_1y_2$$

(1)

In above model r_i, a_i, e_i, d_i and m are the model parameters which assuming only positive values. The prey x_i grows with intrinsic growth rate

r_i and carrying capacity $\frac{r_i}{a_i}$ in the absence of predator. y_1, y_2 are the immature and mature predator at time t. The constant e_i being the search rate, y_1, y_2 represent the death rate of immature and mature predator. The constant m was the rate at which immature predator becomes mature predator.

For a biologically food web model to be logically credible the model must be split into subsystem [7].

The first subsystem is obtained by assuming the absence of the second prey x_2

$$\frac{dx_1}{dt} = x_1(r_1 - a_1x_1) - x_1y_2$$
$$\frac{dy_1}{dt} = e_1\,x_1\,y_2 - (m + d_1)y_1$$
$$\frac{dy_2}{dt} = m\,y_1 - d_1y_2$$

(2)

While the second subsystem is obtained when x_2 absent

$$\frac{dx_1}{dt} = x_1(r_1 - a_1x_1) - x_1y_2$$
$$\frac{dy_1}{dt} = e_1\,x_1\,y_2 - (m + d_1)y_1$$
$$\frac{dy_2}{dt} = m\,y_1 - d_1y_2$$

(3)

Analysis of the system

It's clear that the interaction function of system (1) are continues and have continues partial derivative in R_4^+. The solution of system (1) with non-negative initial values is bounded as shown in the following theorem. Furthermore, the system is said to be dissipative if all population are uniformly limited by their environment [7].

Theorem 1: System (1) is dissipative.

Proof from first equation of system (1) we get

$$\frac{dx_1}{dt} \le x_1(r_1 - a_1x_1)$$

By usual comparison theorem [8], for $x_1(0) = x_{10} > 0$ we have

$x_1(t) \le \frac{r_1}{a_1 + c_1e^{-r_1t}}$ for $t \ge 0$ and $c = \frac{1}{x_{10}}$

Accordingly, as $t \to \infty$ we get $x_1(t) \le \frac{r_1}{a_1}$.

Similarly we get $x_2(t) \le \frac{r_2}{a_2}$

Now let

$$w(t) = e_1x_1(t) + e_2x_2(t) + y_1 + y_2$$
$$w'(t) = e_1x_1'(t) + e_2x_2'(t) + y_1' + y_2'$$
$$\le e_1r_1x_1 + e_2r_2x_2 - d_1y_1 - d_2y_2$$
$$\le e_1x_1 + e_2x_2 - \min(d_1 + d_2)(y_1 + y_2)$$
$$= e_1x_1(1+d) + e_2x_2(1+d) - d(e_1x_1 + e_2x_2 + y_1 + y_2)$$
$$\le \left(\frac{e_1r_1}{a_1} + \frac{e_2r_2}{a_2}\right)(1+d) - dw$$
$$= Q - dw$$
$$\therefore \frac{dw}{dt} \le Q - dw$$

Where $Q = \left(\frac{e_1r_1}{a_1} + \frac{e_2r_2}{a_2}\right)(1+d)$ and $d = \min(d_1 + d_2)$

Therefore $w(t) \le \frac{Q}{d} + Ce^{-dt}$ where C is the constant of integration, so for $t \ge 0$ all species are uniformly bounded, hence system (1) is dissipative.

The subsystem analysis and stability:

In this section the existence of the equilibrium point of subsystem (2) and (3) and the local stability analysis of each one are investigated.

There are at most three non-negative equilibrium point gives as the following:

1. The equilibrium points $E_0 = (0,0,0)\, and\, E_1 = (\frac{r_i}{a_i}, 0, 0)$ always exist.
2. The equilibrium points $E^* = (x_1, y_1, y_2), (i = 1, 2)$ exist if there is a positive solution to the following set of nonlinear equation.

$$(r_i - a_ix_i) - y_2 = 0,$$
$$e_ix_iy_2 - (m + d_1)y_1 = 0,$$
$$my_1 - dy_2 = 0$$

Obviously from above equations we get

$$x_i = \frac{r_i}{a_i + y_2},\ y_1 = \frac{1}{(m+d_1)y_1} * \frac{e_ir_iy_2}{a_i + y_2},\ y_2 = \frac{d_2(m+d_1)a_i}{e_ir_i - d_2(m+d_1)}$$
$$y_2 > 0\, if\ e_ir_i > d_2(m + d_1)$$

