Game theory using by of weights Finding values

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Abstract

In this paper we introduce weighting method where this method is used for solving multi-objective linear programming Problem, such that this method transform the multi – objectives problem into single –objective programming form . Also new proposition is constructed for solving multi-objective functions, where this proposition dependent on weighting method and Game theory.

Introduction

Signal objective programming problems is related to maximizing a single function Subject to a number of constraints. However, it has been increasingly recognized that many real word decision- making problems involve multiple, non commensurable and conflicting objectives which should be considered simultaneously. As an extension,multi– Objective programming (MOP) is defined as a means of optimizing multiple objective functions Subject to a number of constraints, i.e.:-

$$\left.\begin{array}{l} \text{Max } [z1(x), z2(x), \ldots, zp(x)] \\ \text{Subject to} \\ g_j(x) <= 0 \;,\; j=1,2,\ldots,m. \end{array}\right\} \quad \ldots\ldots\ldots.(*)$$

Where $z_k(x)$ are objective functions ,k=1,2,…,p and $g_j(x)<=0$, j=1,2,…,m are system constraints.

When the objectives are in conflict, there is no optimal solution that simultaneously maximizes

All objective functions .for this case, we employ a concept of efficient solution, which means

That it is impossible to improve any one objective without sacrificing on one or more of the Other Objectives. Therefore in this paper we introduce a new method for solving multi- Objectives Problems where this method depends on weighting method and game theory.

§.1 Basic definitions

Later. needed we are introducing some general definitions and theorems that will be

<u>Definition (1,1),[3]:-</u> In single objective function $z_k(X)$,we call X a decision vector , and $x1,x2,\dots,x_n$ decision variables. The set $S=\{X \in R^n : g_j(X) <= 0, j=1,2,\dots,m\}$ is called the feasible set. And the element X is called the feasible solution.

<u>Definition (1,2),[4]:-</u> A feasible solution X* is called the optimal solution of single objective function $z_k(x)$ if and only if $z_k(x^*) >= z_k(x)$ for any feasible solution X .

<u>Definition (1,3),[1]:-</u> A feasible solution X* is efficient if and only if there exists no other feasible Solution X such that $z_k(x) >= z_k(x^*)$,for $k=1,2,\dots,p$, and $z_k(x) > z_k(x^*)$ for at least one k . X* is Also termed a non-dominated solution or pareto-optimal solution.

<u>Definition (1,4),[4]:-</u> A payoff matrix is a decision analysis tool that summarizes pros and cons

Of a decision in tabular form. It list payoffs (negative or positive returns) associated with all

Possible combinations of alternative actions (under the decision maker's control) and external

Conditions (not under the decision maker's control).

<u>Definition (1,5),[4]:-</u> A payoff matrix p of size (m × n) where p=

	1	, 2, ……………, n
1	a 11	a12 ………..a1n
2	a 21	a 22……… a2n
:	:	
m	a m1	am2 … …..amn

Is said to have saddle point if:-

$$\underset{i}{\text{Maximum}}\ a_{ij}\ \underset{j}{\text{Minimum}}\ a_{ij} = \underset{j}{\text{Minimum}}\ \underset{i}{\text{Maximum}}$$

For $i=1,2,\dots,m$, $j=1,2,\dots,n$.

<u>Theorem(1,1),[1]</u> :- Let $z_k(x)$be one of the objective functions of linear vector maximum problem(*).Consider the linear –programming of maximizing $z_k(x)$ subject to $g_j(x) \le 0$,and$X \ge 0$

And let X*be its unique solution. Then X*is an efficient of problem (*).

- ((The proof follows directly from the definition (1. 3) of an efficient solution.)).

<u>Theorem (1.2),[1]</u>:- A vector X* that solves the problem(2.3) for fixed $w_k \ge 0$,for(k=1,2,..,p) is

An efficient solution to the problem (*).

§.2 weighting method and game theory

In this section we shall introduce two methods (weighting method and game theory) that will be needed for later work.

