

Approximate Solution for linear Fredholm Integro-Differential Equation and Integral Equation by Using Bernstein Polynomials method

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Abstract

In this paper, Bernstein polynomials method is used to find an approximate solution for Fredholm integro-Differential equation and integral equation of the second kind. These polynomials are incredibly useful mathematical tools, because they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions. They can be differentiated and integrated easily.

1. Introduction

Several numerical methods for approximating Fredholm integro-differential equation and integral equation are known.

In this paper, we introduce approximate method for solving linear Fredholm integro-differential equation and integral equation by using Bernstein polynomials method. Hence, we begin by giving a general introduction to integral equations.

An integral equation is generally defined as an equation which involves the integral of an unknown function.

A linear integral equation is an integral equation which involves a linear expression of the unknown function.

The Fredholm integral equations are integral equations divided into two kinds, referred to as Fredholm integral equations of the first and the second kind.

They have the following general expression [1][2][6]:

$$y(x) = f(x) + \int_{\Omega} k(x,t)y(t)dt \quad \dots(1)$$

where $k(x,t)$ and $f(x)$ are known function. $k(x,t)$ is called the kernel of integral equation. $y(x)$ is the function to be determined. When Ω is finite interval $[a,b] \subseteq R$ in this case, the Fredholm integral equations of first and second kind will respectively have the following expressions:

$$f(x) = \int_a^b k(x,t)y(t)dt \quad x \in [a,b] \quad \dots(2)$$

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$$y(x) = f(x) + \int_a^b k(x,t)y(t)dt \quad x \in [a,b] \quad \dots(3)$$

There for the fredholm integro-differential equation by[5][8]:

$$y'(x) = f(x) + \int_a^b k(x,t)y(t)dt$$
$$y(a) = y_a \quad a \leq x \leq b \quad \dots(4)$$

In this paper, we introduce approximate method to solve the linear fredholm integro-differential equation and integral equation of the second kind by using Benstein polynomials method.

2. Bernstein Polynomials Method

Polynomials are incredibly useful mathematical tools as they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions.

The Bernstein polynomials of degree n are defined by [3],[4]:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad \text{for } i = 0,1,2,\dots,n \quad \dots(5)$$

where

$\binom{n}{i} = \frac{n!}{i!(n-i)!}$, (n) is the degree of polynomials, (i) is the index of polynomials and (t) is the variable.

The exponents on the (t) term increase by one as (i) increases, and the exponents on the (1-t) term decrease by one as (i) increases.

The Bernstein polynomial of degree (n) can be defined by blending together two Bernstein polynomials of degree (n-1). That is, the nth degree Bernstein polynomial can be written as, [4]:

$$B_k^n(t) = (1-t)B_k^{n-1}(t) + tB_{k-1}^{n-1}(t) \quad \dots(6)$$

Bernstein polynomials of degree (n) can be written in terms of the power basis. This can be directly calculated using the equation (5) and the binomial theorem as follows, [4].

$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k} = \sum_{i=k}^n (-1)^{i-k} \binom{n}{i} \binom{i}{k} t^i$$

Where the binomial theorem is used to Expand $(1-t)^{n-k}$.

The derivatives of the nth degree Bernstein polynomials are polynomials of degree (n-1)

$$\frac{d}{dt} B_k^n(t) = \frac{d}{dt} \binom{n}{k} t^k (1-t)^{n-k} = n(B_{k-1}^{n-1}(t) - B_k^{n-1}(t)) \quad 0 \leq k \leq n \quad \dots(7)$$

3. A Matrix Representation for Bernstein Polynomials

In many applications, a matrix formulation for the Bernstein polynomials is useful. These are straight forward to develop if only looking at a linear combination in terms of dot products. Given a polynomial written as a linear combination of the Bernstein basis functions [3],[4]:

$$B(t) = c_0 B_0^n(t) + c_1 B_1^n(t) + c_2 B_2^n(t) + \dots + c_n B_n^n(t) \quad \dots(8)$$

It is easy to write this as a dot product of two vectors

$$B(t) = \begin{bmatrix} B_0^n(t) & B_1^n(t) & B_2^n(t) & \dots & B_n^n(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \dots(9)$$

which can be converted to the following form:

$$B(t) = \begin{bmatrix} 1 & t & t^2 & \dots & t^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \dots & 0 \\ b_{10} & b_{11} & 0 & \dots & 0 \\ b_{20} & b_{21} & b_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \dots (10)$$

where b_{nn} are the coefficients of the power basis that are used to determine the respective Bernstein polynomials, we note that the matrix in this case lower triangular.

