The link between the Dirac's method and the Hamilton-Jacobi method for second order Hamiltonian constrained systems

Ass. Lec. Ahmed Ali A. ALmuqdadi
College of Basic Education
AL-Mustansiriyah University

Abstract

The link between the Dirac's method and the Hamilton-Jacobi method for second order constrained systems is studied. It is shown that the Dirac's method is always in exact agreement with the Hamilton-Jacobi method. The integrability conditions in the Hamilton-Jacobi method are equivalent to the consistency conditions in the Dirac's method.

1 - Introduction

The Lagrangian formulation for constrained systems was studied by Sundermeyer (1982),[22] Sudrshan and Mukunda (1974),[21] while the Hamiltonian formulation of singular systems is usually made through a formalism developed by Dirac (1950,1964)[1,2]. And another powerful method the Hamilton-Jacobi has been developed for investigating singular systems in the first order (Güler, 1992; Rabei and Güler, 1992, 1995)[4,5,18,19,20]. The formal generalization of Hamilton-Jacobi formalism for singular systems with arbitrarily second-order Lagrangians was developed by Pimentel and Teixeira (1996)[13]

The link between the Dirac's method and the Hamilton-Jacobi method for first order constrained systems was studied by Rabei, 1996 [17].

The Lagrangian function of any physical systems with N degrees of freedom is a function of generalized coordinates $q_i, \dot{q}_i, \ddot{q}_i$ and a parameter t i.e.

$$L \equiv (q_i, \dot{q}_i, \ddot{q}_i, t) \tag{1}$$

The Hessian matrix is defined as

$$W_{ij} \equiv \frac{\partial^2 L}{\partial \ddot{q}_i \partial \ddot{q}_j}, \qquad i, j = 1, ..., N \tag{2}$$

If the rank of this matrix is N, systems which have this property are called regular and their treatments are found in standard mechanics books, systems which have the rank less than N are called singular systems.

The passage from the Lagrangian method to the Hamiltonian method is achieved by introducing the generalized momenta (p_i, π_i) conjugated to the generalized coordinates $(q_i, \dot{q}_i)$ respectively as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) \tag{3}$$

$$\pi_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (4)$$

In this paper we would like to show that the Hamilton-Jacobi method is always in exact agreement with Dirac's method. In section 2 Dirac's method, in section 3 Hamilton-Jacobi method, in section 4 the link between the two methods is discussed, and in section 5 an example of singular with second order Lagrangian is constructed and solved by using the two methods.

2-Dirac's Method

The well-known method to investigate the Hamilton formulation of constrained systems was initiated by Dirac [1,2].In this formulation one defines the total Hamiltonian as:

$$H_T = H_0 + V_\alpha H'_\alpha \qquad \alpha = 1,2,\ldots,m<2(n-1) \qquad (5)$$

where H_0 being the canonical Hamiltonian and determined as

$$H_0 = p_i \dot{q}_i + \pi_i \ddot{q}_i - L \qquad (6)$$

and V_α are arbitrary coefficients.

Due to the singular nature of the Hessian matrix, we have α functionally independent relations of the form

$$H'_\alpha(q_i, p_i, \dot{q}_i, \pi_i) = 0; \qquad \alpha = 1,2,\ldots,m < 2(n-1) \qquad (7)$$

The equations of motion for any function $f(q_i, \dot{q}_i, p_i, \pi_i)$ is given in terms of H_T by Dirac [1] as:

$$\dot{f} \approx \{f, H_T\} = \{f, H_0\} + V_\alpha \{f, H'_\alpha\} \qquad (8)$$

where the symbol { , } denotes the Poisson brackets of two functions $f(q_i, p_i, \dot{q}_i, \pi_i)$ and $g(q_i, p_i, \dot{q}_i, \pi_i)$ by Pimentel [13] is defined as:

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial \dot{q}_i}\frac{\partial g}{\partial \pi_i} - \frac{\partial f}{\partial \pi_i}\frac{\partial g}{\partial \dot{q}_i} \qquad (9)$$

Against the equations of motion from equation (8) are written as:

$$\dot{q}_i = \{q_i, H_T\} = \{q_i, H_0\} + V_\alpha \{q_i, H'_\alpha\} \qquad (10)$$

$$\dot{p}_i = \{p_i, H_T\} = \{p_i, H_0\} + V_\alpha \{p_i, H'_\alpha\} \qquad (11)$$

$$\ddot{q}_i = \{\dot{q}_i, H_T\} = \{\dot{q}_i, H_0\} + V_\alpha \{\dot{q}_i, H'_\alpha\} \qquad (12)$$

$$\dot{\pi}_i = \{\pi_i, H_T\} = \{\pi_i, H_0\} + V_\alpha \{\pi_i, H'_\alpha\} \qquad (13)$$

The consistency condition are given by:

$$\dot{H}'_\alpha = \{H'_\alpha, H_T\} = \{H'_\alpha, H_0\} + V_\alpha \{H'_\alpha, H'_\nu\} \approx 0; \qquad \alpha, \upsilon = 1,2,\ldots,m \qquad (14)$$

These conditions may be either identically satisfied (when we use the primary constraints), determine some of the arbitrary coefficients V_α, or generate new constraints that will be called secondary constraints. The constraints that have null Poisson brackets with all other constraints are called first class constraints otherwise they are called second class ones.

