Carisit's theorem and It's Restrition Dependibg on zermelo's theorem

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Abstract

The purpose of this paper is to establish the relation between fixed point theorem's of Zermelo and Caristi ,the equivalent between them, and show the restriction of Caristi's theorem to continuons function φ can be derived directly from the Zermelo theorem.

١.Introduction

Let X be a non empty set and T be a self-map of X . Let Fix(T) denote the set of all Fixed point of T the converse to Zermelo's fixed point theorem said that if

Fix(T)$\neq \varphi$, then there exists apartial ordering $\preccurlyeq$ such that every chain in (X,$\preccurlyeq$) has a supremum and for all $x \in X$. $x \preccurlyeq Tx$.This result is a converse of Zermelo's fixed point theorem .we also show the equivalent between fixed point theorems of Zermelo and Caristi . Finally ,We discuss relation between Caristi's theorem and it's restriction to mappings satisfying Caristi's condition with a continuous real function φ.

٢-CARISTI / ZERMELO THEOREMS

We begin with show Caristi's theorem and Zermelo's theorem .

Theorem (٢،١) : (Caristi's theorem),[٣ ,p.٥٥]

Let(X,d) be a complete matric space and $\varphi : X \to \mathbb{R}^+$ is lower semicontinuous , suppose T: X⟶ X satisfies :

$$d(x,Tx) \leq \varphi(x) - \varphi(Tx) \quad , x \in X..................(*)$$

Then T has a fixed point .

Proof:

Since $\varphi : X \to R$ is proper ,There exists $u \in X$ with $\varphi(u) < \infty$.

So let X ={ $x \in X : \varphi(x) \leq \varphi(u) - d(u,x)$}

Then X is non empty .We can see that X is invariant under the mapping T(Tx, x) as follows ,Let $x \in X$

$$\varphi(Tx) \leq \varphi(x) - d(x,Tx)$$

$$< \varphi(u) - d(u,x) - d(x,Tx)$$

$$\varphi(Tx) = \varphi(u) - [d(u,x) + d(x,Tx)]$$

$$\leq \varphi(u) - d(u,Tx)$$

Hence, $Tx \in X$

Suppose that $x \neq Tx$, for all $x \in X$.

Then for every $x \in X$,there exists $w \in X$ such that $x \neq w$ and $\varphi(w) + d(x,w) \leq \varphi(x)$ so,by theorem (Ekeland variaonal principle [٣]).We obtain $x_0 \in X$ with $\varphi(x_0) = \inf_{x\in X} \varphi(x)$ for such $x_0 \in X$,we have

$$\begin{aligned} ٠ &< d(x_0,Tx_0) \\ &\leq \varphi(x_0) - \varphi(Tx_0) \\ &< \varphi(Tx_o) - \varphi(Tx_0) \\ &= ٠ \end{aligned}$$

This is a contraction.■

Theorem (٢،٢) (Zermelo's theorem) [٢,P.٥]

Let X be a nonempty abstract set and T be a self-map of X then there exists apartial ordaring $\preccurlyeq$such that every chin in $(X,\preccurlyeq)$has a supremun and T is a progressive with respect to $\preccurlyeq$.

Proposition (٢،١) [٦]

Let $(X,\preccurlyeq)$ be apartially ordered set and $T:X \longrightarrow X$ be a progressive. Then

T=Fix T

Proof

Let $x_0 = T^k_{x_0}$ for some k∈N .Since T is progressive so

$T^{j-1}_{x_0} \preccurlyeq T^J_{X_0}$ for j = ١ , …………, k

Hence $x_0 \preccurlyeq T_{x_0}$ and $T_{x_0} \leq T^K_{x_0} = x_0$.