The general variational matrix of subsystem (2), (3) at the point (x_i, y_1, y_2) is given by

$$v(x_i, y_1, y_2) = \begin{bmatrix} \frac{\partial f_i}{\partial x_i} & \frac{\partial f_i}{\partial y_1} & \frac{\partial f_i}{\partial y_2} \\ \frac{\partial g_1}{\partial x_i} & \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial x_i} & \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{bmatrix}$$

Where

$\frac{\partial f_i}{\partial x_i} = r_i - 2a_i x_i - y_2$ $\frac{\partial g_1}{\partial x_i} = e_i y_2$ $\frac{\partial g_2}{\partial x_i} = 0$

$\frac{\partial f_i}{\partial y_1} = 0$ $\frac{\partial g_1}{\partial x_i} = -(m + d_1)$ $\frac{\partial g_2}{\partial y_1} = m$

$\frac{\partial f_i}{\partial y_2} = -x_i$ $\frac{\partial g_1}{\partial y_2} = e_i x_i$ $\frac{\partial g_1}{\partial y_2} = -d_2$

So it's clear that from the variational matrix at $E_0 = (0,0,0)$, the eigenvalue of E_0 are $\lambda_1 = r_i, \lambda_2 = -(m + d_1), \lambda_3 = -d_2$ then E_0 was unstable point in x_i direction with stable manifold in $y_1 y_2$ plain.

Now the stability analysis for the equilibrium point $E_1 = \left(\frac{r_i}{a_i}, 0, 0\right)$ and $E^* = (x_i, y_1, y_2)$ are given in the following theorem.

Theorem 2: The axial equilibrium point $E_1 = \left(\frac{r_i}{a_i}, 0, 0\right)$ is stable under the following condition $(m + d_1)d_2 > \frac{e_i r_i m}{a_i}$ (4)

Proof: From the variational matrix we can written the Eigen value as follow $(-r_i - \lambda) * \left[\lambda^2 + \lambda(m + d_1 + d_2) + (m + d_1)d_2 - \frac{e_i r_i m}{a_i}\right]$, then the proof follow directly by using Routh-Hurwitz criteria [9] which show that if $(m + d_1)d_2 > \frac{e_i r_i m}{a_i}$ then $E_1 = \left(\frac{r_i}{a_i}, 0, 0\right)$ is locally asymptotically and its unstable otherwise.

Theorem 3: The positive equilibrium point $E^* = (x_i, y_1, y_2)$ is locally stable if $y_2 > a_i$ and $(d_2 + 2m + d_1)(m + d_1)(d_2 + a) > (d_2 + 2m + d_1 + y_2)e_i m x_i$

Proof: From the variational matrix we get $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$ where

$A_1 = d_2 + d_1 + 2m$

$A_2 = (a_i d_2 + d_2(m + d_1) + a_i(m + d_1) - e_i m x_i)$

$A_3 = (a_i d_2(m + d_1) + e_i m x_i(y_2 - a_i))$

Now using Routh-Hurwtiz criteria to get the the proof.

The dynamical analysis of system (1):

In this section the existence and local stability analysis of non-negative equilibrium are investigated. We also discuss the global dynamic of the axial equilibrium using Laybanov function. There are five non-negative equilibrium points. The existence and the stability condition for them are shown as follow.

1. The equilibrium point $E_0 = (0,0,0,0)$ always exists.
2. The equilibrium points $E_1 = (\frac{r_1}{a_1}, 0,0,0), E_2 = (0, \frac{r_1}{a_1}, 0, 0)$ and $E_3 = (\frac{r_1}{a_1}, \frac{r_2}{a_2}, 0, 0)$ always exists. As the prey population grows to the carrying capacity in the

absences of predation. However the predator dies in the absences of its prey.

