# Numerical Solution for Linear Delay Fourth Order Multi-Value Problems

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## Abstract

This paper attempts to study the linear delay fourth order multi-value problems, and modify one of the powerful numerical methods namely the variational technique to solve this kind of problems.

## Introduction

Many problems in solid mechanics, fluid mechanics, heat transfer, electromagnetism and acoustics can be formulated as a linear delay fourth order multi-value problem. In this paper we shall investigate a numerical solution to this type of problems since most practical problems is too complicated to be solved analytically. Thus numerical methods are essential for solving these kinds of problems, also we modify a kind of these problems called the Sturm-Liouville problems that remarked as the linear delay fourth order multi-value problems.

## 1- Fundamental Preliminaries

To make the informations about the theory of the linear delay fourth order multi-value problems as complete as possible, we give some definitions, facts and remarks which give necessary conditions that ensure the existence of a nontrivial solution for such kind of problems.

We start this section by recalling the following definition

**Definition 1.1**

The linear delay fourth order multi-value problems is a problem in which the unknown multi-function and some of its derivatives, evaluated at arguments which are different by fixed function of values.

Consider the following linear delay fourth order Sturm-Liouville multi-value problem:

$$-(p_i(x)y_i''(x))''+(q_i(x)-\sum_{j=1}^{n}\lambda_j r_{ij}(x))y_i(x-\tau)=0 \qquad [1.1]$$

with the boundary conditions:

$$c_{i1}y_i(a)-c_{i2}(py_i'')'(a)=0,\quad d_{i1}y_i(b)-d_{i2}(py_i'')'(b)=0 \qquad ,\ x\in[a-\tau,a]$$

[1.2]

$$e_{i1}y_i'(a)-e_{i2}p(a)y_i''(a)=0,\quad f_{i1}y_i'(b)-f_{i2}p(b)y_i''(b)=0$$

$i,j=1,2,\ldots,n,$ where $p_i$, $p_i'$, $p_i''$, $q_i$ and $r_{ij}$ are given real-valued continuous functions defined on the interval $[a,b]$, $p_i$ and $r_{ij}$ are positive, not both coefficients in one condition are zero, $\tau>0$ is the time delay.

## Definition 1.2

The eigen-values of the problem given by equations [1.1]-[1.2] are called the delay eigen-values, while the eigen-functions are called the delay eigen-functions.

For simplicity fix $n = 2$, therefore the problem given by equations [1.1]-[1.2] reduces to the following:

$$-(p_1(x)y_1''(x))''+(q_1(x)-\sum_{j=1}^{2}\lambda_j r_{1j}(x))y_1(x-\tau)=0 \qquad [1.3,a]$$

$$-(p_2(x)y_2''(x))''+(q_2(x)-\sum_{j=1}^{2}\lambda_j r_{2j}(x))y_2(x-\tau)=0 \qquad [1.3,b]$$

with the boundary conditions:

$$c_{i1}y_i(a)-c_{i2}(py_i'')'(a)=0,\qquad d_{i1}y_i(b)-d_{i2}(py_i'')'(b)=0 \qquad [1.4,a]$$

$$,\ x\in[a-\tau,a]$$

$$e_{i1}y_i'(a)-e_{i2}p(a)y_i''(a)=0,\qquad f_{i1}y_i'(b)-f_{i2}p(b)y_i''(b)=0 \qquad [1.4,b]$$

for $i = 1,2$, and all the assumptions of the problem given by equations [1.1]-[1.2] are also satisfied for the above problem.

The following facts that given by Reid,(1). Hold for the problems given by equations [1.1]-[1.2] and [1.3]-[1.4].

**Fact 1.1**

The linear operator:

$$L_i = -(p_i(x)\frac{d^4}{dx^4} + 2p_i'(x)\frac{d^3}{dx^3} + p_i''(x)\frac{d^2}{dx^2}) + q_i(x)$$

where $i = 1,2,\ldots,n$ is self-adjoint.

**Fact 1.2**

The linear operator $L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$ of the problem given by equations [1.3]-[1.4] where $L_i = -(p_i(x)\frac{d^4}{dx^4} + 2p_i'(x)\frac{d^3}{dx^3} + p_i''(x)\frac{d^2}{dx^2}) + q_i(x)$ for $i$ =1,2 is self-adjoint.

**Fact 1.3**

There are infinite number of eigen-values forming a monotone increasing sequence with $\lambda_{ij} \to \infty$ as $j \to \infty$. Moreover, the eigen-functions corresponding to the eigen-values $\lambda_{ij}$ has exactly $j$ roots on the interval $(a,b)$ for each $i$=1,2,…,$n$**.**

While the following remarks are given by Bhattacharyya and others,((2),(3) and(4)).