3- Hamilton-Jacobi Method

The Hamilton-Jacobi formulation for singular first order systems was developed by Güler [4,5] and developed by Pimentel [13] for second order systems who obtained a set of Hamilton-Jacobi partial differential equations has been constructed as:

$$p_\alpha = -H_\alpha^p(q_i, \dot{q}_i, p_a, \pi_a) \quad (15)$$

$$\pi_\alpha = -H_\alpha^\pi(q_i, \dot{q}_i, p_a, \pi_a) \quad (16)$$

$$H_0' = p_0 + H_0 \quad (17)$$

$$H_\alpha'^p = p_\alpha + H_\alpha^p \quad (18)$$

$$H_\alpha'^\pi = \pi_\alpha + H_\alpha^\pi \quad (19)$$

The equations of motion are written as total differential equations in many variables as follows:

$$dq_i = \frac{dH_0'}{dp_i} dt_0 + \frac{dH_\alpha'^p}{dp_i} dt_\alpha + \frac{dH_\alpha'^\pi}{dp_i} dt_\alpha' \quad (20)$$

$$d\dot{q}_i = \frac{dH_0'}{d\pi_i} dt_0 + \frac{dH_\alpha'^p}{d\pi_i} dt_\alpha + \frac{dH_\alpha'^\pi}{d\pi_i} dt_\alpha' \quad (21)$$

$$dp_i = -\frac{dH_0'}{dq_i} dt_0 - \frac{dH_\alpha'^p}{dq_i} dt_\alpha - \frac{dH_\alpha'^\pi}{dq_i} dt_\alpha' \quad (22)$$

$$d\pi_i = -\frac{dH_0'}{d\dot{q}_i} dt_0 - \frac{dH_\alpha'^p}{d\dot{q}_i} dt_\alpha - \frac{dH_\alpha'^\pi}{d\dot{q}_i} dt_\alpha' \quad (23)$$

These total differential equations are integrable if, and only if, the corresponding system of partial differential equations. For this purpose, defining the linear operators X_α as: by Güler [4]

$$X_\alpha f(q_i, \dot{q}_i, p_i, \pi_i t) = \{H_\alpha', f,\}$$
$$= \frac{\partial H_\alpha'}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial H_\alpha'}{\partial p_i}\frac{\partial f}{\partial q_i} + \frac{\partial H_\alpha'}{\partial \dot{q}_i}\frac{\partial f}{\partial \pi_i} - \frac{\partial H_\alpha'}{\partial \pi_i}\frac{\partial f}{\partial \dot{q}_i} + \frac{\partial f}{\partial t} \quad (24)$$

where $u, s = 0, 1, ..., k-1;$ where k here equal 2 and $i = 1, ..., N$ the equations of motion are integrable if, and only if, the bracket relations

$$[X_\alpha, X_\beta]f = (X_\alpha X_\beta - X_\beta X_\alpha)f = 0 \quad (25)$$

Then one may solve equations (20-23) to find $(q_i, \dot{q}_i, p_i \text{ and } \pi_i)$ as functions of $(t_0, t_\alpha, \text{and } t_\alpha')$. If the relation (25) are not satisfied identically, one may add the bracket relations which cannot be expressible in this form as new operators to obtain a new completes system by Rabie [17].

2 - The linkage of the two methods

In Dirac's method and from the equation (5) we have

$$H_T = H_0 + V_\alpha H_\alpha'^p + \lambda_\alpha H_\alpha'^\pi \quad (26)$$

and from equation(14) we have

$$\dot{H}_{\alpha}'^{p} = \{H_{\alpha}'^{p}, H_T\} = \{H_{\alpha}'^{p}, H_0\} + V_{\alpha}\{H_{\alpha}'^{p}, H_{\upsilon}'^{p}\} + \lambda_{\alpha}\{H_{\alpha}'^{p}, H_{\upsilon}'^{\pi}\} \quad (27)$$

$$\dot{H}_{\alpha}'^{\pi} = \{H_{\alpha}'^{\pi}, H_T\} = \{H_{\alpha}'^{\pi}, H_0\} + V_{\alpha}\{H_{\alpha}'^{\pi}, H_{\upsilon}'^{p}\} + \lambda_{\alpha}\{H_{\alpha}'^{\pi}, H_{\upsilon}'^{\pi}\} \quad (28)$$

In the Hamilton-Jacobi method the equations of motion (20-23) can be written making use of Poisson brackets, we can write these four equations as:

$$dq_i = \{q_i, H_0\}dt + \{q_i, H_{\alpha}'^{p}\}dq_{\alpha} + \{q_i, H_{\alpha}'^{\pi}\}d\dot{q}_{\alpha} \quad (29)$$

$$d\dot{q}_i = \{\dot{q}_i, H_0\}dt + \{\dot{q}_i, H_{\alpha}'^{p}\}dq_{\alpha} + \{\dot{q}_i, H_{\alpha}'^{\pi}\}d\dot{q}_{\alpha} \quad (30)$$

$$dp_i = \{p_i, H_0\}dt + \{p_i, H_{\alpha}'^{p}\}dq_{\alpha} + \{p_i, H_{\alpha}'^{\pi}\}d\dot{q}_{\alpha} \quad (31)$$

$$d\pi_i = \{\pi_i, H_0\}dt + \{\pi_i, H_{\alpha}'^{p}\}dq_{\alpha} + \{\pi_i, H_{\alpha}'^{\pi}\}d\dot{q}_{\alpha} \quad (32)$$