3. The positive equilibrium point $E_4 = (x_1^\bullet, x_2^\bullet, y_1^\bullet, y_2^\bullet)$ exist if there is a positive solution to the following algebraic non-linear equations.

$$(r_1 - a_1 x_1) - y_2 = 0,$$
$$(r_2 - a_2 x_2) - y_2 = 0,$$
$$(e_1 x_1 + e_2 x_2) y_2 - (m + d_1) y_1 = 0,$$
$$m y_1 - d y_2 = 0$$

Thus, by solving the above system we obtain.

$$x_1 = \frac{r_1 - y_2}{a_1}, x_2 = \frac{r_2 - y_2}{a_2}, y_1 \frac{d_2}{m} y_2, y_2 = \frac{\frac{e_1 r_1}{a_1} + \frac{e_2 r_2}{a_2} - \frac{(m + d_1)\mathrm{d}_2}{m}}{\frac{e_1}{a_1} + \frac{e_2}{a_2}},$$

$$x_i, y_i > 0 \; if \; \frac{e_1 r_1}{a_1} + \frac{e_2 r_2}{a_2} > \frac{(m + d_1)\mathrm{d}_2}{m} \; and \; y_2 < r_i \; (i = 1, 2)$$

Now in order to study the behavior of the solution near the equilibrium point, we need to compute the variational matrix of system (1). Assume $v(x_1, x_2, y_1, y_2)$ denote the variational matrix of system (1) at the point (x_1, x_2, y_1, y_2) then

$$v(x_i, y_1, y_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{bmatrix}$$

Where

$$\frac{\partial f_1}{\partial x_1} = r_1 - 2a_1 x_1 - y_2 \qquad \frac{\partial f_2}{\partial x_1} = 0$$
$$\frac{\partial f_1}{\partial x_2} = 0 \qquad \frac{\partial f_2}{\partial x_1} = r_2 - 2a_2 x_2 - y_2$$
$$\frac{\partial f_1}{\partial y_1} = 0 \qquad \frac{\partial f_2}{\partial x_1} = 0$$
$$\frac{\partial f_1}{\partial y_2} = -x_1 \qquad \frac{\partial f_1}{\partial y_2} = -x_1$$
$$\frac{\partial g_1}{\partial x_1} = e_1 y_2 \qquad \frac{\partial g_2}{\partial x_1} = 0$$
$$\frac{\partial g_1}{\partial x_2} = e_2 y_2 \qquad \frac{\partial g_2}{\partial x_2} = 0$$
$$\frac{\partial g_1}{\partial y_1} = -(m + d_1) \qquad \frac{\partial g_2}{\partial y_1} = m$$
$$\frac{\partial g_1}{\partial y_2} = e_1 x_1 + e_2 x_2 \qquad \frac{\partial g_2}{\partial x_2} = -d_2$$

Accordingly, the linear stability analysis about the equilibrium point $E_i = 0,1,2,3$ gives below.

- The equilibrium point E_0 is unstable point in the x_1x_2 plain but its stable in y_1y_2 plain which means that the prey population grows while both the predator population decline.
- The equilibrium point E_1 and E_2 is unstable otherwise under condition (4).
- The equilibrium point E_3 is locally asymptotically stable in R_+^4 if and only if $\frac{(m+d_1)\mathrm{d}_2}{m} > \frac{e_1r_1}{a_1} + \frac{e_2r_2}{a_2}$ hold and it's unstable otherwise.

The variational matrix of system (1) at the positive equilibrium point $E_4 = (x_1^\bullet, x_2^\bullet, y_1^\bullet, y_2^\bullet)$ is

$$v(x_1^\bullet, x_2^\bullet, y_1^\bullet, y_2^\bullet) = \begin{bmatrix} -a_1x_1 & 0 & 0 & -x_1 \\ 0 & -a_2x_2 & 0 & -x_2 \\ e_1y_2 & e_2y_2 & -(m+d_1) & e_1x_1+e_2x_2 \\ 0 & 0 & m & d_2 \end{bmatrix}$$

Then the characteristic equation of $v(x_1^\bullet, x_2^\bullet, y_1^\bullet, y_2^\bullet)$ is

$$A_1 = (\mathrm{a}_1 x_1 + a_2x_2 + d_2 + h),$$
$$A_2 = (a_2x_2h + a_2x_2d_2 + a_1a_2x_1x_2 + hd_2 + a_1hx_1 + a_1d_2x_1 - a_2lmx_2)$$
$$A_3 = (a_2d_2x_2h + a_1a_2x_2h + a_1a_2d_2x_1x_2 + a_1hd_2x_1 + e_2m + y_2x_2 - a_1a_2lmx_1x_2 - e_1mx_1y_2),$$
$$A_4 = (a_1a_2x_1x_2h + a_1e_2y_2x_1x_2m - e_1a_2y_2x_1x_2m)$$
$$with\, h = (m+d_1), and\, l = (e_1x_1 + e_2x_2)$$

Now applying Routh-Hurwitz criteria we get that E_4 is stable if $e_2x_2 > e_1x_1, e_2 > e_1, \frac{d_2}{m} > l, and\, (A_1A_2 - A_3)A_3 - A_1^2A_4 > 0$ and its unstable otherwise.

