Numerical Solution of the System of Linear Volterra Integro – Differential Equations of the First Order using Monte- Carlo Method.

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Abstract

In this paper, a method for solving linear system of Volterra integro – differential equation of the first order numerical presented based on Monte – Carlo techniques. Numerical examples illustrate the pertinent features of the method with the proposed system.

The computer program are written in (TURBO MATLAB) language (Version ٦).

Introduction

It is in general, very difficult to find a useful solution of a linear integro – differential equations of the first order if the solution depends on several variables or if the equation is coupled with other integro – differential equations. Well known methods of solution are mostly in effective because the amount of computation involved is too great, even for the latest machines. In many cases, however, especially in particle transport problems, one can use a statistical procedure – the Monte – Carlo method to find a solution which is sufficient accurate for practical purposes. [٢]

The application of a well known Monte – Carlo method to the solution of an equation with non-negative kernel is already illustrated in the paper [٢,٣,٥].

In this study, the basic ideas of the previous works are developed and applied to the system of the linear Volterra Integro – Differential Equation of the first order.

Consider the following system of linear Volterra Integro– Differential Equation of order *n* is an equation of the form:

$$u^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x) u^{(i)}(x) = f(x) + \lambda \int_a^{b(x)} k(v,t) u(t) dt \text{ -----(١)}$$

Here, K(x, t) , f(x), $p_i(x)$ (h=٠,١,……., n-١) are known functions, u(x) is the known function, and λ is a scalar parameter [٤].

When n=١

$$u'(x) + p(x)u(x) = f(x) + \int_a^x k(x,t)u(t)dt, x \in i = [a,b] \text{--(٢)}$$

With the initial condition u(a) = u_o, where the functions f and p are assumed to be continuous on I and K denotes given continuous functions.

Are the interval [a,b] is divided into n equal subintervals, where *h=(b-a)/n*, $y_o=a$, $y_n=b$ and $y_j=a+j*h$, *j= ٠, ١, ...,n*. we set $x_i=y_j$, *(i= ٠, ١, ..., n)*, $u(x_i)= u_i$, $p(x_i) = p_i$, $f(x_i) = f_i$, , $u(x_i) = u_i$ and $k(x_i, y_j)= k_{ij}$.

Numerical Approximation

The method we propose is based on a similar concept of the So- Called Monte – Carlo method to approximate integro – differential when the integrand is known the method is developed as iteration technical where the approximation is over a finite interval for the unknown function, At each step we have a new partition for the interval; such partition comes from the generation of random numbers on the interval [١].

We can summarize the method as follows:

Step ١: We start with a value in the interval where we want to find the approximation of the function (the unknown function).

Step ٢: We assume that the function is a constant then it can be put outside the integro – differential and we solve a system linear equation.

Step ٣: We generate a random number on the interval we want to find the approximation we assume that the functions are stepwise function. We take the solution of the system of linear equation of step ٢ as the value of the function a round some subinterval. We assume a new unknown value for the function around some other interval. Again the unknown value for the function can be put outside the integro – differential, we sole a new system linear equation to find a new value, then we come back step ٢.

This iteration come from the fact that we generate random numbers on the interval where we want to find the approximation.

Approximation for system of linear Volterra Integro – Differential Equation of the First Order

To find an approximation for the solution u_i=(x), *i*=١,…, m of (٢) on the interval [r, b] where a ≤ r as follows we start with $x_o \in$[r, b], from eq.(١) we known that $u_i(x)$, i=١, ٢, …, m satisfies

$$[u_i'(x) + p_i(x)u_i = f_i(x) + \int_a^x k(x_i, t_j)u(t_j)dt] \quad ٠ \le x \le b \; ….(٣)$$

$$\frac{u_i - u_{i-1}}{h} - p_i u_i = f_i + \int_0^x k(x_i, t_j)u(t_j)dt$$

$$u_i - u_{i-1} - hp_i u_i = hf_i + h\int_0^x k(x_i, t_j)u(t_j)dt$$

$$(1 + hp_i)u_i - u_{i-1} = hf_i + h\int_0^x k_{ij}u_j$$

$$(1 + h)u_i - u_{i-1} = hf_i + h\sum_{j=0}^{i-1} k_{ij}u_j$$

$$(1 + h)u_i - u_{i-1} = hf_i + h[k_{i0}u_0 + k_{i1}u_1 + k_{i2}u_2 + ... + k_{ii-1}u_{i-1}]$$

