The Diagonalization Matrix of the $\otimes(\equiv^* Z_p)$, Where $p$ is an Odd Prime

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Abstract

In this paper we give some concepts that we shall use to determine the diagonalization of the $\otimes(\equiv^* Z_p)$, where $\otimes(\equiv^* Z_p)$ is the tensor product of the matrix of the rational valued character table $Z_p$ by itself $n$-times, $p$ is an odd prime.

1. Introduction

The tensor product of two matrices and the rational character table of $Z_p$ has been given in [2], [5] and [6] respectively. Many studies present new results for finding the rational valued character of $Z_p^{(n)}$ and determination the cyclic decomposition of the factor group $K(G)$, when $G = Z_p^{(n)}$ for $p = 3, 5, 7, 11, 13$ in [6], [8], [7], [9] and [1] respectively.

But in this work we found two matrices $P$ and $Q$ and using some concepts to determine the diagonalization of the $\otimes(\equiv^* Z_p)$ where $\equiv^* Z_p$ is the matrix of the rational character table of $Z_p$, $p$ is an odd prime.

2. Preliminaries

In this section some definitions and basic concepts of tensor product, character theory and the characters table of finite abelian group $Z_p$ are introduced.

Can found these concepts in [2], [3], [5] and [6].

Definition (2-1) [2]
Let $A \in M_n(K), B \in M_m(K)$ we define a matrix $A \otimes B \in M_{nm}(K)$ put:
The Diagonalization Matrix of the $\mathbb{Z}(\equiv \mathbb{Z}_p)$, Where $p$ is an Odd Prime.....

Dunya M. Hamed

$A \otimes B = \begin{bmatrix}
  a_{11}B & a_{12}B & \cdots & a_{1n}B \\
  a_{21}B & a_{22}B & \cdots & a_{2n}B \\
  \vdots & \vdots & \cdots & \vdots \\
  a_{n1}B & a_{n2}B & \cdots & a_{nn}B
\end{bmatrix}_{mn \times mn}$

Where

$A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \cdots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}_{n \times n}$

and

$B = \begin{bmatrix}
  b_{11} & b_{12} & \cdots & b_{1m} \\
  b_{21} & b_{22} & \cdots & b_{2m} \\
  \vdots & \vdots & \cdots & \vdots \\
  b_{m1} & b_{m2} & \cdots & b_{mm}
\end{bmatrix}_{m \times m}$

Thus

$A \otimes B = \begin{bmatrix}
  \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} \\
  \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2k} \\
  \vdots & \vdots & \cdots & \vdots \\
  \alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{kk}
\end{bmatrix}$

Where

$\alpha_{11} = \begin{bmatrix}
  a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1m} \\
  a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2m} \\
  \vdots & \vdots & \cdots & \vdots \\
  a_{11}b_{m1} & a_{11}b_{m2} & \cdots & a_{11}b_{mm}
\end{bmatrix}_{m \times m}$

$\alpha_{1k} = \begin{bmatrix}
  a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1m} \\
  a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2m} \\
  \vdots & \vdots & \cdots & \vdots \\
  a_{1n}b_{m1} & a_{1n}b_{m2} & \cdots & a_{1n}b_{mm}
\end{bmatrix}_{m \times m}$
The Diagonalization Matrix of the $\mathcal{O}(\equiv^* \mathbb{Z}_p)$, Where $p$ is an Odd Prime.....

Dunya M. Hamed

\[
\alpha_{kk} = \begin{bmatrix}
a_{nn}b_{11} & a_{nn}b_{12} & \cdots & a_{nn}b_{1m} \\
a_{nn}b_{21} & a_{nn}b_{22} & \cdots & a_{nn}b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{nn}b_{m1} & a_{nn}b_{m2} & \cdots & a_{nn}b_{mm}
\end{bmatrix}_{m \times m}
\]

and $k = nm$

**Example**

Consider $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & 3 \\ 1 & 2 & 4 \end{bmatrix}_{3 \times 3}$ then:

