Wavelet Polynomials for Solving Linear Fractional Partial Differential Equations

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Abstract

Wavelets methods are commonly used for the numerical solution of partial differential equations. In this paper, we extend the Legendre wavelet polynomial of one variable to Legendre wavelet polynomial of two variables to approximate the solution of fractional partial differential equations. Convergence analysis for the solution is discussed. Total error is computed for the numerical examples to demonstrate the validity of the method.

Keywords: Legendre wavelet method**,** Partial differential equation, Fractional order.

١- Introduction

Ordinary and partial fractional differential equations have been focus of many studies due to their frequent appearance in various applications like in fluid mechanics, biology, physics and engineering. Most fractional ordinary/partial differential equations do not have exact analytical solutions, various numerical and analytic methods have been used to solve these equations. Recently, several numerical methods to solve fractional differential equations have been given such as differential transform method [٧], Taylor collocation method [٨],[٥], homotopy perturbation method [٤] and finite difference method[٢].

In this paper one of the wavelets polynomials (Legendre wavelet) is defined and used to approximate partial differential equations of fractional order, Diethelm method is used to approximate the fractional order. Convergence analysis for the solution is proved, Numerical examples are provided to illustrate the Legendre wavelet method and MathCad ١٤ program is used for computations.

٢- Definitions

Definition ٢,١: The Caputo fractional derivative operator D^{α} of order α is defined in the following form, [٨], [٢]:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt \quad , \qquad \alpha > ٠,$$

where $m-1 < \alpha \le m$, $m \in N$, $x > ٠$.

Some basic properties of the fractional operator are listed below:

- $D^{\alpha}(\lambda f(x)+\mu g(x))=\lambda D^{\alpha} f(x)+\mu D^{\alpha} g(x)$ λ, μ are constants
- $D^{\alpha}D^{\beta} f(x)=D^{\alpha+\beta} f(x)=D^{\beta}D^{\alpha} f(x)$ $\forall$ α, $\beta \in R^{+}$
- $D^{\alpha}C=0$ for any constant C
- $D^{\alpha}x^{n}=\dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}$, for $n\in N$ and $n\geq\alpha$

Definition ٢،٢: For m to be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha>٠$ is defined as, [٦]:

$$D^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_{0}^{x}(t-\tau)^{m-\alpha-1}\frac{\partial^{m}u(x,\tau)}{\partial^{m}t}d\tau, & for \quad m-1<\alpha<m\\ \dfrac{\partial^{m}u(x,\tau)}{\partial t^{m}}, & for \quad \alpha=m\in N\end{cases}$$

Definition٢،٣: Wavelets constitute a family of functions constructed from dilation and translation of a single function called Mother Wavelet. When the dilation parameter *a* and translation parameter *b* vary continuously we have the following family of continuous wavelets as, [٣]:

$$\psi_{a,b}(t)=|a|^{\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \qquad a,b\in R, \quad a\neq 0$$

If we restrict the parameters *a*, *b* to discrete values of $a=a_0^{-k}, b=nb_0a_0^{-k}, a_0>1, b_0>0$ *n* and *k* positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t)=|a|^{\frac{k}{2}}\psi(a_0^{k}t-nb_0),$$

Where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$, then $\psi_{k,n}(t)$ forms an orthonormal basis.

Definition ٢.٤: The following functions, [١]:

$$\psi_{n,m}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{k/2}\,p_m(2^{k}t-\eta), & \frac{\eta-1}{2^{k}}\leq t<\frac{\eta+1}{2^{k}}\\ 0, & otherwise\end{cases} \qquad ---(١)$$

are called Legendre wavelets polynomials where $\eta=2n-1, n=1,...,2^{k-1}, k\in N$, $t\in[0,1]$ and *m* is the order of the Legendre polynomials p_m.