Where $dt_0 = dt,\ dt_{\alpha} = dq_{\alpha},\ dt_{\alpha}' = d\dot{q}_{\alpha}$

And where

$$\frac{\partial H_0'}{\partial p_i} = \frac{\partial H_0}{\partial p_i} = \{q_i, H_0\} \quad (33)$$

$$\frac{\partial H_{\alpha}'^{p}}{\partial p_i} = \frac{\partial H_{\alpha}^{p}}{\partial p_i} = \{q_i, H_{\alpha}^{p}\} \quad (34)$$

$$\frac{\partial H_0'}{\partial \pi_i} = \frac{\partial H_0}{\partial \pi_i} = \{\dot{q}_i, H_0\} \quad (35)$$

$$\frac{\partial H_{\alpha}'^{\pi}}{\partial \pi_i} = \frac{\partial H_{\alpha}^{\pi}}{\partial \pi_i} = \{\dot{q}_i, H_{\alpha}^{\pi}\} \quad (36)$$

And

$$-\frac{\partial H_0'}{\partial q_i} = -\frac{\partial H_0}{\partial q_i} = \{p_i, H_0\} \quad (37)$$

$$-\frac{\partial H_{\alpha}'^{p}}{\partial q_i} = -\frac{\partial H_{\alpha}^{p}}{\partial q_i} = \{p_i, H_{\alpha}^{p}\} \quad (38)$$

$$-\frac{\partial H_0'}{\partial \dot{q}_i} = -\frac{\partial H_0}{\partial \dot{q}_i} = \{\pi_i, H_0\} \quad (39)$$

$$-\frac{\partial H_{\alpha}'^{\pi}}{\partial \dot{q}_i} = -\frac{\partial H_{\alpha}^{\pi}}{\partial \dot{q}_i} = \{\pi_i, H_{\alpha}^{\pi}\} \quad (40)$$

The set equations (20-23) are integrable if, and only if, the conditions (24) can be written as:

$$\begin{aligned} X_0 f(q_i, p_i, \dot{q}_i, \pi_i, t) &= \frac{\partial H_0'}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial H_0'}{\partial p_i}\frac{\partial f}{\partial q_i} + \frac{\partial f}{\partial t} \\ &+ \frac{\partial H_0'}{\partial \dot{q}_i}\frac{\partial f}{\partial \pi_i} - \frac{\partial H_0'}{\partial \pi_i}\frac{\partial f}{\partial \dot{q}_i} + \frac{\partial f}{\partial t} = 0 \end{aligned} \quad (41)$$

$$\begin{aligned} X_{\alpha} f(q_i, p_i, \dot{q}_i, \pi_i, t) &= \frac{\partial H_{\alpha}'^{p}}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial H_{\alpha}'^{p}}{\partial p_i}\frac{\partial f}{\partial q_i} \\ &+ \frac{\partial H_{\alpha}'^{p}}{\partial \dot{q}_i}\frac{\partial f}{\partial \pi_i} - \frac{\partial H_{\alpha}'^{p}}{\partial \pi_i}\frac{\partial f}{\partial \dot{q}_i} = 0 \end{aligned} \quad (42)$$

Where

$$\frac{\partial H'^{p}_{\alpha}}{\partial p_{\alpha}}=1; \quad \frac{\partial H'^{p}_{\alpha}}{\partial q_{\alpha}}=0; \quad \frac{\partial H'_{0}}{\partial q_{\alpha}}=0;$$

$$\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_{\alpha}}=1; \quad \frac{\partial H'_{0}}{\partial \dot{q}_{\alpha}}=0; \quad \frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_{\alpha}}=0.$$

On the other hand, the integrability conditions (25) can be written in a compact form as:

$$\begin{aligned}[X_0, X_\alpha]f = & \frac{\partial}{\partial p_i}\left(\frac{\partial H'_0}{\partial p_i}\frac{\partial H'^{p}_{\alpha}}{\partial q_i}-\frac{\partial H'_0}{\partial q_i}\frac{\partial H'^{p}_{\alpha}}{\partial p_i}+\frac{\partial H'^{p}_{\alpha}}{\partial t}\right)\frac{\partial f}{\partial q_i} \\ & -\frac{\partial}{\partial q_i}\left(\frac{\partial H'_0}{\partial p_i}\frac{\partial H'^{p}_{\alpha}}{\partial q_i}-\frac{\partial H'_0}{\partial q_i}\frac{\partial H'^{p}_{\alpha}}{\partial p_i}+\frac{\partial H'^{p}_{\alpha}}{\partial t}\right)\frac{\partial f}{\partial p_i} \\ & +\frac{\partial}{\partial \pi_i}\left(\frac{\partial H'_0}{\partial \pi_i}\frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_i}-\frac{\partial H'_0}{\partial \dot{q}_i}\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}+\frac{\partial H'^{\pi}_{\alpha}}{\partial t}\right)\frac{\partial f}{\partial \dot{q}_i} \\ & -\frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial H'_0}{\partial \pi_i}\frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_i}-\frac{\partial H'_0}{\partial \dot{q}_i}\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}+\frac{\partial H'^{\pi}_{\alpha}}{\partial t}\right)\frac{\partial f}{\partial \pi_i}\end{aligned} \qquad (43)$$