Furthermore, in the following theorem the global stability condition of $E_3 = (\frac{r_1}{a_1}, \frac{r_2}{a_2}, 0, 0)$ is establish by using suitable Lapanov function.

Theorem 4: Assume that E_3 is locally asymptotically stable, then its globally stable under the same condition $\frac{(m+d_1)\mathrm{d}_2}{m} > \frac{e_1r_1}{a_1} + \frac{e_2r_2}{a_2}$

Proof: consider the following positive definite function

$$U = \int_{x_{10}}^{x_1} \frac{\tau - x_{10}}{e_2\tau} d\tau + \int_{x_{20}}^{x_2} \frac{\tau - x_{20}}{e_1\tau} d\tau + \frac{1}{e_1e_2} y_1 + \frac{(m+d_1)}{e_1e_2d_1} y_2$$

We now compute the derivative of u

$$\frac{x_1 - x_{10}}{e_2}(r_1 - a_1x_1) + \frac{x_2 - x_{20}}{e_1}(r_2 - a_2x_2) + \frac{1}{e_1e_2}\left((e_1x_{10} + e_2x_{20}) - \frac{(m+d_1)d_2}{m}\right)y_2$$

$\therefore U' = 0\, if\ and\ only\ if\ x_1 = x_{10}, and\ x_2 = x_{20}\ and\ \frac{(m+d_1)d_2}{m} = \left(\frac{e_1r_1}{a_1} + \frac{e_2r_2}{a_2}\right)$ and

$U' < 0\, if\ \frac{(m+d_1)d_2}{m} = \left(\frac{e_1r_1}{a_1} + \frac{e_2r_2}{a_2}\right)$

So it's clear that

1. If $x_1 < x_{10}, \Rightarrow (r_1 - a_1x_1) > 0$ *and* $x_2 < x_{20}, \Rightarrow (r_2 - a_2x_2) > 0$ *then* $U' < 0$
2. If $x_1 > x_{10}, \Rightarrow (r_1 - a_1x_1) < 0$ *and* $x_2 > x_{20}, \Rightarrow (r_2 - a_2x_2) < 0$ *then* $U' < 0$

Numerical analysis:

The global dynamical behavior of the non-linear system (2) or (3), in the positive octant, is investigated numerically. A numerical integration for system (2) or (3) is carried out for various choices of biologically feasible parameters value and for different set of initial condition. System (2) or (3) is solved numerically using the matlap (Simulink, odu45). The numerical behavior enhance and convenience the analytic behavior. In theoream (2) under condition (4) the equilibrium point $E_1 = \left(\frac{r_i}{a_i}, 0, 0\right)$ is stable which is shown also in fig. (1) and unstable if condition (4) not hold which is shown in fig. (2)

Fig. (1) show that system (2), (3) stable when $a = 3, e = m = r = 1, d_1 = 0.1$ *and* $d_2 = 0.2$ under condition (4) at the equilibrium point $E_1 = (1/3, 0, 0)$.

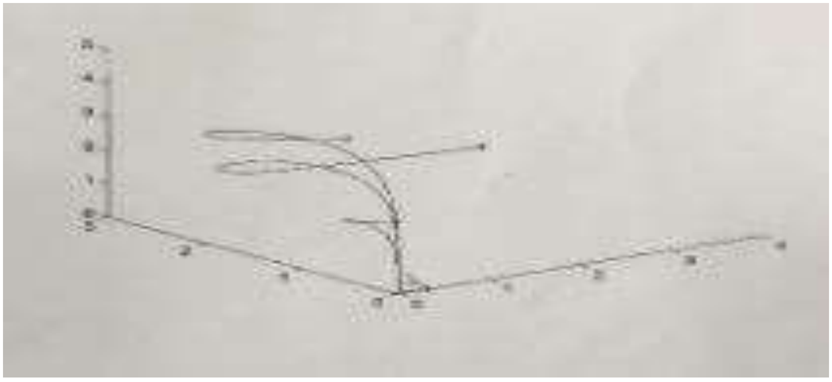


Fig. (2) show that system (2), (3) unstable when $(m + d_1)d_2 < \frac{e_i r_i m}{a_i}$ with $a = 2, e = 1, m = 0.7, r = 1$ *and* $d = 0.2, equ. = (1/3, 0, 0)$ under condition (4) at the equilibrium point $E_1 = (1/3, 0, 0)$.