2.1 Game theory,[5]:-Game theory is defined as mathematical method of decision– making in which a competitive situation is analyzed to determine the optimal course of action for an Interested party, often used in political, economic, and military planning. Where In the game theory, an opponent is referred to as a player. Each player has a number of choices, finite or infinite, called strategies .the out comes or payoffs of a game are summarized as function of the different strategies for each player. A game with two players, where a gain of one player equals a loss to the other is known as two- Person sum game. If the payoff matrix has saddle point then the game is stable otherwise the game is not stable.

Example 1 :- Consider the following payoff matrix which represents player A's gain.

The computations of the minimax and maximin values are show on the matrix

Player A	1	2	3	4	*Row minimum*
1	8	2	9	5	2
2	6	5	7	18	5
3	7	3	-4	10	-4
Column maximum	8	5	9	18	

Then maximin of [2, 5,-4]=5 and minimax of [8,5,9,18]=5 then the above matrix

Has saddle point and value of game is equal to 5.

Example 2 :- Consider the payoff matrix :-

A \ B	1	2	3	4	Row minimum
1	5	-10	9	0	-1 0
2	6	7	8	1	1
	8	7	15	2	2
3	3	4	-1	4	-4
4	4	15	7	8	
Column maximum					

Then the maximin of [-10, 1,2,-1]=2 and minimax of [8,7,15,4]=4 then the above matrix has no saddle point and the matrix is not stable .

2.2 Solution ($n \times m$) Game by linear programming,[5]:-this section illustrates the solution of game Problem that not stable by using linear programming.

Player B

Consider we have ($n \times m$) payoff matrix where p=

	1	2	…… m
1	a11	a12	…. a1m
2	a21	a22	… … a2m
:	:		:
n	an1	an2	…… anm

player A

Suppose that the player A can chose the strategies (rows) 1 or 2 ,…..,n with probabilities p1, p2,..,pn,where ($0 \le pi \le 1$), Also in the same manner if player B chose the strategies (columns)
1 or 2,….,m , with probabilities q1,q2,…,qm, where ($0 \le qi \le 1$) .Now suppose that the value of the game according to player A is equal to v , the aim of player A is maximize v such that($v \ge 0$) therefore the objective function will be maximize v, and we can construct the constraints of the linear programming problem if player B chose first column then player A chose first row(a11) With probability p1, and chose second row (a21) with probability p2 and so on until nth row (an1) with probability p_n ,therefore the expected value for player A is a11p1+a21p2+….+an1pn , such that $a11p1+a_{21}p2+\ldots+a_{n1}pn \ge v$…….(1) , and in the same manner if player B chose second column, then the expected value for player A is a12p1+a22p2+…+an2pn , such that $a12p1+a22p2+\ldots+an2pn \ge v$…….(2),and if B chose jth column Then the expected value for player A $a1jp1+a2jp2+\ldots+anjpn \ge v$…..(3) ,and so on for the another constraints and the last constraint must be p1+p2+….+pn=1 , Then we can find the value of game(v) by solving the following linear programming problem:-

Maximize v
Subject to

$$a11p1+a_{21}p2+\ldots+a_{n1}pn \ge v$$
$$a11p1+a_{21}p2+\ldots+a_{n1}pn \ge v$$
$$a11p1+a_{21}p2+\ldots+a_{n1}pn \ge v$$
$$a_{1m}p1 + a_{2m}p2+\ldots+a_{nm}pn \ge v$$
$$p1+p2+\ldots+pn=1,\ p1 \ge 0,\ldots,pn \ge 0. \qquad \ldots\ldots (2.1)$$

The linear programming formulation above can be simplified by dividing all constraints by v, Therefore problem (2.1) becomes:-

Minimize z=x1+x2+…..+xn
Subject to

$$a_{11}x1+a_{21}x2+\ldots+a_{n1}xn \ge 1 \qquad \ldots\ldots (2.2)$$

$a_{1m}x1+a_{2m}x2+….+a_{nm}xn\geq 1$

Where x1 = p1\v , x2=p2\v ,……., xn=pn\v

And maximize v = minimize 1\v = minimize z, where z=1\v.