$$\begin{aligned}[X_\alpha, X_\beta]f = & \frac{\partial}{\partial p_i}\left(\frac{\partial H'^{p}_{\alpha}}{\partial p_i}\frac{\partial H'_{\beta}}{\partial q_i}-\frac{\partial H'^{p}_{\alpha}}{\partial q_i}\frac{\partial H'_{\beta}}{\partial p_i}\right)\frac{\partial f}{\partial q_i} \\ & -\frac{\partial}{\partial q_i}\left(\frac{\partial H'^{p}_{\alpha}}{\partial p_i}\frac{\partial H'_{\beta}}{\partial q_i}-\frac{\partial H'^{p}_{\alpha}}{\partial q_i}\frac{\partial H'_{\beta}}{\partial p_i}\right)\frac{\partial f}{\partial p_i} \\ & +\frac{\partial}{\partial \pi_i}\left(\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}\frac{\partial H'_{v}}{\partial \dot{q}_i}-\frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_i}\frac{\partial H'_{v}}{\partial \pi_i}\right)\frac{\partial f}{\partial \dot{q}_i} \\ & -\frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}\frac{\partial H'_{v}}{\partial \dot{q}_i}-\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}\frac{\partial H'_{\upsilon}}{\partial \dot{q}_i}\right)\frac{\partial f}{\partial \pi_i}\end{aligned} \qquad (44)$$

Equating these brackets relation to zero leads to the following conditions:

$$\frac{\partial H'_0}{\partial p_i}\frac{\partial H'^{p}_{\alpha}}{\partial q_i}-\frac{\partial H'_0}{\partial q_i}\frac{\partial H'^{p}_{\alpha}}{\partial p_i}=0; \qquad (45)$$

$$\frac{\partial H'_0}{\partial \pi_i}\frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_i}-\frac{\partial H'_0}{\partial \dot{q}_i}\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}=0; \qquad (46)$$

$$\frac{\partial H'^{p}_{\alpha}}{\partial t}=0; \qquad (47)$$

$$\frac{\partial H'^{\pi}_{\alpha}}{\partial t}=0; \qquad (48)$$

$$\frac{\partial H'^{p}_{\alpha}}{\partial p_i}\frac{\partial H'_{\beta}}{\partial q_i}-\frac{\partial H'^{p}_{\alpha}}{\partial q_i}\frac{\partial H'_{\beta}}{\partial p_i}=0\ ; \qquad (49)$$

$$\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}\frac{\partial H'_{v}}{\partial \dot{q}_i}-\frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_i}\frac{\partial H'_{v}}{\partial \pi_i}=0 \qquad (50)$$

These conditions mean that the total differential of the functions $H'_{\alpha}(q_i,\dot{q}_i,p_i,\pi_i)$ should be equal to zero by Muslih [9]:

$$H'_{\alpha}=0 \qquad (51)$$

or

$$dH'^{p}_{\alpha}=0 \qquad (52)$$

$$dH'^{\pi}_{\alpha}=0 \qquad (53)$$

It is known that

$$dH'^{p}_{\alpha}=\frac{\partial H'^{p}_{\alpha}}{\partial q_i}dq_i+\frac{\partial H'^{p}_{\alpha}}{\partial p_i}dp_i \qquad (54)$$

$$dH'^{\pi}_{\alpha}=\frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_i}d\dot{q}_i+\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}d\pi_i \qquad (55)$$

if we substitute for dq_i and dp_i from equations (20),(22) in equation (54) becomes

$$\begin{aligned}dH'^{p}_{\alpha}&=\frac{\partial H'^{p}_{\alpha}}{\partial q_i}\left(\frac{\partial H'_0}{\partial p_i}dt_0+\frac{\partial H'^{p}_{\upsilon}}{\partial p_i}dt_{\upsilon}+\frac{\partial H'^{\pi}_{\upsilon}}{\partial p_i}dt'_{\upsilon}\right)\\&+\frac{\partial H'^{p}_{\alpha}}{\partial p_i}\left(-\frac{\partial H'_0}{\partial q_i}dt_0-\frac{\partial H'^{p}_{\upsilon}}{\partial q_i}dt_{\upsilon}-\frac{\partial H'^{\pi}_{\upsilon}}{\partial q_i}dt'_{\upsilon}\right)\end{aligned} \qquad (56)$$

$$\begin{aligned}dH'^{p}_{\alpha}&=\left(\frac{\partial H'^{p}_{\alpha}}{\partial q_i}\frac{\partial H'_0}{\partial p_i}-\frac{\partial H'^{p}_{\alpha}}{\partial p_i}\frac{\partial H'_0}{\partial q_i}\right)dt_0\\&+\left(\frac{\partial H'^{p}_{\alpha}}{\partial q_i}\frac{\partial H'^{p}_{\upsilon}}{\partial p_i}-\frac{\partial H'^{p}_{\alpha}}{\partial p_i}\frac{\partial H'^{p}_{\upsilon}}{\partial q_i}\right)dt_{\upsilon}\\&+\left(\frac{\partial H'^{p}_{\alpha}}{\partial q_i}\frac{\partial H'^{\pi}_{\upsilon}}{\partial p_i}-\frac{\partial H'^{p}_{\alpha}}{\partial p_i}\frac{\partial H'^{\pi}_{\upsilon}}{\partial q_i}\right)dt'_{\upsilon}\end{aligned} \qquad (57)$$

$$dH'^{p}_{\alpha}=\{H'^{p}_{\alpha},H_0\}dt+\{H'^{p}_{\alpha},H'^{p}_{\upsilon}\}dt_{\upsilon}+\{H'^{p}_{\alpha},H'^{\pi}_{\upsilon}\}dt'_{\upsilon}=0 \qquad (58)$$

This equation, which results from the integrability conditions is equivalent to the consistency conditions (27) in the Dirac's method.