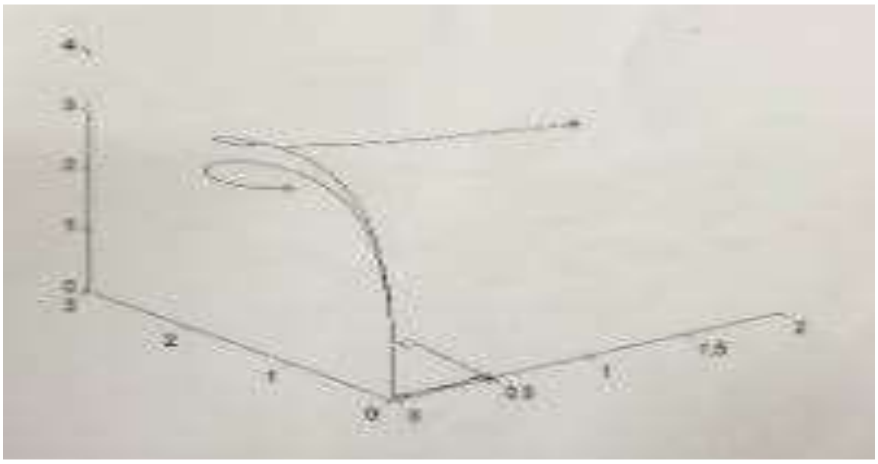


Fig. (3) $a = 2, e = 5, m = 0.4, r = 1 and\ d_1 = d_2 = 0.2$ at the positive equilibrium point (x, y_1, y_2). with initial point =(0.5,0.5,5), (1,1,3)

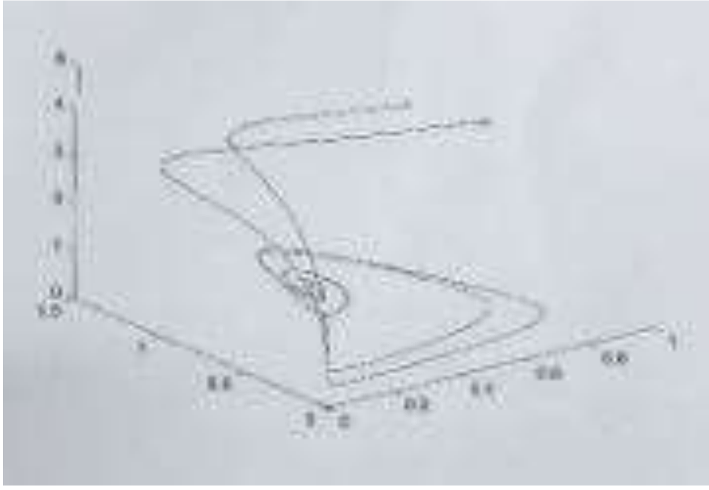
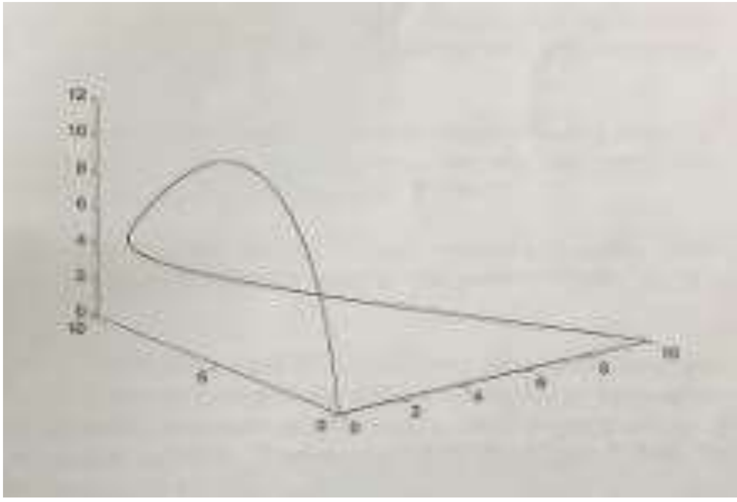


Fig. (4) $a = 0.1, e = 0.7, m = 3, r = 1 and\ d_1 = 0.2 = d_2 = 0.1$ at the positive equilibrium point (x, y_1, y_2).



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