$A \otimes B = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 & 0 \\ -1 & -2 & -4 & 0 & 0 \\ 2 & 1 & 0 & 4 & 2 \\ -1 & -2 & 3 & -2 & -4 \\ 1 & 2 & 4 & 2 & 4 \\ 6 & 6 & 8 \\ \end{bmatrix}_{6 \times 6}$

**Proposition (2-2)**

Let $A, A^\perp$ be two different matrices in $M_n(k)$ and $B, B^\perp$ be two different matrices in $M_m(k)$ then:

1- $(A + A^\perp) \otimes B = (A \otimes B) + (A^\perp \otimes B)$
2- $(A \otimes B) (A^\perp \otimes B^\perp) = AA^\perp \otimes BB^\perp$

**Proof**

(1)

Let $A = (a_{ij})_{n \times n}$, $A^\perp = (a^\perp_{ij})_{n \times n}$ and $B = (b_{ij})_{m \times m}$

Then $(A + A^\perp) = (a_{ij} + a^\perp_{ij})_{n \times n}$

$\Rightarrow (A + A^\perp) \otimes B = ((a_{ij} + a^\perp_{ij})B)_{nm \times nm}$

$= (a_{ij}B + a^\perp_{ij}B)_{nm \times nm}$

And

$(A \otimes B) = (a_{ij}B)_{nm \times nm}$

$(A^\perp \otimes B^\perp) = (a^\perp_{ij}B)_{nm \times nm}$

Thus,

$(A \otimes B) + (A^\perp \otimes B^\perp) = (a_{ij}B)_{nm \times nm} + (a^\perp_{ij}B)_{nm \times nm}$
The Diagonalization Matrix of the $\otimes(\cong * \mathbb{Z}_p)$, Where $p$ is an Odd Prime.....

Dunya M. Hamed

Then

$$(A + A^\dagger) \otimes B = (A \otimes B) + (A^\dagger \otimes B)$$

(2)

Let $A = (a_{ij})_{n \times n}$, $A^\dagger = (a^\dagger_{ij})_{n \times n}$

$B = (b_{ij})_{m \times m}$, $B^\dagger = (b^\dagger_{ij})_{m \times m}$

Then, $AA^\dagger \otimes BB^\dagger = (a_{ij})(a^\dagger_{ij}) \otimes (b_{ij})(b^\dagger_{ij})$

Let $C_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$

Where

$AA^\dagger \otimes BB^\dagger = (C_{ij}) \otimes BB^\dagger = (C_{ij}BB^\dagger)$

$= (\sum_{k=1}^{n} a_{ik} a^\dagger_{kj} BB^\dagger)$

$= [a_{ij}a^\dagger_{ij}B + a_{i2}a^\dagger_{2j}B + \cdots + a_{in}a^\dagger_{nj}B]B^\dagger$

$= [a_{ij}Ba^\dagger_{ij}B^\dagger + a_{i2}Ba^\dagger_{2j}B^\dagger + \cdots + a_{in}Ba^\dagger_{nj}B^\dagger]$

$= \left[ \sum_{k=1}^{n} a_{ik} B a^\dagger_{kj} B^\dagger \right]_{nm \times nm}$

$= \sum_{k=1}^{nm} a^*_{ik} a^\dagger_{kj}$

And

$(A \otimes B) + (A^\dagger \otimes B) = (a_{ij}B)(a^\dagger_{ij}B^\dagger)$

$= (a^*_{ik})(a^\dagger_{kj})$

$= \sum_{k=1}^{nm} a^*_{ik} a^\dagger_{kj}$

Where $a^*_{ik} = (a_{ik}B)$

Definition (2-3)

Let $T$ be a matrix representation of finite group $G$ over the field $F$.

The character $\chi$ of $T$ is the mapping $\chi: G \rightarrow F$ defined by $\chi(g) = \text{Tr}(T(g))$, $\forall g \in G$, where $\text{Tr}(T(g))$ refers to the trace of the matrix $T(g)$.