Which implies that $T_{x_0} = x_0$. ■

Theorem (٢،٣) [٦]

Let X be a nonempty abstract set and T beaself-map of X .if there exists a partial ordering $\preccurlyeq$ such that every chain in $(X,\preccurlyeq)$ has supremum and T is progressive with respect $\preccurlyeq$,then per T=Fix T≠∅

Proof

Fix T≠∅ follows from Zermelo's theorem and Fix T= per T is satisfied by proposition (٢،١) ■.

In fact the inverse of theorem(٢،٣)is a converse to Zermelo's theorem .

Remark (١)

Under the axiom of choise ,the assumption of theorem (١،٣) can be weakened to "each nonempty well-ordered subset has an upper bounded" .This is kneser's [٤] fixed point theorem, In particular, Zermelo's theorem Implies the Banach contraction principle .

The following theorem show the equivalent between theorem (٢،١) and theorem (٢،٢).

Theorem (٢،٤)[١]

Let X be a nonempty sat and T be a self-map of X .the following statements are equivalent:

(i) Theorem (٢،١) (Caristi's theorem)

(ii) Theorem (٢،٢) (Zermelo's theorem)

i⟹ii proof:

Assume (ii) is false.

Then $\forall x \in X \ \exists T(x)$ such that $x \prec T(x)$. It follows that

$$d(x,T(x)) \leq \varphi(x) - \varphi(T(x)) \ , \ x \in X$$

By (i) T must have a fixed point x .

But by a sumption , x<T(x) a contradiction .■

ii⟹ I **proof**:

with X and T as above and φ as in theorem (٢،١), define

$$x \preccurlyeq y \Leftrightarrow d(x,y) \leq \varphi(x) - \varphi(y) \quad , \ x,y \in X$$

such that $\bar{x}$ is a supremum in $(X,\preccurlyeq)$.

but $d(\bar{x}, T(\bar{x})) \leq \varphi(\bar{x}) - \varphi(T(\bar{x}))$.

Hince by maximality , $\bar{x} = T(\bar{x})$.■

Those familiar with logical foundations of mathematics might note that the proof of implication (i)⟹(ii)uses the Axiom of choice ,whereas the proof that (ii)⟹(i) does not .

Corollary (١)

Let (X,d) be a metric space .The following statements are equivalent.

(i) X has a limit point .

(ii)There exists a Caristi mapping T such that T does not satisfy:

$d(x,T(x) \leq \varphi(x) - \varphi(T(x))$ .for any continuous function $\Psi: X \to \mathbb{R}$.

proof (ii) ⟹(i) suppose ,on the contrary ,that X is discrete .Then every function on X is continuous ,which violates (ii) .

(i)⟹ (ii) Let x_0 be a limit point of X. there is on $x_1 \in X$, $x_1 \neq x_0$. Set

$Tx = x_0$ for $x \in \{x_0, x_1\}$ and $Tx = x_1$ for $x \notin \{x_0, x_1\}$,

$\varphi(x_0) = 0$ and $\varphi(x) = d(x,x_1) + d(x_0,x_1)$ for $x \neq x_0$.

Now, $d(x,Tx) = \varphi(x) - \varphi(Tx)$ for all $x \in X$.

Moreover ,φ is l.s.c. suppose that there exists a continuous function Ψ such that $d(x,Tx) \leq \Psi(x) - \Psi(Tx)$ for all $x \in X$.

Hence for $x \notin \{x_0, x_1\}$, $d(x, x_1) \leq \Psi(x) - \Psi(x_1)$.

On the other hand, setting $x = x_1$, we get that $d(x_1, x_0) \leq \Psi(x_1) - \Psi(x_0)$,

by satisfying: $d(x,Tx) \leq \varphi(x) - \varphi(Tx)$.

Hence by satisfied: $d(x_0,x_1) \leq \Psi(x_0) - \Psi(x_1)$.

We may infer that $x_0 = x_1$, which yields a contraction .■

٣-TWO RESTRICTION OF THE CARISTI THEOREM

Under the axiom of choice ,the assumption of theorem (٢،٢)[Zermelo's theorem of fixed point]can be weakened to "each nonempty well-ordered subset has an upper bound "[see ,٤].