Note: -The value of game (v) with respect to player (A) =the value of game (v) with respect to player (B).

2.3 The weighting method,[1] : -

One way of trying to achieve a balance or suitable trade off between multiple objectives is Assign positive weighs to each one, where a weight reflects the importance of the corresponding Objective functions, and choice the values of these weights dependent on decision – maker. The Basic idea of assigning weights to the various objective functions combining these into a single- Objective function and parametrically varying the weights to generate the set of efficient solution.

The problem (*) with weights w_k , for k=1,2,…,p can be stated as follows :-

$$\left.\begin{aligned} &\text{Max } z(x) = w_1z_1(x)+w_2z_2(x)+\ldots+w_pz_p(x)\\ &\text{Subject to}\\ &g_j(x)\leq b_j,\quad j=1,2,\ldots,m\\ &\text{With } 0\leq w_k\leq 1,\ K=1,2,\ldots,P \end{aligned}\right\}\quad \ldots\ldots\ (2.3)$$

§.3:-"Construct mathematical model for solving multi-objective linear programming problems"

3.1:- Introduction

In this section we introduce a new mathematical model for solving multi-objective problem, Where this model depending on weighting method. That is to say:-

Let $Z(x) = w_1d_1^2+w_2d_2^2+\ldots+w_pd_p^2$, Where $d_i^2= (z_i- z_i^*)^2$, for i=1, 2,….., p, and $z_1^*=z_1(X^*1)$, $X^*1=(x_1^*,x_2^*,\ldots,x_n^*)$ is optimal solution of z_1 , $z_2^*=z_2(x^*2)$, $X^*2=(x^*_1,x^*_2,\ldots x^*_n)$ is optimal solution Of z_2 and so on until $z_p^*=z_p(x^*_p)$, $X^*_P=(x^*_1, x^*_2, \ldots,x^*_n)$ is optimal solution of z_p .

Therefore problem (*) can be written as:-

$$\left.\begin{aligned} &\text{Minimize } Z(X) = w_1d_1^2+w_2d_2^2+\ldots+w_pd_p^2\\ &\text{Subject to}\\ &g_j(X)\leq b_j,\ j=1,2,\ldots,m\\ &X\geq 0,\ \text{where } X=(x1,x2,\ldots,xn),\ \text{And } 0\leq w_k\leq 1 \end{aligned}\right\}\ ,\ K=1,2,\ldots,P.\quad \ldots\ldots\ (3.1)$$

3.2 Finding values of weights of problem (3.1) by using game theory:-

In my observation to the weighting method in section (2.3), these weights are often not exact

Because the decision maker chose the value of these weights .Therefore in this section we shall

Deal with possibility of finding the values of weights of problem (3.1) by using game theory. Now in Order to find the values of weights of problem (3.1), First we find the optimal solution of each Objective function by using simplex method [6] , That is to say we solve the following problem :-

$$\left.\begin{array}{l} \sum_{i=1}^{n} c_i^k x_i = \text{maximize } z_k(x) \quad ,\text{for } k=1,2,\ldots\ldots,p \\ \text{Subject to} \\ g_j(X) \le b_j \quad , j=1,2,\ldots,m \end{array}\right\} \quad \ldots\ldots \quad (3.2)$$

where $Z_K(X)$ and $g_j(X)$, $k=1,2,\ldots,p$, $j=1,2,\ldots,m$ are linear functions an d C^K_i , b_j are constants. In this case we shall obtain the following solutions $X^*1=(x_1^*,x_2^*,\ldots,x_n^*)$ where X^*1 is optimal solution of z_1 that is to say $z^*_1=z_1(X^*1)$,….., $X^*_P=(x^*_1, X^*_2, \ldots,x^*_n)$ is optimal solution of z_p that is to say $z^*_p=z_p(X^*p)$. The next step we construct a payoff matrix p_{rs} [2] , for $r=1,2,\ldots,p$,$s=1,2,\ldots,p$.