The matrix of derivatives of Brenstein polynomials

$$B'(t) = \begin{bmatrix} 0 & 1 & 2t & \dots & nt^{n-1} \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \dots & 0 \\ b_{10} & b_{11} & 0 & \dots & 0 \\ b_{20} & b_{21} & b_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \dots (10a)$$

4. Solution of Fredholm Integro-differential Equation with Bernstein Polynomials

In this section, Bernstein polynomials to find the approximate solution for Fredholm integro-differential equation, will be introduced [4],[5]and[7].

Let us reconsider the Fredholm integro-differential equation of the second kind in equation (4).

$$y'(x) = f(x) + \int_a^b k(x,t)y(t)dt \quad x \in [a, b] \quad \dots (11)$$

And

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Let $y(t) = B(t) = [B_0^n(t) \ B_1^n(t) \ B_2^n(t) \dots B_n^n(t)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$ by using equation (9)

$y'(t) = n(B_{k-1}^{n-1}(t) - B_k^{n-1}(t))$ by using equation (7)

A applying the Bernstein polynomials method for equation (11), we get the following formula.

$$[0 \ 1 \ B_{1-1}^{n-1}(t) \dots B_n^{n-1}(t)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = f(x) + \int_a^b k(x,t) [B_0^n(t) \ B_1^n(t) \dots B_n^n(t)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt \quad \dots(12)$$

by using equations (10) and (10a), which can be converted to the following form:

$$[0 \ 1 \ 2t \dots nt^{n-1}] \begin{bmatrix} b_{00} & 0 & \dots & 0 \\ b_{10} & b_{11} & \dots & 0 \\ b_{20} & b_{21} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{n0} & b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = f(x) + \int_a^b k(x,t) [1 \ t \dots t^n] \begin{bmatrix} b_{00} & 0 & \dots & 0 \\ b_{10} & b_{11} & \dots & 0 \\ b_{20} & b_{21} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{n0} & b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt \quad \dots (13)$$

now to find all integration in equation(13). Then in order to determine c_0, c_1, \dots, c_n , we need n equations; Now Choice $x_i, i=1,2,3,\dots,n$ in the interval [a, b], which gives (n) equations. Solve the (n) equations by Gauss elimination to find the values c_0, c_1, \dots, c_n . The following algorithm summarizes the steps for finding the approximate solution for the second kind of linear Fredholm integro-differential equation .

5. Solution of Fredholm integral equation with Bernstein polynomials

In this section Bernstein polynomials to find the approximate solution for Fredholm integral equation, will be introduced.

Let us reconsider the Fredholm integral equation of the second kind.

$$y(x) = f(x) + \int_a^b k(x,t)y(t)dt \quad \dots (14)$$

A applying the Bernstein polynomials method for equation (14) by using equation (8),(10), we get the following formula.

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$$y(t) = \begin{bmatrix} B_0^n(t) & B_1^n(t) & B_2^n(t) & \dots & B_n^n(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

If n=2

$$B(t) = \begin{bmatrix} c_0 B_0^2(t) & c_1 B_1^2(t) & c_2 B_2^2(t) \end{bmatrix} = y(t)$$

$$y(t) = \begin{bmatrix} B_0^2(t) & B_1^2(t) & B_2^2(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} \binom{2}{0} t^0 (1-t)^{2-0} & \binom{2}{1} t^1 (1-t)^{2-1} & \binom{2}{2} t^2 (1-t)^{2-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} (1-t)^2 & 2t(1-t) & t^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1-2t+t^2 & 2t-2t^2 & t^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \quad \dots(15)$$