Clearly $\chi(1) = n$, which is called the degree of $\chi$. Also characters of degree 1 are called linear characters.
The Diagonalization Matrix of the $\otimes(\cong Z_p)$, Where $p$ is an Odd Prime…

Dunya M. Hamed

**Example**

In symmetric group $S_3 = <x, y: x^2 = y^3 = 1, xy = y^2x>$, define the representation $T: S_3 \to GL(2, \mathbb{C})$ such that:

$T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $T(y) = \begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix}$, where $w = e^{2\pi i/3}$

The character $\chi$ of $T$ is:

$\chi(T(x)) = 0 + 0 = 0$, $\chi(T(y)) = w + w^2 = -1$.

**Definition (2-4)**

The character afforded by irreducible representation is called **irreducible character**, otherwise it is called **compound character**.

**Example**

Linear characters are irreducible character.

**Definition (2-5)**

A class function on a group $G$ is a function $f: G \to \mathbb{C}$ which is constant on conjugacy classes, that is $f(x^{-1}yx) = f(y), \; \forall x, y \in G$. If all values of $f$ are in $\mathbb{Z}$, then it is called **$\mathbb{Z}$-valued class function**.

**Lemma (2-6)**

Characters are class function.

**Proof**

Let $T$ be matrix representation and $\chi$ character of $T$,

Then,

$\chi(x^{-1}yx) = \text{Tr}(T(x^{-1}yx))$

$= \text{Tr}(T(x^{-1})T(y)T(x))$

$= \text{Tr}(T(x^{-1})T(x)T(y))$

$= \text{Tr}(T(y)) = \chi(y)$

**Theorem (2-7) [3]**

A finite abelian group $G$ of order $n$ has exactly $n$ distinct characters.

**The character table of finite abelian group (2-8)**

For a finite abelian group $G$ of order $n$ a complete information about the irreducible characters of $G$ is displayed in a table called the **character table** of $G$.

We list the elements of $G$ in the $1^{st}$ row, we put $\chi_i(x^j) = \chi_i^j, 1 \leq i \leq n - 1$
The Diagonalization Matrix of the $\otimes(\cong Z_p)$, Where $p$ is an Odd Prime.....

Dunya M. Hamed

Table 1

<table>
<thead>
<tr>
<th>$C_g$</th>
<th>1</th>
<th>$x$</th>
<th>$x^2$</th>
<th>...</th>
<th>$x^{n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C_g</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$</td>
<td>C_G(C_g)</td>
<td>$</td>
<td>n</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>$(\chi_2)^1$</td>
<td>$(\chi_2)^2$</td>
<td>...</td>
<td>$(\chi_2)^{n-1}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\chi_n$</td>
<td>1</td>
<td>$(\chi_n)^1$</td>
<td>$(\chi_n)^2$</td>
<td>...</td>
<td>$(\chi_n)^{n-1}$</td>
</tr>
</tbody>
</table>

Where $|C_g|$ = order of conjugacy class of $g$ and $|C_G(C_g)|$ = order of centralizer of $g$ in $G$.

If $G = Z_n$, the cyclic group of order $n$, and let $w = e^{\frac{2\pi i}{n}}$ be a primitive $n$-th root of unity then the general formula of the character table of $Z_n$ is:

Table 2

<table>
<thead>
<tr>
<th>$C_g$</th>
<th>1</th>
<th>$Z$</th>
<th>$Z^2$</th>
<th>...</th>
<th>$Z^{n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C_g</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$</td>
<td>C_G(C_g)</td>
<td>$</td>
<td>n</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>$w$</td>
<td>$w^2$</td>
<td>...</td>
<td>$w^{n-1}$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>$w^2$</td>
<td>$w^3$</td>
<td>...</td>
<td>$w^{n-2}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\chi_n$</td>
<td>1</td>
<td>$w^{n-1}$</td>
<td>$w^{n-2}$</td>
<td>...</td>
<td>$w$</td>
</tr>
</tbody>
</table>

Example

The group $Z_5$ consists the elements $1, z, z^2, z^3, z^4, (z^5 = 1)$.