Where $p_{rs}=$ (S) (r)

	X *1	x*2 ……………….,	x*p
Z1	Z^*_{11}	z_{12} ……………………… .,	z_{1p}
Z2	Z_{21}	z^*_{22} ……………………….,	z_{2p}
:	:	:	:
:	:	:	:
zp	Z_{p1}	z_{p2} ……………………….,	z^*_{pp}

Such that $Zrs=Zr(X^*_S)$, $r\neq s$,for $r=1,2,\ldots,p$,$s=1,2,\ldots,p$, and if r=s then $Zrs=Z^*_S$, for $s=1,2,\ldots,p$.

Now from matrix prs we define drs ,where $drs= Z^*_r-Z_{rs}$ [2]….(3.3), for $r=1,2,\ldots p,s=1,2,\ldots,p$.

And drs=0 if (r=s), Now from equation (3.3) we calculate the following elements:-

$d_{11}, d_{12},\ldots,d_{1p}$,and so on until $d_{p1},d_{p2},\ldots,d_{pp}$.

In this case we construct another payoff matrix $p_{\tilde{r}\tilde{s}}$ for $\tilde{r} = 1,2,\ldots,p$ and $\tilde{s}=1,2,\ldots,p$

Such that $p_{\tilde{r}\tilde{s}} =$

	1	2	……	p $(\tilde{s})$
1	0	d^2_{12}	……	d^2_{1p}
2 $(\tilde{r})$	d^2_{21}	0	……	d^2_{2p}
:	:			:
p	d^2_{p1}	d^2_{p2}	……	0

Where $d^2_{rs} = (d_{rs})^2$,for r=1,2,…,p ,s=1,2,…,p .

Then from matrix $p_{\tilde{r}\tilde{s}}$ and by using the method in section (2.2) with replace maximizes v by minimize v, (because we want to reduce the deviation between the objective functions and its optimal solution), such that $a_{11}\, p_1+a_{21}\, p_2+\ldots+a_{p1}\, p_n \le v$, and so on for the another constraints, therefore in this case we solve the following linear programming problem:-

$$\left.\begin{array}{l} \text{Maximize } Z(X)= x_1+x_2+\ldots+x_p \\ \text{S.t} \\ 0.x1+d^2_{21}x2+\ldots+d^2_{p1}xn \le 1 \\ d^2_{12}x1+0.x2+\ldots+d^2_{p2}xn \le 1 \\ : \qquad : \qquad : \\ d^2_{1p}x1+d^2_{2p}x2+\ldots+0.xn \le 1 \end{array}\right\} \quad \ldots(3.4)$$

where xi=pi/v → pi=xi.v , where v≠0 ,and $0 \le pi \le 1$, for i=1,2,….,p .
and minimize v=maximize 1/v =maximize z , where z= 1/v .Finally we solve the problem (3.4) by using simplex method in this case we get $p_1,p_2,\ldots,p_p$, Now we set $w_1= p_1$, $w_2= p_2$,…,$w_p= p_p$, then we substitute these values of weights in problem (3.1) and solve it by using simplex method .

Now we shall illustrate the above method with the following example:-

Example :- Consider we have the following problem:-

$$\text{Maximize } Z(X) = \left\{\begin{array}{l} Z_1(X) =0.4X1+0.3X2 \\ Z_2(X) = X1+ 0.5X2 \\ Z_3(X) = 0.5X1+2X2 \end{array}\right.$$

S.t

$400 \le X2 + X1$ ……………………(**)

$2X1+X2 \le 500$

With $X1 \ge 0$, $X2 \ge 0$.

<u>*Solution*</u>: - In order to solve problem (**) we must find the optimal solution of each objective function, in this case we use the simplex method which give us the following optimal solutions:-

X^*1= (100,300), Z^*_1 (X^*1) = 130.
X^*2= (250, 0) , Z^*_2 (X^*2) = 250.
X^*3= (0, 400) , Z^*_3 (X^*3) = 800.