**Remark 1.1**

All the eigen-values are real.

**Remark 1.2**

The eigen-functions of the problems given by equations [1.1]-[1.2] and [1.3]-[1.4] are orthogonal to the weight functions $r_{ij}$ where $i=j$.

**Remark 1.3**

The eigen-functions are complete and normalized in $L^2[a,b]$.

Also the folowing fact is given by Asmer, (5)

**Fact 1.4**

Each eigen-value of the problem given by equations [1.1]-[1.2] and [1.3]-[1.4] corresponds only one eigen-function in $L^2[a,b]$.

## 2- Variational Technique

The calculus of variation plays an important role n solving linear equations of the form $Ly = f$ where $L$ is a linear operator corresponding to the problem under consideration, $f$ is the known non-homogeneous term and $y$ is the unknown term. In this section, we extend the use of the variational technique to solve the linear delay fourth order multi-value problems using the approach given by Magri, (6) and (7).

This method starts by rewriting equation [1.3,a] as $L_1 y_1 = 0$, where

$$L_1 = -(p_1(x)\frac{d^4}{dx^4} + 2p_1'(x)\frac{d^3}{dx^3} + p_1''(x)\frac{d^2}{dx^2}) + q_1(x) - \lambda_1 r_{11}(x) - \lambda_2 r_{12}(x)$$

It is easy to check that, this operator is linear. On the other hand choose the bilinear form: $< u(x), v(x) > = \int_a^b u(x)\, L\, v(x)$ which makes the operator $L$ symmetric with respect to it.

Thus, Magri's theorem can be applied. So, the critical points of the functional:

$$F(\lambda_1, \lambda_2, y_1) = \frac{1}{2}\int_a^b [-(p_1(x) y_1''(x))'' + q_1(x)$$

$$-(\lambda_1 r_{11}(x) + \lambda_2 r_{12}(x)) y_1(x-\tau)]^2 dx$$

[2.1]

are the solutions of the problem given by equations [1.3,a]-[1.4,a]. This is the variational formulation to the above problem.

To solve the variational formulation given by equation [2.1], one must approximate the unknown function $y_1$ as a linear combination of $n$ linearly independent functions $\{\phi_i(x)\}^n_{i=1}$. That is, write

$$y_1(x) = \sum_{i=1}^{n} c_i \phi_i(x)$$

But, this approximated solution must satisfy the boundary conditions given by equations [1.4,a] to get a new approximated solution. By substituting this approximated solution into equation [1.3,a] one can get:

$$F(\lambda_1, \lambda_2, \vec{c}) = \frac{1}{2}\int_a^b [-(p_1(x)\sum_{i=1}^{n} c_i \phi_i''(x))'' + q_1(x)$$

$$-(\lambda_1 r_{11}(x) + \lambda_2 r_{12}(x))\sum_{i=1}^{n} c_i \phi_i(x-\tau)]^2 dx$$

where $\vec{c}$ is a vector of $n - 4$ of $c_i$, $i \in \{1,2,\ldots,n\}$.

To find the critical points of the above functional, set $\frac{\partial F}{\partial \lambda_1}=\frac{\partial F}{\partial \lambda_2}=\frac{\partial F}{\partial c_i}=0$ to get a system of $n-2$ nonlinear equations with $n-2$ unknowns, that can be solved easily.

To illustrate this method see the following example:

**Example 2.1**

Consider the linear delay fourth order Sturm-Liouville two-value problem:

$$-(e^{-x}\frac{d^4y_1(x)}{dx^4}-2e^{-x}\frac{d^3y_1(x)}{dx^3}+e^{-x}\frac{d^2y_1(x)}{dx2})$$

$$+(3\sin x+4x-\lambda_1\sin x-\lambda_2 x)y_1(x-1)=0$$

[2.2,a]

$$-(\sin x\frac{d^4y_1(x)}{dx^4}+2\cos x\frac{d^3y_1(x)}{dx^3}-\sin x\frac{d^2y_1(x)}{dx2})$$

$$+(3x+4x^2-\lambda_1 x-\lambda_2 x^2)y_1(x-1)=0 \qquad [2.2,b]$$

with the boundary conditions:

$(e^{-x}y_1'')'(1)=0$ $\qquad (e^{-x}y_1'')'(2)=0$

$y_1'(1)=0$ $\qquad y_1'(2)=0$

[2.3,a]

$, x\in[0,1]$

$(\sin xy_2'')'(1)=0$ $\qquad (\sin xy_2'')'(2)=0$

$y_2''(1)=0$ $\qquad y_2''(2)=0$

[2.3,b]

Here, we use the variational method to solve this problem.

