Numerical Treatment for n^th Order Linear Functional-Differential Equations Using Nyström's Method

Ahmed Showqi Jaber
College of Science
University of Al-Mustansiriyah
Asra'a Hadi Hassan
Atheer Jawad Kadhim
Department of Applied Sciences
University of Technology

Abstract

The paper presents a proposed method with algorithms written in Matlab language for solving a system of linear functional-differential equations numerically using fifth-order-six-stage Nyström’s method. This method has been developed to find the numerical evolutions for three types (retarded, neutral and mixed) of n^th order linear functional differential equations. Comparison between the numerical and exact results has been given for three numerical examples for solving three types of linear functional differential equations. Finally, the results are arranged in tabulated form and suitable graphing is given for the examples.

1. Introduction

The functional differential equation "FDE" is an differential equation in which function having delay arguments where it is an equation in an unknown function y(t) and some of its derivatives are evaluated at arguments that differ in any of fixed number of values $\tau_1, \tau_2, ..., \tau_n$. FDE has been developed over twenty years ago. It has been much effort devoted to study n^th order linear *functional- differential equations* of the form:

$$F(t, y(t), y(t-\tau_1), ..., y(t-\tau_k), y'(t), y'(t-\tau_1), ..., y'(t-\tau_k), y^{(n)}(t), y^{(n)}(t-\tau_1), ..., y^{(n)}(t-\tau_k)) = 0 \quad ... (1)$$

where F is a given function and $\tau_1, \tau_2, ..., \tau_k$ are given positive numbers called “difference argument” [1].

In some literature, eq.(1) is called a differential equation with deviating argument [2,3], or an equation with time lag [4].

In this work, fifth order six-stage Nyström’s method has been used to find the numerical solution of n^th-order linear functional differential equations which given with their types in the following section.

2. Functional-Differential Equation :

A functional-differential equation (FDE) is a difference equation in which various derivatives of the function y(t) can be present. FDE arises in many realistic models of problems in science, engineering and medicine, only in the last few years has much effort in behavior of solution of linear functional differential equation [5,6].

The n^{th} order functional differential equation is given by eq.(1). The main difference between functional differential equation and ordinary differential equation is the kind of initial condition that should be used in functional differential equation differs from ordinary differential equation so that one should specify in functional differential equations an initial function on some interval of length τ, say $[t_0 - \tau, t_0]$ and then try to find the solution of equation (1) for all $t \geq t_0$ [7,8].

The functional-differential equation is classified into three types [1,6,7] :-

- Equation (1) is called a *Neutral* type if the highest-order derivative of unknown function appears both with and without difference argument.
- Equation (1) is called *Retarded* type if the derivatives of unknown function appear without difference argument.
- All other FDE (1) with *mixed* types i.e. a combination of the previous two types.

3. Nyström’s Method:

Nyström’s method provides efficient mean for the solution of the many problem arising in various fields of science and engineering. It uses only the information from the last step computed; therefore, it is called single-step method [7,9].

Nyström [10,11] uses this method to find the numerical solution of linear ordinary differential equations.

Consider the following first order linear differential equation:

$$\frac{dy}{dt} = f(t, y) \quad \text{with initial condition} \quad y(t_0) = y_0 \qquad \ldots (2)$$

In order to solve the above equation numerically, a fifth order six-stage method which is valid for a linear differential equation is derived by Nyström [9,10] as:

$$\left.\begin{aligned}
&y_{n+1}-y_n=\frac{h}{192}\left(23N_1+125N_3-81N_5+125N_6\right) \quad ,\\
&where\\
&N_1=f(t_n,y_n),\\
&N_2=f(t_n+\tfrac{1}{3}h,y_n+\tfrac{1}{3}hN_1),\\
&N_3=f\left(t_n+\tfrac{2}{5}h,y_n+\tfrac{1}{25}h(4N_1+6N_2)\right),\\
&N_4=f\left(t_n+h,y_n+\frac{h}{4}(N_1-12N_2+15N_3)\right),\\
&N_5=f\left(t_n+\tfrac{2}{3}h,y_n+\frac{h}{81}(6N_1+90N_2-50N_3+8N_4)\right)\\
&N_6=f\left(t_n+\tfrac{4}{5}h,y_n+\frac{h}{75}(6N_1+36N_2+10N_3+8N_4)\right)
\end{aligned}\right\} \quad \ldots (3)$$

In this work the idea of Nyström's method has been used to treat both linear functional differential equation as well as system of linear functional differential equations numerically.