If we substitute equation(15) in to equation(14), we obtain,

$$\begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = f(x) + \int_a^b k(x,t) \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} dt$$

$$c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2 = f(x) + \lambda \int_a^b k(x,t) [c_0(1-t)^2 + 2c_1t(1-t) + c_2t^2] dt$$

And after performing the integration.

$$c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2 = f(x) + \lambda \left(\int_a^b k(x,t)c_0(1-t)^2 dt + 2 \int_a^b k(x,t)c_1t(1-t) dt + \int_a^b k(x,t)c_2t^2 dt \right)$$

$$= f(x) + c_0 \int_a^b k(x,t)(1-2t+t^2) dt + 2c_1 \int_a^b k(x,t)(t-t^2) dt + c_2 \int_a^b k(x,t)t^2 dt \quad \dots(16)$$

now to find all integration in equation(16). Then in order to determine c_0, c_1 and c_2 , we need three equation; Now Choice $x_i, i=1,2,3$ in the interval $[a, b]$, which gives three equations. Solve the three equation by Gauss elimination to

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find the values c_0, c_1 and c_2 . The following algorithm summarizes the steps for finding the approximate solution for the second kind of Fredholm integral equation.

6. Algorithm (BPM)

Input: $(f(t), k(t, s), y(s), a, b)$,

Output: polynomials of degree n

Step1:

Choice n the degree of Bernstein polynomials

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad \text{for } i = 0, 1, 2, \dots, n$$

Step2:

Put the Bernstein polynomials in linear Fredholm integro differential equation and integral equation of second kind.

$$y'(t) = \begin{bmatrix} 0 & 1 & 2t & \dots & nt^{n-1} \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \dots & 0 \\ b_{10} & b_{11} & 0 & \dots & 0 \\ b_{20} & b_{21} & b_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = f(x) + \int_a^b k(x, t) [B_0^n(t) B_1^n(t) \dots B_n^n(t)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt$$

$$B_i^n(x) = f(x) + \int_a^b k(x, t) B_i^n(t) dt$$

Step3:

$$\text{Compute } \int_a^b k(x, t) \begin{bmatrix} 1 & t & \dots & t^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \dots & 0 \\ b_{10} & b_{11} & 0 & \dots & 0 \\ b_{20} & b_{21} & b_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt$$

$$\text{Compute } \begin{bmatrix} 0 & 1 & 2t & \dots & nt^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \dots & 0 \\ b_{10} & b_{11} & 0 & \dots & 0 \\ b_{20} & b_{21} & b_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$\text{Compute } \int_a^b k(x, t) B_i^n(t) dt$$

Step4:

Compute c_0, c_1, \dots, c_n , where $x_i, i = 1, 2, 3, \dots, n$, $x_i \in [a, b]$

End.

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Example1:

Consider the following linear Fredholm integro-differential equation of the second kind:

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)dt$$

With initial condition $y(0) = 0$, and with the exact solution $y(x) = xe^x$

Now to derive the solution by using the Bernstein polynomials method, we can use the following scheme:

When Bernstein polynomials algorithm is applied both sides in example. And choice the degree of Bernstein polynomials $n=2$, we get:

$$-2a_0(1-x) + 2a_1(1-x) - 2c_1x + 2c_2x = xe^x + e^x - x + \int_0^1 x[c_0(1-t)^2 + 2c_1t(1-t) + c_2t^2]dt$$

$$-2c_0(1-x) + 2c_1(1-x) - 2c_1x + 2c_2x = xe^x + e^x - x + \left(2c_0 \int_0^1 x(1-t)^2 dt + 2c_1 \int_0^1 xt(1-t) dt + c_2 \int_0^1 xt^2 dt \right)$$

And after performing the integration.

$$-2c_0(1-x) + 2c_1 - 4c_1x + 2c_2x = xe^x + e^x - x + \left[\frac{c_0}{3} + c_1 + \frac{c_2}{3} \right]x - \frac{2}{3}c_1$$

$$\left(-2 + \frac{2}{3}x\right)c_0 + \left(\frac{8}{3} - 5x\right)c_1 + \left(\frac{5}{3}x\right)c_2 = xe^x + e^x - x$$