And if we substitute $d\dot{q}_i$ and $d\pi_i$ from equations (21),(23) in equation(55) becomes

$$\begin{aligned}dH'^{\pi}_{\alpha}&=\frac{\partial H'^{\pi}_{\alpha}}{\partial \dot{q}_i}\left(\frac{\partial H'_0}{\partial \pi_i}dt_0+\frac{\partial H'^{p}_{\upsilon}}{\partial \pi_i}dt_{\upsilon}+\frac{\partial H'^{\pi}_{\upsilon}}{\partial \pi_i}dt'_{\upsilon}\right)\\&+\frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_i}\left(-\frac{\partial H'_0}{\partial \dot{q}_i}dt_0-\frac{\partial H'^{p}_{\upsilon}}{\partial \dot{q}_i}dt_{\upsilon}-\frac{\partial H'^{\pi}_{\upsilon}}{\partial \dot{q}_i}dt'_{\upsilon}\right)\end{aligned} \qquad (59)$$

$$dH_{\alpha}'^{\pi} = \left(\frac{\partial H_{\alpha}'^{\pi}}{\partial \dot{q}_i}\frac{\partial H_0'}{\partial \pi_i} - \frac{\partial H_{\alpha}'^{\pi}}{\partial \pi_i}\frac{\partial H_0'}{\partial \dot{q}_i}\right)dt_0 + \left(\frac{\partial H_{\alpha}'^{\pi}}{\partial \dot{q}_i}\frac{\partial H_{\upsilon}'^{p}}{\partial \pi_i} - \frac{\partial H_{\alpha}'^{\pi}}{\partial \pi_i}\frac{\partial H_{\upsilon}'^{p}}{\partial \dot{q}_i}\right)dt_{\upsilon} + \left(\frac{\partial H_{\alpha}'^{\pi}}{\partial \dot{q}_i}\frac{\partial H_{\upsilon}'^{\pi}}{\partial \pi_i} - \frac{\partial H_{\alpha}'^{\pi}}{\partial \pi_i}\frac{\partial H_{\upsilon}'^{\pi}}{\partial \dot{q}_i}\right)dt_{\upsilon}' \qquad (60)$$

$$dH_{\alpha}'^{\pi} = \{H_{\alpha}'^{\pi}, H_0'\}dt + \{H_{\alpha}'^{\pi}, H_{\upsilon}'^{p}\}dt_{\upsilon} + \{H_{\alpha}'^{\pi}, H_{\upsilon}'^{\pi}\}dt_{\upsilon}' = 0 \qquad (61)$$

This equation, which results from the integrability conditions is equivalent to the consistency conditions (28) in the Dirac's method. In other words, the integrability conditions in the Hamilton-Jacobi method are equivalent to the consistency conditions in the Dirac's method.

3 – An example

The procedure described in section 2 will be demonstrated by an example of singular Lagrangian given by:

$$L = \frac{1}{2}(\ddot{q}_1^{\,2} + \ddot{q}_2^{\,2}) + \dot{q}_3\ddot{q}_3 - \frac{1}{2}\dot{q}_3^2 \qquad (62)$$

The momenta canonically conjugate to the coordinates q_1, q_2, q_3 and $\dot{q}_1, \dot{q}_2, \dot{q}_3$ from equations (3) and (4).

$$p_1 = -\dddot{q} \qquad (63)$$

$$p_2 = -\dddot{q}_2 \qquad (64)$$

$$p_3 = 0 = -H_3^p \qquad (65)$$

$$\pi_1 = \ddot{q}_1 \qquad (66)$$

$$\pi_2 = \ddot{q}_2 \qquad (67)$$

$$\pi_3 = \dot{q}_3 = -H_3^{\pi} \qquad (68)$$

The Hamiltonian H_0 from equation (6) is defined as

$$H_0 = p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 + \pi_1\ddot{q}_1 + \pi_2\ddot{q}_2 + \pi_3\ddot{q}_3 - L \qquad (69)$$

or

$$H_0 = p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{1}{2}(\pi_1^2 + \pi_2^2) - \frac{1}{2}\pi_3^2 \qquad (70)$$