٣- Legendre wavelet method

In this section Legendre wavelet method is proposed for solving the following one dimensional fractional problem :

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}-\frac{\partial^{2}u(x,t)}{\partial x^{2}}=f(x,t), \qquad a\leq x\leq b, \quad 0\leq t\leq T \qquad ---(٢)$$

subject to the boundary conditions:

$$u(0,t)=h_1(t), \quad u(1,t)=h_2(t) \qquad ---(٣)$$

and initial conditions:

$$u(x,0) = g(x) \qquad \text{---(٤)}$$

where α is a parameter describing the fractional derivative, $0 < \alpha \le 1$
The Legendre wavelets series is given by:

$$u(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t)$$

where $\psi_{n,m}(t)$ is given by equation (١). We approximate the solution *u(t)* by the truncated series

$$u_{k,M}(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) \qquad \text{---(٥)}$$

Now we will expand Legendre wavelets series of one variable *t* in equation (٥) to Legendre wavelets series of two variables *x* , *t* to approximate eq.(٢) as follows:
Let *k* = ١, then equation (٥) will be $c_{10}\psi_{10} + c_{11}\psi_{11} + ... + c_{1,M-1}\psi_{1,M-1}$ and write it as

$$c_0\psi_0 + c_1\psi_1 + ... + c_{M-1}\psi_{M-1} = \sum_{m=0}^{M-1} c_m\psi_m(t) \text{ which is denot by } u_M(t)$$

Then Legendre wavelets series of two variables *x* , *t* is given by:

$$u_M(x,t) = \sum_{m=0}^{M-1}\sum_{r=0}^{M-1} c_{rm}\phi_r(x)\psi_m(t) \qquad \text{---(٦)}$$

where $\phi(x)$ and $\psi(t)$ are Legendre wavelet functions, c_{rm} unknown coefficients to compute.

The method starts by dividing the *x*- interval into *n* subintervals to get the grid points $x_i = a + i\Delta x$, where $\Delta x = \dfrac{b-a}{n}$ and $i = 0,1,...,n$; also *t*- interval is divide into *s* subintervals to get the grid points $t_i = j\Delta t$, where $\Delta t = \dfrac{T}{s}$ and $j = 0,1,...,s$;
Substitute eq.(٦) in eq.(١) for $i = 1,...,n-1$ and $j = 1,...,s-1$ to get:

$$\sum_{m=0}^{M-1}\sum_{r=0}^{M-1} c_{rm}\phi_r(x_i)D^{\alpha}\psi_m(t_j) - \sum_{m=0}^{M-1}\sum_{r=0}^{M-1} c_{rm}D^2\phi_r(x_i)\psi_m(t_j) = f(x_i,t_j) \qquad \text{---(٧)}$$

Substitute eq.(٦) in the boundary and initial conditions eqs.(٢),(٣),(٤) to get:

$$\sum_{m=0}^{M-1}\sum_{r=0}^{M-1} c_{rm}\phi_r(0)\psi_m(t_j) = h_1(t_j)\ , \qquad j = 0,1,...,s \qquad \text{---(٨)}$$

$$\sum_{m=0}^{M-1}\sum_{r=0}^{M-1} c_{rm}\phi_r(1)\psi_m(t_j) = h_2(t_j)\ , \qquad j = 0,1,...,s \qquad \text{---(٩)}$$

$$\sum_{m=0}^{M-1}\sum_{r=0}^{M-1} c_{rm}\phi_r(x_i)\psi_m(0) = g(x_i), \qquad i = 0,1,...,n \qquad \text{---(١٠)}$$

Diethelm method, [١١] is using to approximate $D^{\alpha}\psi_m(t_j)$ in eq.(٧):

$$D^{\alpha}\psi_m(t_j)=\int_0^{x_i}\frac{\psi_m(s)-\psi_m(0)}{(t_j-s)^{1+\alpha}}ds=\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\int_0^1\frac{\psi_m(t_j-t_jw)-\psi_m(0)}{w^{1+\alpha}}dw$$

$$\approx\sum_{v=0}^{L}\sigma_v(\psi_m(t_j-t_jv/L)-\psi_m(0)) \quad , \ j=1,...,s-1$$

where σ_v are given by

$$\alpha(1-\alpha)L^{-\alpha}\frac{\Gamma(-\alpha)}{t_j^{-\alpha}}\sigma_v=\begin{cases}-1, & if \ v=0\\ 2v^{1-\alpha}-(v-1)^{1-\alpha}-(v+1)^{1-\alpha}, & if \ v=1,2,...,L-1\\ (\alpha-1)v^{-\alpha}-(v-1)^{1-\alpha}+v^{1-\alpha}, & if \ v=L\end{cases} \quad \text{---(١١)}$$