Then in order to determine c_0, c_1 and c_2 , we need three equation; Now Choice $x_i, i=1,2$ in the interval $[0, 1]$, with substitution in the initial condition in the equation $y'(x) = -2a_0(1-x) + 2a_1 - 4c_1x + 2c_2x$ which gives three equations.

$$c_0 = 0$$

$$\frac{-7}{6}c_0 + \frac{1}{6}c_1 + \frac{5}{6}c_2 = \frac{3}{2}e^{0.5} - \frac{1}{2}$$

$$\frac{-1}{3}c_0 - \frac{7}{3}c_1 + \frac{5}{3}c_2 = 2e - 1$$

solve the three equation by Gauss elimination to find the values c_0, c_1 and c_2 to get

$$c_0 = 0$$

$$c_1 = -0.1839$$

$$c_2 = 2.4045$$

then the solution of linear Fredholm integro-differential equation of the second kind is:

$$y(x) = (c_0 - 2c_1 + c_2)x^2 - 2((c_0 - c_1)x + c_0)$$

$$y(x) = 2.7723x^2 - 0.3678x$$

Approximated solution for some values of (x) by using Bernstein polynomials method and exact values $y(x) = xe^x$ of Example, depending on least square error

(L.S.E), $Error = \sum_{k=1}^m (y_{Exact}(x) - y_{Approximation}(x))^2$ are presented in Table(1) and figure(1).

Example(2):

Consider the following Fredholm integral equation of the second kind:

$$u(x) = e^{-x} - \int_0^1 xe^t u(t) dt$$

Which has the exact solution

$$u(x) = e^{-x} - \frac{x}{2}$$

Here

$$f(x) = e^{-x}, \quad \lambda = -1, \quad k(x,t) = xe^t$$

When Bernstein polynomials algorithm is applied, Table(2) and Fig(2) presents the comparison between the approximate solutions using Bernstein polynomials method and exact values $u(x) = e^{-x} - \frac{x}{2}$ depending on least square error (L.S.E).

7. Conclusion

Integro-differential equations and integral equation are usually difficult to solve analytically. It is required to obtain the approximate solutions.

In This paper presents the use of the Bernstein polynomials method, for solving linear Fredholm integro-differential equation and integral equation of the second kind. From solving numerical example the following points have been identified:

1. This method can be used to solve the all kinds of linear Fredholm integro-differential equation and integral equation.
2. It is clear that using the Bernstein polynomial basis function to approximate when the nth degree of Bernstein polynomial is increases the error is decreases.
3. We can see also from Figure(1) and Figure(2) that the approximation is good. And when comparisons approximate solution with exact solution that the Bernstein polynomial method is very effective and convenient.

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Table (1) The results of Example(1) using (BPM) algorithm.