3.1. The Hamilton – Jacobi Method

Let us now apply the Hamilton – Jacobi treatment described above to this example. Making use of (17), (18) and (19), the set of Hamilton – Jacobi partial differential equations are calculated as:

$$H_0' = p_0 + H_0 = p_0 + p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{1}{2}(\pi_1^2 + \pi_2^2) - \frac{1}{2}\pi_3^2 \qquad (71)$$

$$H_3'^{p} = p_3 + H_3^p = 0 \qquad (72)$$

$$H_3'^{\pi} = \pi_3 - \dot{q}_3 = 0 \qquad (73)$$

The equations of motion (20-23) are obtained as total differential equations in many variable as follows:

$$dq_1 = \dot{q}_1 dt \qquad (74)$$

$$dq_2 = \dot{q}_2 dt \qquad (75)$$

$$dq_3 = dq_3 \qquad (76)$$

$$dp_1 = 0 \qquad (77)$$

$$dp_2 = 0 \qquad (78)$$

$$dp_3 = 0 \qquad (79)$$

$$d\dot{q}_1 = \pi_1 dt \qquad (80)$$

$$d\dot{q}_2 = \pi_2 dt \qquad (81)$$

$$d\dot{q}_3 = -\pi_3 dt + d\dot{q}_3 \qquad (82)$$

$$d\pi_1 = -p_1 dt \qquad (83)$$

$$d\pi_2 = -p_2 dt \qquad (84)$$

$$d\pi_3 = d\dot{q}_3 \qquad (85)$$

These total differential equations are integrable if condition (52) and (53) is satisfied

$$dH'_0 = 0 \qquad (86)$$

$$dH'^{\pi}_3 = 0 \qquad (87)$$

$$dH'_0 = \frac{\partial H'_0}{\partial q_i} dq_i + \frac{\partial H'_0}{\partial p_i} dp_i + \frac{\partial H'_0}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial H'_0}{\partial \pi_i} d\pi_i \qquad (88)$$

$$dH'_0 = \frac{\partial H'_0}{\partial q_1} dq_1 + \frac{\partial H'_0}{\partial q_2} dq_2 + \frac{\partial H'_0}{\partial q_3} dq_3 + \frac{\partial H'_0}{\partial p_1} dp_1 + \frac{\partial H'_0}{\partial p_2} dp_2 + \frac{\partial H'_0}{\partial p_3} dp_3 + \frac{\partial H'_0}{\partial \dot{q}_1} d\dot{q}_1 + \frac{\partial H'_0}{\partial \dot{q}_2} d\dot{q}_2 + \frac{\partial H'_0}{\partial \dot{q}_3} d\dot{q}_3 + \frac{\partial H'_0}{\partial \pi_1} d\pi_1 + \frac{\partial H'_0}{\partial \pi_2} d\pi_2 + \frac{\partial H'_0}{\partial \pi_3} d\pi_3 \qquad (89)$$

if we substitute for the set of equations (74-85) in equation (89) we get:

$$dH'_0 = -\pi_3 \qquad (90)$$

Since dH'_0 is not identically zero, we have a new constraint:

$$H'^{\pi}_1 = -\pi_3 = 0 \Rightarrow \pi_3 = 0 \qquad (91)$$

From equation (68)we get:

$$\dot{q}_3 = 0 \qquad (92)$$

from equation (73) we have:

$$dH'^{\pi}_3 = d\pi_3 - d\dot{q}_3 = 0 \qquad (93)$$

substitute equation (85) in equation (93) we get:

$$dH'^{\pi}_3 = d\dot{q}_3 - d\dot{q}_3 = 0 \qquad (94)$$

Thus, the set of equations (74-85) are integrable.

Then, one can rewrite equations (74-85) in the form:

$$dq_1 = \dot{q}_1 dt \qquad (95)$$

$$dq_2 = \dot{q}_2 dt \tag{96}$$
$$dq_3 = dq_3 \tag{97}$$
$$dp_1 = 0 \tag{98}$$
$$dp_2 = 0 \tag{99}$$
$$dp_3 = 0 \tag{100}$$
$$d\dot{q}_1 = \pi_1 dt \tag{101}$$
$$d\dot{q}_2 = \pi_2 dt \tag{102}$$
$$d\dot{q}_3 = 0 \tag{103}$$
$$d\pi_1 = -p_1 dt \tag{104}$$
$$d\pi_2 = -p_2 dt \tag{105}$$
$$d\pi_3 = 0 \tag{106}$$

3.2. The Dirac's Method

The total Hamiltonian from equation (26) is defined as:

$$H_T = H_0 + V_3 H_3'^{p} + \lambda_3 H_3'^{\pi} \tag{107}$$

where $H_0, H_3'^{p}$ and $H_3'^{\pi}$ are the three primary constraints defined in equations (71), (72) and (73) and we substitute these equations in equation (107) we get:

$$H_T = p_0 + p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{1}{2}(\pi_1^2 + \pi_2^2) - \frac{1}{2}\pi_3^2 + V_3 p_3 + \lambda_3(\pi_3 - \dot{q}_3) \tag{108}$$