Then eq. (٧) becomes

$$\sum_{m=0}^{M-1}\sum_{r=0}^{M-1}\sum_{v=0}^{L}\sigma_v(\psi_m(t_j-t_jv/L)-\psi_m(0))c_{rm}\phi_r(x_i)-\sum_{m=0}^{M-1}\sum_{r=0}^{M-1}c_{rm}D^2\phi_r(x_i)\psi_m(t_j)=f(x_i,t_j) \quad \text{---(١٢)}$$

where $i=1,...,n-1$, $j=1,...,s-1$

Combine eqs.(٨),(٩),(١٠) and (١٢) to obtain system which can be solved to get the unknown coefficients c_{rm}.

In the following theorem Legendre wavelets solution converges to exact solution

Theorem ٣,١: Legendre wavelets series of two variables *x,t* gave in eq.(٦) converges towards exact solution *u(x,t)*.

Proof:

By eq.(٦) $c_{rm}=<u_M(x,t),\phi_r(x)\psi_m(t)>$ where <•> represents an inner product and $\phi_r(x),\psi_m(t)$ form orthonormal basis, let $\phi_r(x)\psi_m(t)=\lambda(x,t)$, $u_M(x,t)=u(x,t)$ and let $\alpha_j=<u(x,t),\lambda(x,t)>$. Define S_n be the partial sum of $\alpha_j\lambda(x,t)$, i.e., $S_n=\sum_{j=1}^{n}\alpha_j\lambda(x,t)$. To prove S_n is a Cauchy sequence in Hilbert space, let S_m be arbitrary partial sum with $n\geq m$.

$$<u(x,t),S_n>=<u(x,t),\sum_{j=1}^{n}\alpha_j\lambda(x,t)>$$

$$=\sum_{j=1}^{n}\alpha_j<u(x,t),\lambda(x,t)>$$

$$=\sum_{j=1}^{n}\left|\alpha_j\right|^2$$

$$\left\|S_n-S_m\right\|^2=\left\|\sum_{j=m+1}^{n}\alpha_j\lambda(x,t)\right\|^2$$

$$=<\sum_{j=m+1}^{n}\alpha_j\lambda(x,t),\sum_{j=m+1}^{n}\alpha_j\lambda(x,t)>=\sum_{j=m+1}^{n}\left|\alpha_j\right|^2, \quad \text{for } n>m$$

Hence $\left\| \sum_{j=m+1}^{n} \alpha_j \lambda(x,t) \right\|^2 \to 0$ as $n,m \to \infty$ and $\{S_n\}$ is a Cauchy sequence and it converges to s.

To prove $u(x,t) = s$.

$$< s - u(x,t), \lambda(x,t) > = < s, \lambda(x,t) > - < u(x,t), \lambda(x,t) >$$
$$= < \lim_{n\to\infty} S_n, \lambda(x,t) > - \alpha_j$$
$$= \alpha_j - \alpha_j = 0$$

Hence $s = u(x,t)$ and $\sum_{j=1}^{n} \alpha_j \lambda(x,t)$ converges to $u(x,t)$ and this complete the proof.

٤- Numerical Examples

To demonstrate the effectiveness of the proposed method we consider here two test examples of one dimensional fractional problem. Software MathCad ١٤ is used to get the numerical results.