When i=١

$$(1 + h)u_1 - u_0 = hf_1 + hk_{10}u_0 + hk_{11}u_1$$

$$(1 + h - hk_{11})u_1 = hf_1 + hk_{10}u_0 + u_0$$

When i=٢

$$(1 + h)u_2 - u_1 = hf_2 + h[k_{20}u_0 + k_{21}u_1 + k_{22}u_2]$$

$$(1 + h - hk_{22})u_2 - (1 + hk_{21})u_1 = hf_2 + hk_{20}u_0$$

i=٣

$$(1 + h)u_3 - u_2 = hf_3 + h[k_{30}u_0 + k_{31}u_1 + k_{32}u_2 + k_{33}u_3]$$

$$(1 + h - hk_{33})u_3 - (1 + hk_{32})u_2 - hk_{31}u_1 = hf_3 + hk_{30}u_0$$

$$\vdots$$

AX= B [٤]

$$\begin{bmatrix} -1 + p_1 h - hk_{11} & 0 & 0 & 0 \\ -1 - hk_{21} & 1 + p_2 h - hk_{22} & 0 & 0 \\ -hk_{31} & -1 - hk_{32} & 1 + p_3 h - hk_{33} & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} hf_1 + hk_{10}u_0 + u_0 \\ hf_2 + k_{20}u_0 \\ hf_3 + k_{30}u_0 \end{bmatrix}$$

Algorithm (AQ)

In this algorithm we expose the step for solve VIDE$_s$ of ١st order using Monte-Carlo method.

Step ١

a- put h= (b-a)/n , n∈N.

b- Set u(a)=u٠ (which is initial condition) is given

c- Compute u_i' by using

$$u_i' = \frac{u_i - u_{i-1}}{h}$$

Step ٢

Using step -١ in equation (٣) to find u_i, (i=١,٢,….., n)

$$(1 + hp_i k_{ii})u_i - u_{i-1} = hf_i + h[k_{i0}u_0 + k_{i1}u_1 + ... + k_{ii-1}u_{i-1}]$$

Illustrative Example:

In this section, two examples are presented for demonstrating the methods and a depending on the ,least square errors.

Example ١: [٧]

Consider the following VIDE

$$u'(x) = -\frac{1}{2} - \frac{5}{6}x + \int_0^x (x\,t + 1)u(t)dt \quad \text{٠} \le x \le \text{١}$$

The exact solution is $u_{\text{١}}(x)$= ١+ x and $u_{\text{٢}}(x)$= ١-٠,٥x-٠,٤١٦٦٧$x^{\text{٢}}$

Take n = ١٠ and x_i= $a+ih$, i=٠, ١, ….., n

The numerical results obtained for example ١ are shown in Table ١ and Table ٢.

Example ٢

Consider the following Volterra Integro – Differential Equation:

$$u'(x) = 2x - \frac{5}{6}x^4 + \int_0^x (x + 2t)u(t)dt \quad x \ge \text{٠}$$

Table (٢) presents results from a computer program that solves this problem over the interval x=٠ to x=١ with u١(x) = x$^{\text{٢}}$, and u٢(x) = x$^{\text{٢}}$ -١,٦٦٦٧x$^{\text{٥}}$ for which the analytical solution h=٠,١ [٧]

The numerical results obtained for example ٢ are shown in Table ٣ and Table ٤

Table١: Comparison between the exact solution (١+x) and the numerical solution $u_{١}(x)$ of example ١ with different value of h.

Table (١)

$u_{١}(x)$=١+x					
x	***Exact solution***	***h=٠,٠٥***	***h=٠,٠٣***	***h=٠,٠١***	***h=٠,١***
٠	١	٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠
٠,١	١,١	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٥١٣٤	-٠,٠٦٤٨٨٦٩١١
٠,٢	١,٢	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠١٠٤٥٧	-٠,١٥٤٢٠٩٨٦٩
٠,٣	١,٣	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠١٦٠٢٣	-٠,٢٨٣٠٩٧٠٤٥
٠,٤	١,٤	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٢١٨٩٢	-٠,٤٧٦٨٥٥٣٧٦
٠,٥	١,٥	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٢٨١٢٧	-٠,٧٧٨٥١٦٦٣٩
٠,٦	١,٦	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٣٤٧٩٤	-١,٢٧٠٢٩٩٥٨١
٠,٧	١,٧	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٤١٩٦٧	-٢,٠٩٦٤٣٣٩٠٩
٠,٨	١,٨	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٤٩٧٢٥	-٣,٥٤٠٩٤٠٢٠١
٠,٩	١,٩	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٥٨١٥٣	-٦,١٦٧٨٠٤٣٧٨٠
١,٠	٢,٠	-٠,٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٦٧٣٤٧	-١١,١٤٥٢٠٥١٠٧
L.S.E		٢,٢٦٢٦٨٣٨٢٣e+١١٤	٣,٠٧٨٦٣٢٢٥e+٠٣٣	١,٢٨٦٧٥٣٠٠e+٠٠٨	٣,٠٣٤٧٨٨٨١٥e+٠٠٢
R.T		١٧,٣٠٠٠٠٠٠٠٠	١٦,٦٩٠٠٠٠٠٠٠	١٦,٩١٠٠٠٠٠٠٠	٠,٣٩٠٠٠٠٠٠٠