These primary constraints satisfy the consistency condition (14) identically:

$$\begin{aligned}\dot{H}_0' = \{H_0', H_T\} = & \frac{\partial H_0'}{\partial q_1}\frac{\partial H_T}{\partial p_1} + \frac{\partial H_0'}{\partial q_2}\frac{\partial H_T}{\partial p_2} + \frac{\partial H_0'}{\partial q_3}\frac{\partial H_T}{\partial p_3} \\ & - \frac{\partial H_0'}{\partial p_1}\frac{\partial H_T}{\partial q_1} - \frac{\partial H_0'}{\partial p_2}\frac{\partial H_T}{\partial q_2} - \frac{\partial H_0'}{\partial p_3}\frac{\partial H_T}{\partial q_3} \\ & + \frac{\partial H_0'}{\partial \dot{q}_1}\frac{\partial H_T}{\partial \pi_1} + \frac{\partial H_0'}{\partial \dot{q}_2}\frac{\partial H_T}{\partial \pi_2} + \frac{\partial H_0'}{\partial \dot{q}_3}\frac{\partial H_T}{\partial \pi_3} \\ & - \frac{\partial H_0'}{\partial \pi_1}\frac{\partial H_T}{\partial \dot{q}_1} - \frac{\partial H_0'}{\partial \pi_2}\frac{\partial H_T}{\partial \dot{q}_2} - \frac{\partial H_0'}{\partial \pi_3}\frac{\partial H_T}{\partial \dot{q}_3}\end{aligned} \tag{109}$$

$$\dot{H}_0' = -\pi_3 = 0 \tag{110}$$

Will lead to the following secondary constraint:

$$H_1'^{\pi} = -\pi_3 = 0 \tag{111}$$

Imposing the condition $H_1'^{\pi} = 0$ we arrive at the result

$$\lambda_3 = 0 \tag{112}$$

And the primary constraint (73) we have

$$\dot{H}_3'^{\pi} = \{H_3'^{\pi}, H_T\} = \frac{\partial H_3'^{\pi}}{\partial q_1}\frac{\partial H_T}{\partial p_1} + \frac{\partial H_3'^{\pi}}{\partial q_2}\frac{\partial H_T}{\partial p_2} + \frac{\partial H_3'^{\pi}}{\partial q_3}\frac{\partial H_T}{\partial p_3}$$
$$- \frac{\partial H_3'^{\pi}}{\partial p_1}\frac{\partial H_T}{\partial q_1} - \frac{\partial H_3'^{\pi}}{\partial p_2}\frac{\partial H_T}{\partial q_2} - \frac{\partial H_3'^{\pi}}{\partial p_3}\frac{\partial H_T}{\partial q_3}$$
$$+ \frac{\partial H_3'^{\pi}}{\partial \dot{q}_1}\frac{\partial H_T}{\partial \pi_1} + \frac{\partial H_3'^{\pi}}{\partial \dot{q}_2}\frac{\partial H_T}{\partial \pi_2} + \frac{\partial H_3'^{\pi}}{\partial \dot{q}_3}\frac{\partial H_T}{\partial \pi_3} \quad (113)$$
$$- \frac{\partial H_3'^{\pi}}{\partial \pi_1}\frac{\partial H_T}{\partial \dot{q}_1} - \frac{\partial H_3'^{\pi}}{\partial \pi_2}\frac{\partial H_T}{\partial \dot{q}_2} - \frac{\partial H_3'^{\pi}}{\partial \pi_3}\frac{\partial H_T}{\partial \dot{q}_3}$$

$$\dot{H}_3'^{\pi} = 0 \quad (114)$$

Thus, no further constraints arise.
Making the equations of motion (10-13) we have:

$$\dot{q}_1 = \dot{q} \quad (115)$$
$$\dot{q}_2 = \dot{q}_2 \quad (116)$$
$$\dot{q}_3 = V_3 \quad (117)$$
$$\dot{p}_1 = 0 \quad (118)$$
$$\dot{p}_2 = 0 \quad (119)$$
$$\dot{p}_3 = 0 \quad (120)$$
$$\ddot{q}_1 = \pi_1 \quad (121)$$
$$\ddot{q}_2 = \pi_2 \quad (122)$$
$$\ddot{q}_3 = -\pi_3 + \lambda_3 \quad (123)$$
$$\dot{\pi}_1 = -p_1 \quad (124)$$
$$\dot{\pi}_2 = -p_2 \quad (125)$$
$$\dot{\pi}_3 = \lambda_3 \quad (126)$$

From equations (111) and (112), then, one can rewrite the equations (115-126) in the form:

$$\dot{q}_1 = \dot{q} \quad (127)$$
$$\dot{q}_2 = \dot{q}_2 \quad (128)$$
$$\dot{q}_3 = V_3 \quad (129)$$
$$\dot{p}_1 = 0 \quad (130)$$
$$\dot{p}_2 = 0 \quad (131)$$
$$\dot{p}_3 = 0 \quad (132)$$
$$\ddot{q}_1 = \pi_1 \quad (133)$$
$$\ddot{q}_2 = \pi_2 \quad (134)$$
$$\ddot{q}_3 = 0 \quad (135)$$
$$\dot{\pi}_1 = -p_1 \quad (136)$$

$$\dot{\pi}_2 = -p_2 \qquad (137)$$

$$\dot{\pi}_3 = 0 \qquad (138)$$

These equations are the same as equations (95-106).

Conclusion

The most common method for investigating the Hamiltonian treatment of constrained systems with first order was initiated by Dirac[1,2]. The main feature of this method is to consider primary constraints first. All the other constraints are obtained using consistency conditions has been developed by Pimentel[14]. The second method was developed by Pimentel with second order is the Hamilton-Jacobi treatment of singular systems. The equations of motion are written as total differential equations in many variable. The coordinates corresponding to dependent momenta considered as parameters.