Example (٤.١):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{2} x^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad , \ 0 < x < 1 \ , \quad 0 < \alpha \le 1, \ t > ٠$$

subject to the boundary conditions: $u(0,t) = 0$, $u(1,t) = e^t$

and the initial condition: $u(x,0) = x^2$

the exact solution for $\alpha = 0.9$, is given by: $u(x,t) = x^2 e^t$

To find σ_v: since $L \in N$, let L = ٥ in eq.(١١) , then $v = 0,1,2,3,4$ and

$$\sigma_0(t) = \frac{-1}{t^\alpha \alpha(1-\alpha) L^{-\alpha} \Gamma(-\alpha)} \ , \quad \sigma_1(t) = \frac{2 - 2^{(1-\alpha)}}{t^\alpha \alpha(1-\alpha) L^{-\alpha} \Gamma(-\alpha)} \quad ,\ldots, \ \sigma_4(t) = \frac{2 - 4^{(1-\alpha)} - 3^{(1-\alpha)} - 5^{(1-\alpha)}}{t^\alpha \alpha(1-\alpha) L^{-\alpha} \Gamma(-\alpha)}$$

Truncate the series in eq.(٧) to M = ٣, we have ٩ unknown coefficients c_{rm}

$c_{00}, c_{01}, c_{02}, c_{10}, c_{11}, c_{12}, c_{20}, c_{21}, c_{22}$ to compute.

Let Δ_1 be a partition for the x-axis, such that: $\Delta_1 : 0 = x_0 < x_1 < x_2 < x_3 = 1$

then $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$, the mesh points for the x-axis.

let Δ_2 be a partition for the t-axis such that: $\Delta_2 : 0 = t_0 < t_1 < t_2 < t_3 = 0.03$

then $t_0 = 0, t_1 = 0.01, t_2 = 0.02, t_3 = 0.03$ the mesh points for t-axis.

By solving the linear system of ١٦ equations and ٦ unknown coefficients we get

$c_{00} = 0.162, c_{01} = 0.106, c_{02} = 0.08, c_{10} = -0.215, c_{11} = -0.222, c_{12} = 0, c_{20} = -0.09, c_{21} = -0.091, c_{22} = 0$

Then the approximate solution is:

$$u_M(x,t) = [0.162\phi_0(x) + 0.106\phi_1(x) + 0.08\phi_2(x)]\psi_0(t) + [-0.215\phi_0(x) - 0.222\phi_1(x)]\psi_1(t) + [-0.09\phi_0(x) - 0.091\phi_1(x)]\psi_2(t)$$

The total error between the approximate solution $u_M(x,t)$ and exact solution $u(x,t)$ for

$x = 0, 0.1, \ldots, 1$ and $t = 0, 0.01, \ldots, 0.1$ is given by

$$\sum_x [\sum_t (u_M(x,t) - u(x,t))^2] = 2.362 \times 10^{-3}$$

Table(١) illustrate the absolute error between the exact solution and Legendre wavelet approximate solution.

Table (١): The approximate and the exact solutions of example (٤.١):

x	t	$u_M(x,t)$	$u(x,t)$	$\mid u_M(x,t) - u(x,t) \mid$
٠	٠,٠١	-٠,١٢٣٤ E-٥	٠	٠.١٢٣٤ E-٥
١/٣	٠,٠١	٠,١١٤	٠,١١٢	٠.١٥٠١ E-٤
٢/٣	٠,٠١	٠,٤٥	٠,٤٤٩	٠.١٣٣١ E-٤
١	٠,٠١	١,٠٠٩	١,٠١	٠.٦٣١٨ E-٥
٠	٠,٠٢	-٠.٢٢٤٣ E-٤	٠	٠.٢٢٤٣ E-٤
١/٣	٠,٠٢	٠,١١٥	٠,١١٣	٠.١٩٧١ E-٤
٢/٣	٠,٠٢	٠.٤٥٧	٠,٤٥٣	٠.٣٢١٢ E-٤
١	٠,٠٢	١,٠٢٢	١,٠٢	٠.١٤٨٢ E-٤
٠	٠,٠٣	-٠.٤١٩٨ E-٤	٠	٠.٤١٩٨ E-٤
١/٣	٠,٠٣	٠.١١٧	٠,١١٤	٠.٢٣٣٥ E-٤
٢/٣	٠,٠٣	٠.٤٦٣	٠,٤٥٨	٠.٤٦٧١ E-٤
١	٠,٠٣	١,٠٣٣	١,٠٣	٠.٢٨١٢ E-٤