3.1 The Solution of a First Order Linear Functional Differential Equation Using Nyström Method:

In this subsection Nyström method has been used to find the numerical solution for a 1st order linear functional differential equation.

Consider the first order linear functional differential equation (FDE) :-

$$y'(t)=f(t,y(t),y(t-\tau),y'(t-\tau)) \quad , \quad t\in[t_0,\infty) \qquad \ldots (4)$$

with initial function

$$y(t)=\phi(t) \quad \text{for} \quad t_0-\tau\le t\le t_0$$

eq.(4) may be solved if we use the initial function as:

$$y'(t)=f(t,y(t),\phi(t-\tau),\phi'(t-\tau)) \qquad \ldots (5)$$

with initial condition

$$y(t_0)=\phi(t_0)$$

For solving the above linear FDE by Nyström method, eq.(3) is written as:

$$\begin{aligned}
&y(t_{j+1})=y(t_j)+\frac{h}{192}\left(23N_1+125N_3-81N_5+125N_6\right),\\
&where\\
&N_1=f\left(t_j,y(t_j),\phi(t_j-\tau),\phi'(t_j-\tau)\right),\\
&N_2=f\left(t_j+\tfrac{1}{3}h,y(t_j)+\tfrac{1}{3}hN_1,\phi((t_j+\tfrac{1}{3}h)-\tau),\phi'((t_j+\tfrac{1}{3}h)-\tau)\right),\\
&N_3=f\left(t_j+\tfrac{2}{5}h,y(t_j)+\tfrac{1}{25}h(4N_1+6N_2),\phi((t_j+\tfrac{2}{5}h)-\tau),\phi'((t_j+\tfrac{2}{5}h)-\tau)\right),\\
&N_4=f\left(t_j+h,y(t_j)+\frac{h}{4}(N_1-12N_2+15N_3),\phi((t_j+h)-\tau),\phi'((t_j+h)-\tau)\right),\\
&N_5=f\left(t_j+\tfrac{2}{3}h,y(t_j)+\frac{h}{81}(6N_1+90N_2-50N_3+8N_4),\phi((t_j+\tfrac{2}{3}h)-\tau),\phi'((t_j+\tfrac{2}{3}h)-\tau)\right)\\
&N_6=f\left(t_j+\tfrac{4}{5}h,y(t_j)+\frac{h}{75}(6N_1+36N_2+10N_3+8N_4),\phi((t_j+\tfrac{4}{5}h)-\tau),\phi'((t_j+\tfrac{4}{5}h)-\tau)\right).
\end{aligned} \qquad \ldots (6)$$

for each $j = 0,1,....,m$, where $(m + 1)$ is the number of points ($t_0, t_1, ..., t_m$).

The numerical solution using fifth order ***N****yström* ***m****ethod* of the ***f****irst* order linear ***FDE*** in eq.(4) can be summarized by the following algorithm :

NM-FFDE Algorithm :

Step 1: Input the step length h and the initial value t_0 .

Step 2: Set j=0

Step 3: Compute: $N_1 = f(t_j, y(t_j), \phi(t_j - \tau), \phi'(t_j - \tau))$

Step 4: Compute :

$$N_2 = f(t_j + \tfrac{1}{3}h, y(t_j) + \tfrac{1}{3}hN_1, \phi(t_j + \tfrac{1}{3}h - \tau), \phi'(t_j + \tfrac{1}{3}h - \tau))$$

Step 5: Compute :

$$N_3 = f\left(t_j + \tfrac{2}{5}h, y(t_j) + \tfrac{1}{25}h(4N_1 + 6N_2), \phi(t_j + \tfrac{2}{5}h - \tau), \phi'(t_j + \tfrac{2}{5}h - \tau)\right)$$

Step 6: Compute :

$$N_4 = f\left(t_j + h, y(t_j) + \frac{h}{4}(N_1 - 12N_2 + 15N_3), \phi(t_j + h - \tau), \phi'(t_j + h - \tau)\right)$$

Step 7: Compute :

$$N_5 = f\left(t_j + \tfrac{2}{3}h, y(t_j) + \frac{h}{81}(6N_1 + 90N_2 - 50N_3 + 8N_4), \phi(t_j + \tfrac{2}{3}h - \tau), \phi'(t_j + \tfrac{2}{3}h - \tau)\right)$$