Now let $Z(X) = w_1d^2_1+ w_2d^2_2+ w_3d^2_3$

$$= w_1 (z_1(x) – z^*_1)^2 + w_2 (z_2 (x) - z^*_2)^2 + w_3 (z_3(x) - z^*_3)^2$$

$$= w1 (0.4x_1+0.3x_2 – 130)^2 + w_2(x_1+0.5x_2- 250)^2 + w_3 (0.5x_1+2x_2- 800)^2$$

Then problem (**) can be written as:-

Minimize $Z(X) = w1 (0.4x_1+0.3x_2 – 130)^2 + w_2(x_1+0. 5x_2- 250)^2 + w_3 (0. 5x_1+2x_2- 800)^2$

……………………(***) S.t

X1+ X2≤400
2X1+X2≤500
With X1≥0, X2≥0.

In order to solve problem (***) we must find the value of each weight of problem (***) , Therefore the first step we construct the payoff matrix prs , for r=1,2,3 ,s=1,2,3.

Where prs= (s) / (r)

	X*1	x*2	x*3
Z1	130	100	120
Z2	250	250	200
Z3	650	125	800

Now from matrix (prs) we define drs=Z^*_r-Z_{rs} , where r=1,2,3 and s=1,2,3 .in this case we construct the matrix pr~s~ ,for r~=1,2,3 and s~=1,2,3 . Where this matrix contains the following elements:-

$d_{11}=z^*_1-z_{11}$ =130-130=0, d12=$z^*_1-z_{12}$ =130-100=30, $d_{13}=z^*_1-z_{13}$ =130-120=10, and so on for another elements.

Then pr~s~= (s~) / (r~)

	X*1	x*2	x*3
Z1	0	$(30)^2$	$(10)^2$
Z2	0	0	$(50)^2$
Z_3	$(150)^2$	$(675)^2$	0

Then by using our method in section (3.2) we solve the following problem:-

Maximize Z(X) = x1+x2+x3

Subject to

$(150)^2\, x_3 \leq 1$

$(30)^2 x_1+(675)^2 x_3 \leq 1$

$(10)^2 x_1+(50)^2 x_2 \leq 1$

$X_1, X_2, X_3 \geq 0$.

Where minimize v=maximize 1\v=maximize z, and pi=xi. v ,i=1,2,3. Then we solve the above problem by using simplex method we have z=0.015 then v=1\0.015 and X1=0.0011 → w1=p1=0.0011*1/0.015= 0.0733, x2= 0.0004→w2= p2=0.0267 and x3=0→w3=p3=0.subsitute these values in problem (***) we get:-

Minimize $Z(x) =0.0733(0.4x1+0.3x2 – 130)^2 +0.0267(x1+0.5x2- 250)^2$

Subject to

X1+X2≤400 2X1+X2≤500

With X1≥0, X2≥0.

Then by using simplex method we have x*1=99.9860, x*2=299.9320 and z*= 0.0001. Then we have z^*_1=129.974, z^*_2=249.952, and z^*_3=649.857.

3.3 Conclusion :-

In this paper we presented a new idea through finding the values of weights depending on game theory where these values of weights in the past are estimate by the decision maker. Also we found that the method used in this research can be used when we have large number of objective Functions, because it is possible to find the acceptable solution for the decision maker when the size of payoff matrix (prs) small.

3.4:- Reference

[1]:- BAODING Liu, "Theory and practice of uncertain programming", (Third edition), (2009), china.

[2]:- Dong Y, Hu X X and Chen J X," Game Theory Analysis on Multi-Objective Decision", Operations Research and Management Science, 2003, 12(6): 35-39.

[3]:- Gass, S.,"Linear programming method and applications", (1985), N.Y.USA.

[4]:- Hamdy Taha, "operation research an introduction ", (6th edition), (1997), prentice – hall, Inc, New Jersey.

[5]:- M.D.Davis, "Introduction of the theory of Games ", (1970), New York.

[6]:- Wany L.Winston , " Operation research : Application and Algorithms",(2005),Boston ,USA.