Example (٤.٢):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) \quad , \ 0 < x < 1 \ , \quad 0 < \alpha \leq 1, \ t > ٠$$

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x)$$

subject to the boundary conditions: $u(0,t) = 0$, $u(1,t) = 0$

and the initial condition: $u(x,0) = 0$

the exact solution for $\alpha = 0.1$, is given by: $u(x,t) = t^2 \sin(2\pi x)$

As in example (٤.١) we find the unknown coefficients c_{rm}:

$$c_{00} = -1.522 \times 10^{-3}, c_{01} = -0.402, c_{02} = 0, c_{10} = 0, c_{11} = -0.351, c_{12} = 0, c_{20} = 0, c_{21} = -0.092, c_{22} = 0$$

The approximate solution is:

$$u_M(x,t) = [-0.402\phi_1(x)]\psi_0(t) + [-0.351\phi_1(x)]\psi_1(t) + [-0.092\phi_1(x)]\psi_2(t)$$

The total error between the approximate solution $u_M(x,t)$ and exact solution $u(x,t)$ for

$x = 0, 0.1, \ldots, 1$ and $t = 0, 0.01, \ldots, 0.1$ is given by

$$\sum_x [\sum_t (u_M(x,t) - u(x,t))^2] = 1.855 \times 10^{-3}$$

Table(٢) illustrate the absolute error between the exact solution and Legendre wavelet approximate solution.

Table (٢): The approximate and the exact solutions of example (٤.٢):

x	t	$u_M(x,t)$	$u(x,t)$	$\lvert u_M(x,t) - u(x,t)\rvert$
٠	٠,٠١	-٠,٥٠٠٨ E-٥	٠	٠,٥٠٠٨ E-٥
١/٣	٠,٠١	٠,٨٥٣٧ E-٧	٠,٨٦٦ E-٦	٠,٧٨٠٧ E-٦
٢/٣	٠,٠١	٠,٣٤٦ E-٥	-٠,٨٦٦ E-٦	٠,٤٣٢٦ E-٥
١	٠,٠١	٠,٥١١٥ E-٥	٠	٠,٥١١٥ E-٥
٠	٠,٠٢	-٠,١٧٨٧ E-٥	٠	٠,١٧٨٧ E-٥
١/٣	٠,٠٢	٠,١١٥٣ E-٥	٠,٣٤٦٤ E-٥	٠,٢٣١١ E-٥
٢/٣	٠,٠٢	٠,٢٣٧٨ E-٥	-٠,٣٤٦٤ E-٥	٠,٥٨٤٢ E-٥
١	٠,٠٢	٠,١٨٨٧ E-٥	٠	٠,١٨٨٧ E-٥
٠	٠,٠٣	٠,٥٧٠٣ E-٥	٠	٠,٥٧٠٣ E-٥
١/٣	٠,٠٣	٠,٣٦٣٥ E-٥	٠,٧٧٩٤ E-٥	٠,٤١٥٩ E-٥
٢/٣	٠,٠٣	-٠,١٤٠٩ E-٦	-٠,٧٧٩٤ E-٥	٠,٧٦٥٣ E-٥
١	٠,٠٣	-٠,٥٦٢٥ E-٥	٠	٠,٥٦٢٥ E-٥

Conclusions

In this work we derive Legendre wavelet method of two variables for solving fractional partial differential equations also demonstrated the convergence analysis of the method. Absolute error in tables(١), (٢) and total error for numerical examples show the validity of the method.

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متعددة حدود ويفليت لحل المعادلات التفضلية الجزئية الكسرية الخطية

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الخلاصة

طرق ويفليت تستخدم عادة لأجل الحل العددي للمعادلات التفاضلية الجزئية. في هذا البحث نوسع متعددة حدود ويفليت ليجندر ذات متغير واحد الى متعددة حدود ويفليت ليجندر ذات متغيرين اثنين لتقريب حل في المعادلات التفاضلية الجزئية الكسرية. البحث ناقش تحليل التقارب للحل. الخطأ الكلي تم حسابه في الامثلة العددية لأظهار فاعلية الطريقة.