Step 8: Compute :

$$N_6 = f\left(t_j + \tfrac{4}{5}h, y(t_j) + \frac{h}{75}(6N_1 + 36N_2 + 10N_3 + 8N_4), \phi(t_j + \tfrac{4}{5}h - \tau), \phi'(t_j + \tfrac{4}{5}h - \tau)\right)$$

Step 9: Compute :

$$t_{j+1} = t_0 + (j+1)h$$

$$y(t_{j+1}) = y(t_j) + \frac{h}{192}\left(23N_1 + 125N_3 - 81N_5 + 125N_6\right)$$

Step 10: Put $j = j+1$

Step 11: If $j = m$ then stop.
Else go to (step 3)

3.2 The Solution of a System of The First Order Linear Functional Differential Equations Using Nyström Method:

Consider the following system of the first order linear functional differential equations:

$$\frac{dy_i(t)}{dt} = f_i(t, y_1(t), ..., y_n(t), y_1(t - \tau_1), ..., y_n(t - \tau_n), y_1'(t - \tau_1), ..., y_n'(t - \tau_n)), \quad ... (7)$$

where $t \in [t_0, \infty)$ and f_i, i=1,2,…,n denotes the ith linear functions.

The initial functions of eq.(7) are:

$$\begin{aligned} y_1(t) &= \phi_1(t) \quad for \quad t_0-\tau_1 \le t \le t_0 \\ &\vdots \\ y_n(t) &= \phi_n(t) \quad for \quad t_0-\tau_n \le t \le t_0 \end{aligned} \qquad \dots (8)$$

Equation (7) can be solved by using Nyström method if the initial functions in eq.(8) were used as follows:

$y_i'(t) = f_i(t, y_1(t),\dots, y_n(t), \phi_1(t-\tau_1),\dots,\phi_n(t-\tau_n), \phi_1'(t-\tau_1),\dots,\phi_n'(t-\tau_n))$, i=1,…,n... (9)

with initial conditions

$$y_1(t_0)=\phi_1(t_0),\dots, y_n(t_0)=\phi_n(t_0).$$

For treating a system of a first order of linear FDE's by Nyström method, eq.(3) is written as:

$$y_i(t_{j+1}) = y_i(t_j) + \frac{h}{192}\left(23N_{1i} + 125N_{3i} - 81N_{5i} + 125N_{6i}\right) \qquad \dots (10)$$

where

$$N_{1i} = f_i(t_j, y_1(t_j),\dots, y_n(t_j), \phi_1(t_j-\tau_1),\dots,\phi_n(t_j-\tau_n), \phi_1'(t_j-\tau_1),\dots,\phi_n'(t_j-\tau_n))$$

$$N_{2i} = f_i\begin{pmatrix} t_j+\frac{1}{3}h, y_1(t_j)+\frac{1}{3}hN_{11},\dots, y_n(t_j)+\frac{1}{3}hN_{1n}, \phi_1(t_j+\frac{1}{3}h-\tau_1),\dots, \\ \phi_n(t_j+\frac{1}{3}h-\tau_n), \phi_1'(t_j+\frac{1}{3}h-\tau_1),\dots,\phi_n'(t_j+\frac{1}{3}h-\tau_n) \end{pmatrix}$$

$$N_{3i} = f_i\begin{pmatrix} t_j+\frac{2}{5}h, y_1(t_j)+\frac{h}{25}(4N_{11}+6N_{21}),\dots, y_n(t_j)+\frac{h}{25}(4N_{1n}+6N_{2n}), \\ \phi_1(t_j+\frac{2}{5}h-\tau_1),\dots,\phi_n(t_j+\frac{2}{5}h-\tau_n), \phi_1'(t_j+\frac{2}{5}h-\tau_1),\dots,\phi_n'(t_j+\frac{2}{5}h-\tau_n) \end{pmatrix}$$

$$N_{4i} = f_i\begin{pmatrix} t_j+h, y_1(t_j)+\frac{h}{4}(N_{11}-12N_{21}+15N_{31}),\dots, y_n(t_j)+\frac{h}{4}(N_{1n}-12N_{2n}+15N_{3n}), \\ \phi_1(t_j+h-\tau_1),\dots,\phi_n(t_j+h-\tau_n), \phi_1'(t_j+h-\tau_1),\dots,\phi_n'(t_j+h-\tau_n) \end{pmatrix}$$