First, we approximate the unknown function $y_1$ as a polynomial of degree zero: i.e.; $y_1(x) = c_1$. Then, this solution is automatically satisfy the boundary conditions given by equations [2.3,a]. Moreover, the critical points of the functional:

$$F(\lambda_1,\lambda_2,c_1)=-\frac{9}{4}c_1^2\cos1\sin1-4c_1^2\sin1+4c_1^2\lambda_1\cos1+\frac{59}{12}c_1^2-\frac{4}{3}c_1^2\lambda_2$$
$$+12c_1^2\sin1-\frac{1}{4}c_1^2\lambda_1^2\cos1\sin1+\frac{1}{4}c_1^2\lambda_1^2-12c_1^2\lambda_1^2-12c_1^2\cos1+$$
$$c_1^2\lambda_1\lambda_2\cos1-c_1^2\lambda_1\lambda_2\sin1+\frac{3}{2}c_1^2\lambda_1\cos1\sin1-\frac{3}{2}c_1^2\lambda_1+\frac{1}{6}c_1^2\lambda_1\lambda_2-3c_1^2\lambda_2\sin1$$
$$+3c_1^2\lambda_2\cos1$$

are the solutions of the problem given by equations [2.2,a]-[2.3,a]. Thus, set

$\dfrac{\partial F}{\partial\lambda_1}=\dfrac{\partial F}{\partial\lambda_2}=\dfrac{\partial F}{\partial c_1}=0$ to get the following system of nonlinear equations:

$$-\frac{9}{2}c_1\cos1\sin1-8c_1\lambda_1\sin1+8c_1\lambda_1\cos1+\frac{59}{6}c_1-\frac{8}{3}c_1\lambda_2+24c_1\sin1-$$
$$\frac{1}{2}c_1\lambda_1^2\cos1\sin1+\frac{1}{2}c_1\lambda_1^2-24c_1\cos1+2c_1\lambda_1\lambda_2\sin1-2c_1\lambda_1\lambda_2\cos1+$$
$$3c_1\lambda_1\cos1\sin1-3c_1\lambda_1+\frac{1}{3}c_1\lambda_2^2-6c_1\lambda_2\sin1+6c_1\lambda_2\cos1=0$$

$$-4c_1^2\sin1+4c_1^2\cos1-\frac{1}{2}c_1^2\cos1\sin1+\frac{1}{2}c_1^2\lambda_1+c_1^2\lambda_2\sin1-c_1^2\lambda_2\cos1+$$
$$\frac{3}{2}c_1^2\cos1\sin1-\frac{3}{2}c_1^2=0$$
$$-\frac{4}{3}c_1^2+c_1^2\lambda_1\sin1-c_1^2\lambda_1\cos1+\frac{1}{3}c_1^2\lambda_2-3c_1^2\sin1+3c_1^2\cos1=0$$

which has the nontrivial solution $c_1\neq 0$, $\lambda_1=3$ and $\lambda_2=4$. Thus, by substituting the values of $\lambda_1$ and $\lambda_2$ into equation [2.2,b], one can get:

$$-(\sin x\frac{d^4y_1(x)}{dx^4}+2\cos x\frac{d^3y_1(x)}{dx^3}-\sin x\frac{d^2y_1(x)}{dx2})=0$$

it is clear that a polynomial of degree one is a solution of the above differential equation with the boundary conditions given by equation [2.3,b].

Second, if we approximate $y_1$ as a polynomial of degree one, two, three, four, five then one can get the same above results. More generally, if $y_1 = \sum_{i=1}^{n} c_i x^{i-1}$ then one can get ((3,4), ($c_1$, $d_1+d_2x$)) is the double eigen-pair of the problem given by equations[2.2]-[2.3]. That is, $y_1(x-1) = c_1$ and $y_2(x-1) = d_1 + d_2(x-1)$ where $x \in [1,2]$.

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# الحل العددي للمسائل الخطية التباطؤيه مزدوجة القيم الذاتية ذات الرتبة الرابعة

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## ألخلاصه

خصص هذا البحث لدراسة المسائل الخطية التباطؤيه مزدوجة القيم الذاتية ذات الرتبة الرابعة، و تطوير واحدة من الطرق العددية القوية التي تسمى الطريقة التغايرية لحل هذا النوع من المسائل.