Table٢: Comparison between the exact solution ١-٠,٥x-٠,٤١٦٦٧$x^{٢}$ and the numerical solution $u_{٢}(x)$ of example ١ with different value of h.

Table (٢)

$u_{١}(x)$= ١-٠,٥x-٠,٤١٦٦٧$x^{٢}$					
x	***Exact solution***	***h=٠,٠٥***	***h=٠,٠٣***	***h=٠,٠١***	***h=٠,١***
٠	١	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠
٠,١	٠,٩٤٥٨٣٣٣٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٥١٣٤	-٠,٠٦٤٨٨٦٩١١
٠,٢	٠,٨٨٣٣٣٣٢٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠١٠٤٥٧	-٠,١٥٤٢٠٩٨٦٩
٠,٣	٠,٨١٢٤٩٩٧٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠١٦٠٢٣	-٠,٢٨٣٠٩٧٠٤٥
٠,٤	٠,٧٣٣٣٣٢٨٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٢١٨٩٢	-٠,٤٧٦٨٥٥٣٧٦
٠,٥	٠,٦٤٥٨٣٢٥٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٢٨١٢٧	-٠,٧٧٩٥١٦٦٣٩
٠,٦	٠,٥٤٩٩٩٨٨٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٣٤٧٩٤	-١,٢٧٠٢٩٩٥٨١
٠,٧	٠,٤٤٥٨٣١٧٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٤١٩٦٧	-٢,٠٩٦٤٣٣٩٠٩
٠,٨	٠,٣٣٣٣٣١٢٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٤٩٧٢٤	-٣,٥٤٠٩٤٠٢٠١
٠,٩	٠,٢١٢٤٩٧٣٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٥٨١٥٣	-٦,١٦٧٨٠٤٣٧٨
١,٠	٠,٠٨٣٣٣٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٠٠٠٠٠٠	-٠,٠٠٠٠٦٧٣٤٧	-١١,١٧٥٢٠٥١٠٧٨
L.S.E		٢,٢٦٢٦٨٣٨٢٣e+١١٤	٣,٠٧٨٦٣٢٢٣٥e+٠٣٣	١,٢٨٥٢٦٠٤٥١e+٠٠٨	-١,٩٨٣٦٩٣٨٨٤e+٠٠٢
R.T		١٧,١٩٠٠٠٠٠٠٠	١٧,١٩٠٠٠٠٠٠٠	١٧,١٩٠٠٠٠٠٠٠	٠,٣٣٠٠٠٠٠٠٠

Table٣: Comparison between the exact solution $X^{٢}$ and the numerical solution $u_{١}(x)$ of example ٢ with different value of h.

$u_{١}(x)=x^{٢}$					
x	***Exact solution***	***h=٠,٠٥***	***h=٠,٠٣***	***h=٠,٠١***	***h=٠,١***
٠	٠	٠	٠	٠	٠
٠,١	٠,٠١	٠,٠٠٠٠٠٠٠٠٠٠	٠.٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٢٠٠	٠,٠٢٠٦٠٩٩٦٥
٠,٢	٠,٠٤	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٦٠٠	٠,٠٦٥٢١٣٨٦٢
٠,٣	٠,٠٩	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠١٢٠٢	٠,١٤٣٠٠٤٧٥٩