$$N_{5i} = f_i\begin{pmatrix} t_j+\frac{2}{3}h, y_1(t_j)+\frac{h}{81}(6N_{11}+90N_{21}-50N_{31}+8N_{41}),\dots, y_n(t_j)+\frac{h}{81}(6N_{1n}+90N_{2n}-50 \\ N_{3n}+8N_{4n}),\dots,\phi_1(t_j+\frac{2}{3}h-\tau_1),\dots,\phi_n(t_j+\frac{2}{3}h-\tau_n), \phi_1'(t_j+\frac{2}{3}h-\tau_1),\dots,\phi_n'(t_j+\frac{2}{3}h-\tau_n) \end{pmatrix}.$$

$$N_{6i} = f_i\begin{pmatrix} t_j+\frac{4}{5}h, y_1(t_j)+\frac{h}{75}(6N_{11}+36N_{21}+10N_{31}+8N_{41}),\dots, y_n(t_j)+\frac{h}{75}(6N_{1n}+36N_{2n}+10 \\ N_{3n}+8N_{4n}),\dots,\phi_1(t_j+\frac{4}{5}h-\tau_1),\dots,\phi_n(t_j+\frac{4}{5}h-\tau_n), \phi_1'(t_j+\frac{4}{5}h-\tau_1),\dots,\phi_n'(t_j+\frac{4}{5}h-\tau_n) \end{pmatrix}$$

for each i=1,2,…,n and j=0,1,…, m .

The following (NM-SYFDE) algorithm summarizes the steps for finding the numerical solution by using ***N**yström **m**ethod* for a ***sy**stem* of ***FDE*** in eq.(7).

NM-SYFDE Algorithm :

Step 1: Input the step length h and the initial value t_0.

Step 2: Set j=0

Step 3: For each i=1,2,…,n compute:

$$N_{1i} = f_i(t_j, y_1(t_j),\dots, y_n(t_j), \phi_1(t_j-\tau_1),\dots,\phi_n(t_j-\tau_n), \phi_1'(t_j-\tau_1),\dots,\phi_n'(t_j-\tau_n))$$

Step 4: $\forall\, i = 1,2,\ldots,n$ compute :

$$N_{2i} = f_i\left(\begin{array}{c} t_j + \tfrac{1}{3}h, y_1(t_j) + \tfrac{1}{3}hN_{11}, \ldots, y_n(t_j) + \tfrac{1}{3}hN_{1n}, \phi_1(t_j + \tfrac{1}{3}h - \tau_1), \ldots, \\ \phi_n(t_j + \tfrac{1}{3}h - \tau_n), \phi_1'(t_j + \tfrac{1}{3}h - \tau_1), \ldots, \phi_n'(t_j + \tfrac{1}{3}h - \tau_n) \end{array}\right)$$

Step 5: $\forall\, i = 1,2,\ldots,n$ compute :

$$N_{3i} = f_i\left(\begin{array}{c} t_j + \tfrac{2}{5}h, y_1(t_j) + \dfrac{h}{25}(4N_{11} + 6N_{21}), \ldots, y_n(t_j) + \dfrac{h}{25}(4N_{1n} + 6N_{2n}), \\ \phi_1(t_j + \tfrac{2}{5}h - \tau_1), \ldots, \phi_n(t_j + \tfrac{2}{5}h - \tau_n), \phi_1'(t_j + \tfrac{2}{5}h - \tau_1), \ldots, \phi_n'(t_j + \tfrac{2}{5}h - \tau_n) \end{array}\right)$$

Step 6: $\forall\, i = 1,2,\ldots,n$ compute :

$$N_{4i} = f_i\left(\begin{array}{c} t_j + h, y_1(t_j) + \dfrac{h}{4}(N_{11} - 12N_{21} + 15N_{31}), \ldots, y_n(t_j) + \dfrac{h}{4}(N_{1n} - 12N_{2n} + 15N_{3n}), \\ \phi_1(t_j + h - \tau_1), \ldots, \phi_n(t_j + h - \tau_n), \phi_1'(t_j + h - \tau_1), \ldots, \phi_n'(t_j + h - \tau_n) \end{array}\right)$$

Step 7: $\forall\, i = 1,2,\ldots,n$ compute :

$$N_{5i} = f_i\left(\begin{array}{c} t_j + \tfrac{2}{3}h, y_1(t_j) + \dfrac{h}{81}(6N_{11} + 90N_{21} - 50N_{31} + 8N_{41}), \ldots, y_n(t_j) + \dfrac{h}{81}(6N_{1n} + 90N_{2n} - 50 \\ N_{3n} + 8N_{4n}), \ldots, \phi_1(t_j + \tfrac{2}{3}h - \tau_1), \ldots, \phi_n(t_j + \tfrac{2}{3}h - \tau_n), \phi_1'(t_j + \tfrac{2}{3}h - \tau_1), \ldots, \phi_n'(t_j + \tfrac{2}{3}h - \tau_n) \end{array}\right)$$