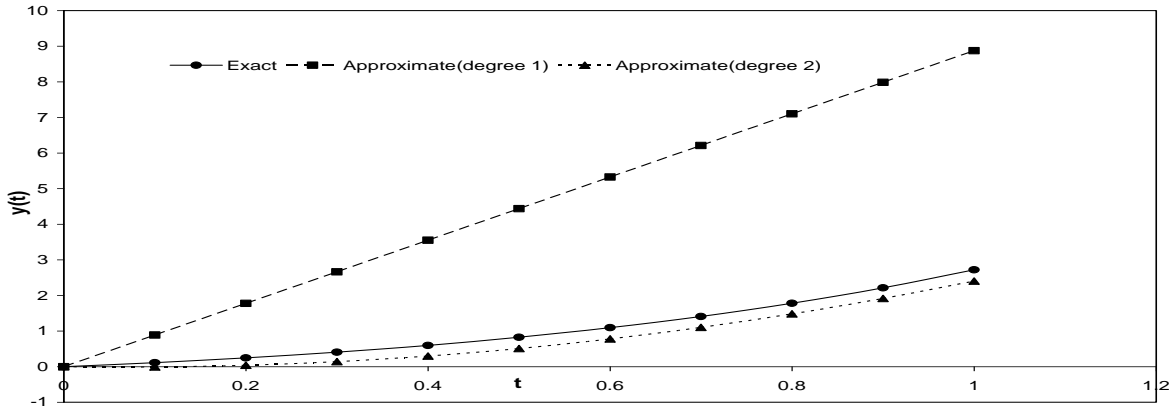
T	<i>Exact</i> <i>y(t)</i>	<i>Approximate</i> <i>y(t)</i> of degree(n=1)	<i>Approximate</i> <i>y(t)</i> of degree(n=2)
0	0	0	0
0.1	0.1105	0.8873	-0.0091
0.2	0.2443	1.7746	0.0373
0.3	0.4050	2.6620	0.1392
0.4	0.5967	3.5493	0.2964
0.5	0.8244	4.4366	0.5092
0.6	1.0933	5.3239	0.7773
0.7	1.4096	6.2112	1.1010
0.8	1.7804	7.0986	1.4800
0.9	2.2136	7.9859	1.9145
1	2.7183	8.8732	2.4045
L.S.E		170.244	0.790595
$Error = \sum_{k=1}^{10} (y_{Exact}(t) - y_{Approximaton}(t))^2$			

Table (2) The results of Example(2) using (BPM) algorithm.

<i>x</i>	<i>Exact</i>	<i>BPM Of degree n=2</i>
0	1	1
0.1	0.8548	0.8590
0.2	0.7187	0.7241
0.3	0.5908	0.5955
0.4	0.4703	0.4730
0.5	0.3565	0.3568
0.6	0.2488	0.2467
0.7	0.1466	0.1428
0.8	0.0493	0.0452
0.9	-0.0434	-0.0463
1	-0.1321	-0.1316
L.S.E		1.0e-3

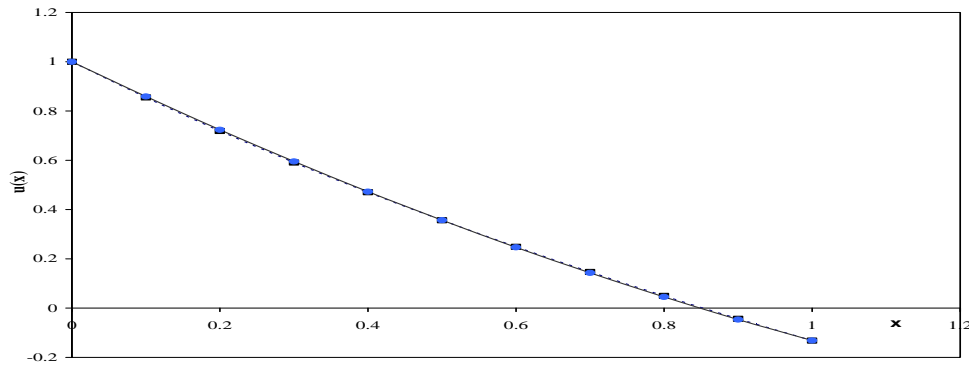
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Figure(1)

Approximate and Exact solution of linear Fredholm integro-differential equation of example(1)



Figure(2)

Approximate and Exact solution of linear Fredholm integral equation of example(2)

حل تقريبي لمعادلة فريدهولوم التفاضلية الخطية والمعادلة التكاملية باستخدام

متعددة حدود برنشتن

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ماجستير رياضيات التطبيقية

الجامعة المستنصرية/ كلية الهندسة

الخلاصة:

في هذا البحث استعملت طريقة متعددة حدود برنشتن لإيجاد الحل التقريبي لمعادلة فريدهولوم التفاضلية الخطية والتكاملية من النوع الثاني. وأن متعددات الحدود تستعمل بشكل كبير في الطرق الرياضية وذلك لبساطة تعريفها، والحل بهذه الطريقة يتقارب بسرعة وبخطوات قليلة. كما ان متعدد حدود برنشتن يمكن تفاضلها وتكاملها بسهولة.