The conclusion is the Hamilton-Jacobi method always in exact agreement with Dirac method. The equations of motion (29-32) are equivalent to equations (10-13). The integrability conditions (25) are satisfied if and only if the total differential of H'_α is equal to zero. In other words the integrability conditions lead us to consistency conditions.

References

1- Alain J. Brizard, 2007 **An Introduction To Lagrangian Mechanics**, Department of Chemistry and Physics Saint Michael's College, Colchester, VT 05439.

2- A.M. Bloch and A. Rojo, (2008),**Quantization of a nonholonomic system,** Phys. Rev. Letters 101030404.

3- Dirac, P. A. M. 1950. **Generalized Hamiltonian Dynamics**. Canadian Journal of Mathematical Physics,(2):129-148.

4- Dirac, P. A. M. 1964. **Lectures on Quantum Mechanics,** Belfer Graduate School of Science, Yeshiva University, New York.

5- Eyad H. Hasan, 2004. **Hamilton–Jacobi Treatment of Constrained Systems with Second-Order Lagrangians,** Ph. D. Sc Thesis, University of Jordan.

6- Güler, Y. 1987. **Hamilton- Jacobi Theory of Continuous Systems.** Il Nuovo Cimento,100B (2): 251-266.

7- Güler, Y. 1992. **Canonical Formulation of Singular Systems.** Il Nuovo Cimento, 107B(12): 1389-1395.

8- Güler, Y. 1992. **Integration of Singular Systems.** Il Nuovo Cimento,107B (10): 1143-1149.

9- Janthan, M. 1991. **On Dirac's Methods for Constrained Systems.** Physics Letters B,256(2):245-249.

10- Mukunda, N. 2006. **Lagrange and Classical Mechanics** Indian Institute of Science, Bangalore 560 012, India.

11- Mukunda, N. 1976. **Symmetries and Constraints in Generalized Hamiltonian Dynamics.** Annals of Physics, 99:408-433.

12- Muslih, S. I. 2002. **Quantization of Singular Systems with Second-Order Lagrangians.** Mathematical Physics. 0010020(1):1-12.

13- Muslih, S. I. and Güler, Y. 1995. **On the integrability Conditions of Constrained Systems.** Il Nuovo Cimento,110B(3):307-315.

14- M. Crampin and T. Mestdag, 2008 **The inverse problem for invariant Lagrangians** on a Lie group,J. of Lie Theory 18 471–502.

15- Nawafleh, K. 1998. **Constrained Hamiltonian Systems: Preliminary Study**, M.Sc. Thesis, University of Jordan.

16- Nesterenko, V. V. 1994. **On squaring the Primary Constraints in a Generalized Hamiltonian Dynamics.** Physics Letters B,327:50-55.

17- Nesterenko, V. V.1989. **Singular Lagrangians with Higher Derivatives.** Journal of Physics A. Mathematical and General. 22(10):1673-1687.

18- Pimentel, B.M. and Teixeira, R.G.1996. **Hamilton-Jacobi Formulation for Singular Systems with second-Order Lagrangians.** Il Nuovo Cimento,111B:805-817

19- Podolsky B. and Schwed P. 1948. **Review Modern Physics.20**:40

20- Pons, J.M. 1988. **Ostrogradsky's Theorem for Higher-Order Singular Lagrangians.** Letter of Mathematical Physics. 17 (3):182-189.

21- Rabie, E. M. 1995. **On the Lagrangians Formulation of Singular Systems.** Turkish Journal of Physics, (19):1580-1585.

22- Rabie, E. M. 1996. **On Hamiltonian Systems with Constraints.** Hadronic Journal,(19):597-605

23- Rabie, E. M. and Güler, Y.1992. **An Investigation of Singular Systems.** Turkish Journal of Physics,(16):297-306.

24- Rabie, E. M. and Güler, Y.1992.**Hamilton-Jacobi Treatment of Second-Class Constraints.** Physical Review A,(64):3513-3515.

25- Rabie ,E. M. and Güler, Y.1995. **An Investigation of Dirac Conjecture on Constraint Systems.** Il Nuovo Cimento,110B(8):893-896.

26- Sudrshan, E.C. and Mukunda, N. 1974.**Classical Dynamics: A Modern Perspective,** (John Wiley and Sons Inc., N.Y.)

27- Sundermeyer,K.1982. **Lectures Notes in Physics**, Springer-Verlag, Berlin.

28- Smetanov, Dana 2006. **On second Order Hamiltonian System,** Archivum Mathematicum; Vol. 42, p341 Academic Journal.

الربط بين طريقة ديراك وطريقة هاميلتون-جاكوبي للأنظمة الهاميلتونية المقيدة من الدرجة الثانية

أحمد علي عبد السادة المقدادي

قسم العلوم
كلية التربية الأساسية
الجامعة المستنصرية

الخلاصة

أن الربطَ بين طريقةِ ديراك وطريـقةِ هاملتون – جاكوبي للأنظمة المقيدة للرتبـةِ الثانيـة قد دُرست. وهنا سوف نبين بأنَّ طريقةَ ديراك دائماً تتفق معَ طريقةِ هاملتون – جاكوبي. وأنَّ الشروط القابلة للتكامل في طريقةِ هاملتون –جاكوبي تكافئُ الشروطُ التناسقية في طريقةِ ديراك.