The following result is independent of axiom of choice.

Theorem (٣،١) [٥]

The Zermelo's theorem implies the restricition of Caristi's theorem to continuous function φ . more precisely, if T is a self-map of a complete metric space (X,d) such that

$d(x,Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$.

with a continuous function $\varphi : X \longrightarrow R^+$,then T and X endowed with the condition

$x \preccurlyeq y$ if $d(x,y) \leq \varphi(x) - \varphi(y)$.

satisfy the assumption of theorem (٢،٢)

proof:

Let $\preccurlyeq$ be the partial ordering defined on X by $x \preccurlyeq y$ if $d(x,y) \leq \varphi(x) - \varphi(y)$.

by condition:

$d(x,Tx) \leq \varphi(x) - \varphi(y)$ for all $x \in X$.

Clearly that T is a progressive map on $(X,\preccurlyeq)$.

Let C be anonempty well-ordered sub set of $(X,\preccurlyeq)$.

We treat C as a net ,let $\{x_\sigma\}\ \sigma \in C$ is couchy ,hance convergent to some x_0,and $x_\sigma \preccurlyeq x_0$.

Now to show that $x_0 = \sup C$.

By the convergence we get

$d(x_\sigma, x_0) \leq \varphi(x_\sigma) - \varphi(x_0)$ for all $\sigma \in C$

That mean x_0 is anupper bound of C .

Let x be an arbitrary upper of C. Then

$d(x_0,x) \leq d(x_0, x_\sigma) + d(x_\sigma,x) \leq d(x_0, x_\sigma) + \varphi(x_0) - \varphi(x)$

So by taking limit and by continuity φ we obtain that:

$d(x_0,x) \leq \varphi(x_0) - \varphi(x)$.

So $x_0 = \sup C$.■

In fact the Zermelo theorem implies the Banach contraction principle.

Now to prove that the theorem (٢،٢) applies

Let T: $X \longrightarrow X$ be a Banach contraction with a constant $\propto \in$(٠،١).

Let $\varphi(x)=(1-\propto)^{-1} d(x,Tx)$ for $x \in X$.

Since φ is continuous so every non-empty well-ordered subset in $(X,\preccurlyeq)$has a supremum and T is progressive $(X,\preccurlyeq)$.■

Thus ,the following theorem show that, when the metric is fixed, Caristi's theorem is more general than its version involving a continuous function convered by Zermelo's theorem .

Theorem (٣,٢)

Let (X,d) be a matric space. If X has a limit point ,then there exists a mapping T on X such that none of iterates of T Satisfies ($d(x,Tx\leq\varphi(x)-\varphi(Tx))$ for any continuous function $\Psi:X\longrightarrow R$.

Proof

Let x_0 be a limit point of X .There exists a sequence $(x_0)_{n=1}^{\infty}$ such that $x_n\rightarrow x_0, x_n\neq x_0$ and $x_i\neq x_j$ if $i\neq j$. Consider an arbitrary partition of $\mathbb{N}$:

$$\mathbb{N}=\bigcup_{j\in\mathbb{N}}\{k_n^{(j)}:n\in\mathbb{N}\}$$

where $k_n^{(j)}<k_{n+1}^{(j)}$ and $k_n^{(j)}\neq k_m^{(i)}$ if $i\neq j$. For convenience denote $y_n^{(j)}=x_{k_n}^{(j)}$ for $j,n\in\mathbb{N}$ and $y_o^{(j)}=x_0$ for $j\in\mathbb{N}$.