٠,٤	٠,١٦	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٢٠٠٦	٠,٢٧٤٥٧٤٥٥٦
٠,٥	٠,٢٥	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٣٠١٥٣	٠.٥٠٣٦٥١٠٣٩
٠,٦	٠,٣٦	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٤٢٣٢	٠,٩٢٣٤٢٢٤٣٧
٠,٧	٠,٤٨	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٥٦٦٢	١,٧٣٨٣٤٢١١١
٠,٨	٠,٦٤	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٧٣١٢	٣,٤١٥٥٥٩١٧٤
٠,٩	٠,٨١	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٩١٨٨	٧,٠٧٢٧١٠٠٦٨
١,٠	١,٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠١١٣٠٢	١٥,٥١٨٢٤٧٨٦٩
L.S.E		٦,٩١٨٠٢٠٩٩٧e+٠٨٣	١,٠٦٣٧٥٣٦٥٨e+٠٣٦	٩,١٤٧٠٠٥٥٧٨e+٠٠٧	٢,٥٩٦٦١٦٤٣e+٠٠٢
R.T		١٧,٥٢٠٠٠٠٠٠٠	١٨,٣٥٠٠٠٠٠٠٠	١٧,١٩٠٠٠٠٠٠٠	٠,٣٣٠٠٠٠٠٠٠

Table ٤: Comparison between the exact solution x٢-١,٦٦٦٧x٥ and the numerical solution u٢(x) of example ٢ with different value of h.

Table (٤)

u٢(x)=x٢- ١,٦٦٦٧x٥					
x	Exact solution	h=٠,٠٥	h=٠,٠٣	h=٠,٠١	h=٠,١
٠	٠,٠٠٠	٠,٠٠	٠,٠٠	٠,٠٠	٠,٠٠٠
٠,١	٠,٠٠٩٩٨٣٣٣٣	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٢٠٠٠	٠,٠٢٠٦٠٩٩٦٥
٠,٢	٠,٠٣٩٤٦٦٦٥٦	٠,٠٠٠٠٠٠٠٠٠٠	٠.٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٦٠٠٤	٠,٠٦٥٢١٣٨٦٢
٠,٣	٠,٠٨٥٩٤٩٩١٩	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠١٢٠٢٠	٠,١٤٣٠٠٤٧٥٩
٠,٤	٠,١٤٢٩٣٢٩٩٢	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٢٠٠٦٢	٠,٢٧٤٥٧٤٥٥٦
٠,٥	٠,١٩٧٩١٥٦٢٥	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٣٠١٥٣	٠,٥٠٣٦٥١٠٣٩
٠,٦	٠,٢٣٠٣٩٧٤٠٨	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٤٢٣٢٧	٠,٩٢٣٤٢٢٤٣٧
٠,٧	٠,٢٠٩٨٧٧٧٣١	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٥٦٦٢٩	١,٧٣٨٣٤٢١١١
٠,٨	٠,٠٩٣٨٥٥٧٤٤	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٧٣١٢٣	٣,٤١٥٥٥٩١٧٤
٠,٩	-٠,١٧٤١٦٩٦٨٣	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٩١٨٨	٧,٠٧٢٧١٠٠٦٨
١,٠	-٠,٦٦٦٧٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠٠٠٠٠٠٠	٠,٠٠٠٠١١٣٠٢١	١٥,٥١٨٢٤٧٨٦٩
L.S.E		٦,٩١٨٠٢٠٩٩٧e+٠٨٣	١,٠٦٣٧٥٣٦٥٨e+٠٣٦	٩,١٥٥٤٢٤١٠٩e+٠٠٧	٣,٢٨٤٣٤٨٣٩١e+٠٠٢
R.T		١٧,١٩٠٠٠٠٠٠٠	١٧,١٩٠٠٠٠٠٠٠	١٧,٥٢٠٠٠٠٠٠٠	٠,٣٨٠٠٠٠٠٠٠

Conclusions

According to the numerical results which obtaining from the illustrative example, we conclude that for sufficiently small h we get a good accuracy, since by reducing step size length the least square error and the running time will be reduced.

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الحل العددي لنظام من معادلات فولتيرا التكاملية – التفاضلية الخطية ذات الرتبة الاولى باستخدام طريقة (مونتيكارلو)

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الخلاصة

في هذا البحث توجد طريقة لحل معادلة فولتيرا التكاملية – التفاضلية الخطية ذات الرتبة الاولى عددياً قدمت بالاعتماد على طريقة (تقنية مونت كارلو). الامثلة العددية وضحت الاشكال ذات العلاقة بطريقة النظام المقترح والبرامج الحسابية كتبت بلغة ماتلاب (Version ٦)