Step 8: $\forall\, i = 1,2,\ldots,n$ compute :

$$N_{6i} = f_i\left(\begin{array}{c} t_j + \tfrac{4}{5}h, y_1(t_j) + \dfrac{h}{75}(6N_{11} + 36N_{21} + 10N_{31} + 8N_{41}), \ldots, y_n(t_j) + \dfrac{h}{75}(6N_{1n} + 36N_{2n} + 10 \\ N_{3n} + 8N_{4n}), \ldots, \phi_1(t_j + \tfrac{4}{5}h - \tau_1), \ldots, \phi_n(t_j + \tfrac{4}{5}h - \tau_n), \phi_1'(t_j + \tfrac{4}{5}h - \tau_1), \ldots, \phi_n'(t_j + \tfrac{4}{5}h - \tau_n) \end{array}\right).$$

Step 9: $\forall\, i = 1,2,\ldots,n$ compute :

$$t_{j+1} = t_0 + (j+1) \times h$$

$$y_i(t_{j+1}) = y_i(t_j) + \frac{h}{192}\left(23N_{1i} + 125N_{3i} - 81N_{5i} + 125N_{6i}\right)$$

Step 10: Put $j = j+1$

Step 11: If $j = m$ then stop.
Else go to (step 3)

3.3 The Solution of n^th Order Linear Functional Differential Equations Using Nyström Method:

The general form of n^th order linear functional differential equation can be written as:

$$y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t), y(t-\tau), y'(t-\tau), \ldots, y^{(n-1)}(t-\tau)), \quad t \ge t_0 \quad \ldots (11)$$

with initial functions:

$$\left.\begin{aligned} y(t) &= \phi(t) \\ y'(t) &= \phi'(t) \\ &\vdots \\ y^{(n-1)}(t) &= \phi^{(n-1)}(t) \end{aligned}\right\} \quad for \quad t_0-\tau \le t \le t_0$$

where $\phi(t)$ and its first (n-1) derivatives $\phi'(t),\ldots,\phi^{n-1}(t)$ are continuous on the interval $[t_0-\tau, t_0]$

Obviously, the nth order equation (11) with difference argument may be replaced by a system of nth-equation of first order functional differential equations as follows:

Let

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ &\vdots \\ x_{n-1}(t) &= y^{(n-2)}(t) \\ x_n(t) &= y^{(n-1)}(t) \end{aligned}$$

Then, the following system of first order equations can be obtained as:

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= x_3(t) \\ &\vdots \\ x_{n-1}'(t) &= x_n(t) \\ x_n'(t) &= f(t, x_1(t),\ldots, x_n(t), x_1(t-\tau),\ldots, x_n(t-\tau)) \end{aligned}$$

The above system of the first order linear FDE's can be treated numerically by using Nyström method as it is prescribed in section (3.2).

4. Numerical Examples :

Example (1):

Consider the following 1st order linear retarded functional differential equation :

$$y'(t) = y(t) + y(t-\tfrac{1}{2}) + \ln(y(t-\tfrac{1}{2})) - t - e^{t} - e^{0.5} + 1 \qquad t \ge 0$$

with initial function : $y(t) = e^{t+0.5} \qquad -0.5 \le t \le 0$.

The exact solution of the above linear FDE is: $y(t) = e^{t} + e^{0.5} - 1 \qquad 0 \le t \le 0.5$.

When the algorithm (NM-FFDE) is applied, table (1) presents the comparison between the exact and numerical solution using Nyström method for m=10, h=0.05, $t_j = t_0 + jh$, $t_0 = 0$, $j = 0,1,\ldots,m$ *.and m=100, h=0.005, depending on least square error (L.S.E.).*

Table (1) The solution of FDE for Ex.(1).

t	*Exact*	*Nyström Method (NM-FFDE) algorithm y(t)*	
		h=0.05	*h=0.005*
0	**1.6487**	**1.6487**	**1.6487**
0.05	**1.7000**	**1.7000**	**1.7000**
0.10	**1.7539**	**1.7539**	**1.7539**
0.15	**1.8106**	**1.8106**	**1.8106**
0.20	**1.8701**	**1.8701**	**1.8701**
0.25	**1.9327**	**1.9327**	**1.9327**
0.30	**1.9986**	**1.9986**	**1.9986**
0.35	**2.0678**	**2.0678**	**2.0678**
0.40	**2.1405**	**2.1405**	**2.1405**
0.45	**2.2170**	**2.2170**	**2.2170**
0.50	**2.2974**	**2.2974**	**2.2974**
L.S.E.		**0.372e-18**	**0.413e-27**

Figure (1) shows the solution of retarded linear FDE which was given in example (1) by using Nyström's method (NM-FFDE algorithm) with the exact solutions.