Then $y_n^{(j)}\rightarrow x_0$ as $n\rightarrow\infty$. Hence , by passing to subsequences if necessary,we may assume that for all $j,n\in\mathbb{N}$,

$d(y_n^{(j)},x_0)<1$ and $d(y_n^{(j)},y_m^{(j)})<1/2^n$ for $m>n$ ……….(١)

We define a mapping T . Set $Tx_0=x_0$ and for $j\in\mathbb{N}$,

$Ty_n^{(j)}=y_{n-1}^{(j)}$ for $n=1,\dots,j$ and $Ty_n^{(j)}=y_j^{(j)}$ for $n>j$.

Denote $X_0=\{x_0\}\cup\{y_n^{(j)}:j,n\in\mathbb{N}\}$ and set $Tx=x$ for $x\in X\backslash X_0$.

We define a function φ. Set $\varphi(x_0)=0$ and for $j\in\mathbb{N}$,

$$\varphi(y_n^{(j)})=\sum_{i=1}^{n}d(y_{i-1}^{(j)},y_i^{(j)})\ \text{for } n=1,\dots,j,$$

$$\varphi(y_n^{(j)})=\sum_{i=1}^{j}d(y_{i-1}^{(j)},y_i^{(j)})+d(y_n^{(j)},y_j^{(j)})\ \text{for } n>j.$$

It is easily seen that by (١) , for all $n\in\mathbb{N}$ and $j>1$,

$$\varphi(y_n^{(j)})\leq d(x_0,y_1^{(j)})+\sum_{i=2}^{j}d(y_{i-1}^{(j)},y_i^{(j)})+1/2^j$$

$$<1+\sum_{i=2}^{j}1/2^{i-1}+1/2^j<1+\sum_{i=1}^{\infty}1/2^i=2,$$

While $\varphi(y_1^{(j)}=d\left(x_0,y_1^{(j)}\right)<1$ so that $\varphi\left(y_n^{(j)}\right)<2$ for all $j,n\in\mathbb{N}$. For $x\in X\backslash X_0$, set $\varphi(x)=2$.

Now we show that φ is lsc . Since for all $x\in X, \varphi(x)\geq 0=\varphi(x_0)$, φ is lower semi continuose at x_0. Since , in fact ,

$X_0 = \{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ and $x_n \to x_0$, we get that X_0^d, the derived set of X_0.
Since the set $X \backslash X_0$ is open and φ is constant on it, φ is continuous at each point $x \in X \backslash X_0$.
Now suppose, there exist a $\kappa \in \mathbb{N}$ and a continuous function $\psi : X \mapsto \mathbb{R}$ such that $(*)$ holds with T^k substituted for T . Observe that for all $n > \kappa$,
$T^\kappa y_n^{(\kappa)} = y_1^{(\kappa)}$ so by $(*)$, $d(y_n^{(\kappa)}, T^\kappa y_n^{(\kappa)}) = d(y_n^{(\kappa)}, y_1^{(\kappa)}) \leq \psi(y_n^{(\kappa)}) - \psi(y_1^{(\kappa)})$.
Hence taking the limit with $n \to \infty$ yields
$d(x_0, y_1^{(\kappa)}) \leq \psi(x_0) - \psi(y_1^{(\kappa)})$ ………..(2)
On the other hand ,by $(*)$ we get $d(y_1^{(\kappa)}, T^\kappa y_1^{(\kappa)}) = d(y_1^{(\kappa)}, x_0) \leq \psi(y_1^{(\kappa)}) - \psi(x_0)$.
Hence and by (٢) we obtain that $y_1^{(\kappa)} = x_0$, which yields a contradiction . ■

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مبرهنة كراستي وابتكارها بالاعتماد على مبرهنة زرملو

انسام غازي نصيف

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الخلاصة

الغرض من هذا البحث هو بيان اسس العلاقة بين مبرهنتيي (كراستي و زرملو) للنقطة الصامدة وكذلك التكافؤ بين المبرهنتين وبيان ابتكار مبرهنة كراستي للدالة المستمرة يمكن اشتقاقها مباشرة من مبرهنة زرملو .