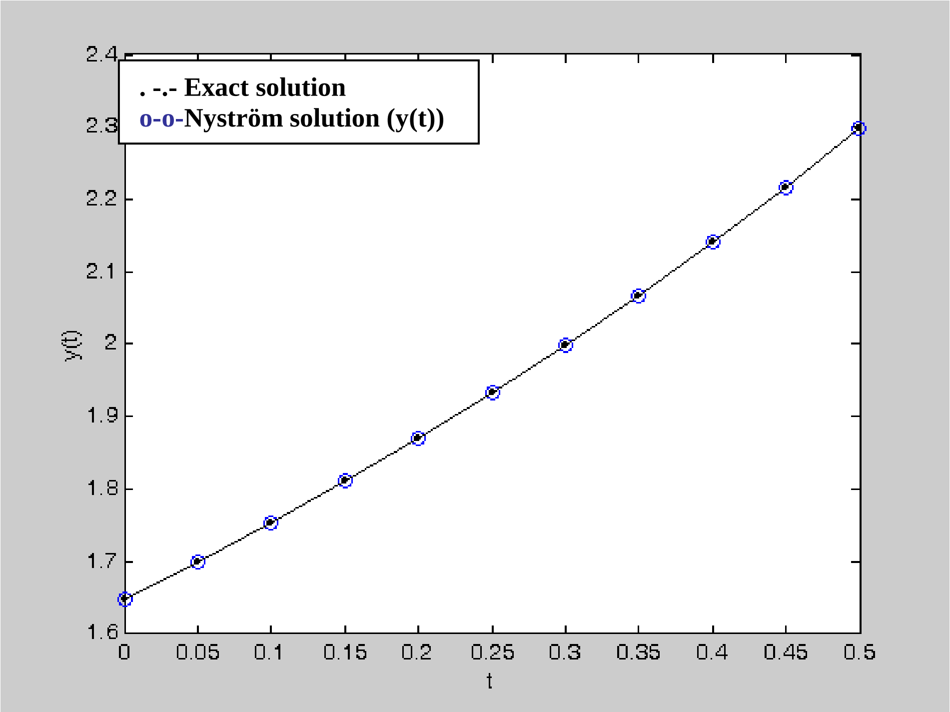

Fig.(1) The comparison between the exact and Nyström solution for linear retarded FDE in Ex.(1).

Example (2):

Consider the following system of first order linear neutral functional differential equations:

$$y_1'(t) = \frac{1}{2} y_1'(t-1) + y_2(t) + \ln(y_2'(t-1)) - e^t - e + 2 \qquad t \ge 0$$

$$y_2'(t) = y_2'(t-1) + y_1'(t-2) - y_1'(t-\tfrac{1}{2}) + 3 \qquad t \ge 0$$

where the initial functions are:

$$y_1(t) = t^2 \qquad -2 \le t \le 0$$

$$y_2(t) = e^{t+1} \qquad -1 \le t \le 0$$

The exact solutions are:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t^2 \\ e^t + e - 1 \end{pmatrix} \qquad 0 \le t \le 1 \quad .$$

Example (2) is solved by using Nyström's method. Table (2) gives a summary of the numerical solution, exact solution and the least square errors (L.E.S.) by applying (NM-SYFDE) algorithm for m=10, h=0.1, $t_j = t_0 + jh$, $t_0 = 0,\ j = 0,1,\ldots,m$.

Table (2) The solution of linear FDE's for example (2)

t	*Exact* $y_1(t)$	*Nyström* $y_1(t)$	*Exact* $y_2(t)$	*Nyström* $y_2(t)$
0	0.0000	0.0000	2.7183	2.7183
0.1	٠.0100	٠.0100	2.8235	2.8235
0.2	٠.0400	٠.0400	2.9397	2.9397
0.3	٠.0900	٠.0900	٣,٠٦٨١	٣,٠٦٨١
0.4	٠.1600	٠.1600	٣,٢١٠١	٣,٢١٠١
0.5	٠.2500	٠.2500	٣,٣٦٧٠	٣,٣٦٧٠
0.6	٠.3600	٠.3600	٣,٥٤٠٤	٣,٥٤٠٤
0.7	٠.4900	٠.4900	٣,٧٣٢٠	٣,٧٣٢٠
0.8	٠.6400	٠.6400	٣,٩٤٣٨	٣,٩٤٣٨
0.9	٠.8100	٠.8100	٤,١٧٧٩	٤,١٧٧٩
1	1.0000	1.0000	4.4366	4.4366
L.S.E		0.208e-16	L.S.E	0.313e-18

Figure (2) shows the solution of linear FDE's which was given in example (2) by using Nyström's method (NM-SYFDE algorithm) with the exact solutions.

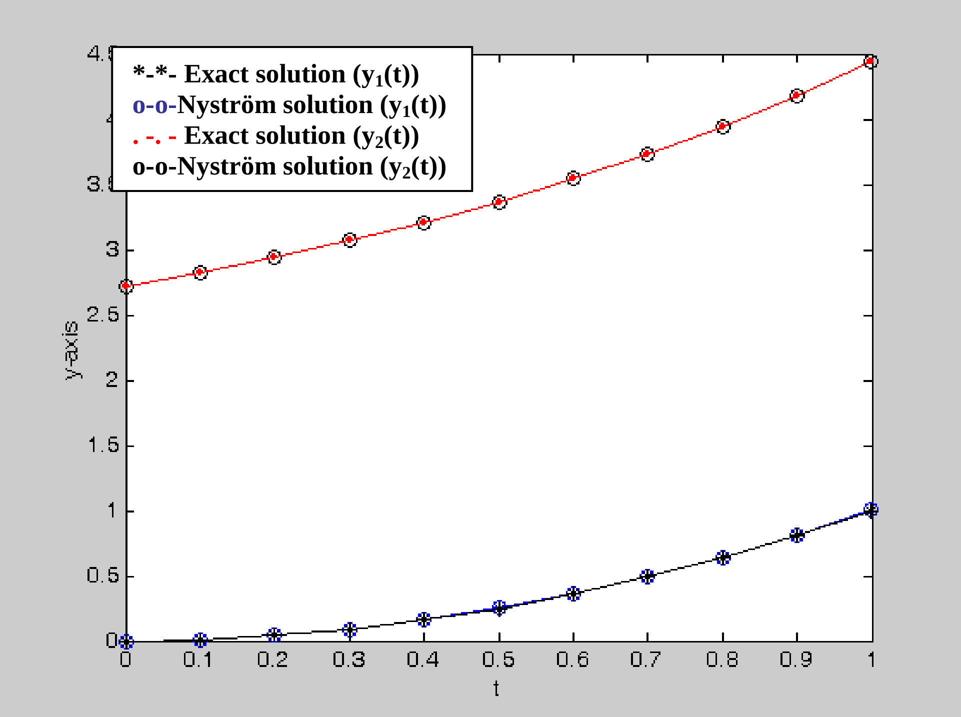

Fig.(2) The comparison between the exact and Nyström solution for linear neutral FDE's in Ex.(2).

Example (3):

Consider the linear mixed third order functional-differential equation:-

$$y^{(3)}(t) = -2y'(t) + y'(t-\tfrac{\pi}{2}) - y''(t-\pi) + y(t-1) - \sin t\cos 1 + \cos t(\sin 1 + 1) \qquad t \ge 0$$

with initial functions :

$$\left.\begin{aligned} y(t) &= \cos(t-\tfrac{\pi}{2}) \\ y'(t) &= -\sin(t-\tfrac{\pi}{2}) \\ y''(t) &= -\cos(t-\tfrac{\pi}{2}) \end{aligned}\right\} \quad t \le 0$$

and exact solution: $y(t) = \sin t \qquad 0 \le t \le 1$

The above mixed linear functional differential equation can be replaced by a system of three first order FDE equations as:

$$x_1'(t) = x_2(t), \qquad t \ge 0$$

$$x_2'(t) = x_3(t), \qquad t \ge 0$$

$$x_3'(t) = -2x_2(t) + x_2(t-\tfrac{\pi}{2}) - x_3(t-\pi) + x_1(t-1) - \sin t\cos 1 + \cos t(\sin 1 + 1) \qquad t \ge 0$$

with initial conditions:

$$\left.\begin{aligned} x_1(t) &= \cos(t-\tfrac{\pi}{2}) \\ x_2(t) &= -\sin(t-\tfrac{\pi}{2}) \\ x_3(t) &= -\cos(t-\tfrac{\pi}{2}) \end{aligned}\right\} \quad t \le 0$$

which has the exact solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \end{pmatrix} \quad 0 \le t \le 1$$

The Nyström's method was calculated for m=10, h=0.1 and m=100, h=0.01 using (NM-SYFDE) algorithm. Table (3) presents the comparison between the numerical and exact solution depending on the least square error (L.S.E.), where $t_j = jh$, $j = 0,1,\ldots,m$.

Table (3) The solution of linear FDE's for example (3)

t	Exact $x_1(t)$	Nyström Method $x_1(t)$		Exact $x_2(t)$	Nyström Method $x_2(t)$		Exact $x_3(t)$	Nyström Method $x_3(t)$	
		h=0.1	h=0.01		h=0.1	h=0.01		h=0.1	h=0.01
0	0.0000	0.0000	0.0000	1.0000	1.0000	1.0000	-0.0000	-0.0000	-0.0000
0.1	0.0998	0.0998	0.0998	0.9950	0.9950	0.9950	-0.0998	-0.0998	-0.0998
0.2	0.1987	0.1987	0.1987	0.9801	0.9801	0.9801	-0.1987	-0.1987	-0.1987
0.3	0.2955	0.2955	0.2955	0.9553	0.9553	0.9553	-0.2955	-0.2955	-0.2955
0.4	0.3894	0.3894	0.3894	0.9211	0.9211	0.9211	-0.3894	-0.3894	-0.3894
0.5	0.4794	0.4794	0.4794	0.8776	0.8776	0.8776	-0.4794	-0.4794	-0.4794
0.6	0.5646	0.5646	0.5646	0.8253	0.8253	0.8253	-0.5646	-0.5646	-0.5646
0.7	06441	0.6441	0.6441	0.7648	0.7648	0.7648	-0.6441	-0.6441	-0.6441
0.8	0.7172	0.7172	0.7172	0.6967	0.6967	0.6967	-0.7172	-0.7172	-0.7172
0.9	0.7833	0.7833	0.7833	0.6216	0.6216	0.6216	-0.7833	-0.7833	-0.7833
1	0.8415	0.8415	0.8415	0.5403	0.5403	0.5403	-0.8415	-0.8415	-0.8415
L.S.E.		0.11e-17	0.76e-27	L.S.E.	0.61e-17	0.53e-26	L.S.E.	0.19e-19	0.13e-28

Conclusion:

Nyström method has been presented for solving three types (retarded, neutral and mixed) of linear functional differential equations. The results show a marked improvement in the least square errors (L.S.E). From solving some numerical examples the following points are listed:

1- Nyström method gives a better accuracy and consistent to the solution of n[th] order linear functional-differential equation by reducing the equation to a system of first order linear FDE's.

2- Nyström method gives qualified way for solving first order linear functional differential equation as well as system of linear functional differential equations.

3- The good approximation depends on the size of h, if h is decreased then the number of points increases and the L.S.E. approaches to zero.

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معالجة عـــددية للمعـادلات التفاضلـية – الدالـية الخطـية

من الرتبة n باسـتخدام طريـــقة نـيسترم

احمد شــوقي جابر — *كلية العلوم* — *الجامعة المستنصرية*

اسراء هـــادي حسن — *قسم العلوم التطبيقية* — *الجامعة التكنولوجية*

أثـــير جـــواد كاظم — *قسم العلوم التطبيقية* — *الجامعة التكنولوجية*

المستخلص :

يقدم البحث طريقة مقترحة مع اشتقاق خوارزميات جديدة تمت برمجتها بلغة (Matlab) لإيجاد الحل العددي للمعادلات التفاضلية الدالية الخطية باستخدام طريقة نيسترم من الرتبة الخامسة. حيث تمت معالجة منظومة من المعادلات التفاضلية الدالية الخطية عددياً باستخدام طريقة نيسترم من الرتبة الخامسة وقد طورت هذه الطريقة لإيجاد النتائج العددية لثلاثة انواع من المعادلات التفاضلية الدالية من الرتبة n و المتضمنة (التراجعية ، المتعادلة والمختلطة) . كما تمت مقارنة النتائج العددية و الحقيقية لثلاثة انواع من هذه المعادلات من خلال بعض الأمثلة و الرسوم التوضيحية وقد تم الحصول على نتائج جيدة.