Let $w = e^{\frac{2\pi i}{5}}$, then

The character table of $Z_5$ is:

<table>
<thead>
<tr>
<th>$C_g$</th>
<th>1</th>
<th>$z$</th>
<th>$z^2$</th>
<th>$z^3$</th>
<th>$z^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C_g</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$</td>
<td>C_G(C_g)</td>
<td>$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>$w$</td>
<td>$w^2$</td>
<td>$w^3$</td>
<td>$w^4$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>$w^2$</td>
<td>$w^3$</td>
<td>$w$</td>
<td>$w^4$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>$w^3$</td>
<td>$w^4$</td>
<td>$w$</td>
<td>$w^2$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>1</td>
<td>$w^4$</td>
<td>$w^3$</td>
<td>$w^2$</td>
<td>$w$</td>
</tr>
</tbody>
</table>
The Diagonalization Matrix of the $\otimes(\equiv^* Z_p)$, Where $p$ is an Odd Prime

Dunya M. Hamed

The Rational Valued Character of $Z_p$ (2-9)

The general formula of the rational valued character table of $Z_p$ is:

$$
\equiv^* Z_p = \begin{bmatrix}
1 & x \\
1 & p-1 \\
\phi_1 & 1 & 1 \\
\phi_2 & p-1 & -1
\end{bmatrix}
$$

Table 3

3. The Diagonal Matrix of the Tensor Product for $\equiv^* Z_p$

In this section we give some concepts that we shall use to determinate $D(\otimes(\equiv^* Z_p))$, where $p$ is an odd prime.

**Definition (3-1) [3]**

A **rational valued** character $\theta$ of $G$ is a character whose values are in $Z$, that is $\theta(x) \in Z, \forall x \in G$.

**Theorem (3-2) [4]**

Let $M$ be an $m \times n$ matrix with entries in a principal domain $R$. then there exist matrices $P, Q, D$ such that:
1. $P$ and $Q$ are invertible.
2. $QMP^{-1} = D$.
3. $D$ is diagonal matrix.
4. If we denote $D_{ii}$ by $d_i$ then there exists a natural number $r$, $0 \leq r \leq \min (m, n)$ such that $j > r$ implies $d_j = 0$ and $j \leq r$ implies $d_j \neq 0$ and $1 \leq j \leq r$ implies $d_j$ divides $d_{j+1}$.

**Definition (3-3) [4]**

Let $M$ be a matrix with entries in a principal domain $R$, be equivalent to a matrix $D = \text{diag} \{d_1, d_2, \ldots, d_r, 0, \ldots, 0\}$ such that $d_j/d_{j+1}$ for $1 \leq j \leq r$, we call $D$ the **invariant factor matrix** of $M$ and $d_1, d_2, \ldots, d_r$ the **invariant factors** of $M$.

**Theorem (3-4) [4]**

Let $M$ be a matrix with entries in a principal domain $R$, then the invariant factors are unique.

**Lemma (3-5) [5]**

Let $A$ and $B$ are two non-singular matrices of degree $n$ and $m$ respectively over a principal domain $R$, and let

$P_AQ_A = D(A) = \text{diag} \{d_1(A), d_2(A), \ldots, d_n(A)\}$.
The Diagonalization Matrix of the $\otimes(\equiv^* Z_p)$, Where $p$ is an Odd Prime.....

Dunya M. Hamed

$P_2 \otimes B_2 = D(B) = \text{diag} \{d_1(B), d_2(B), \ldots, d_n(B)\}$,

be the invariant factor matrices of $A$ and $B$ then,

$(P_1 \otimes P_2) \cdot (A \otimes B) \cdot (Q_1 \otimes Q_2) = D(A) \otimes D(B)$

and from this the invariant factor matrix of $A \otimes B$ can be written down.

Let $H$ and $L$ be $P_1$ and $P_2$ – groups respectively, where $P_1$ and $P_2$ are distinct prime. We know that

$\equiv (H \times L) = \equiv (H) \otimes \equiv (L)$.

Since $\text{g.c.d} (P_1, P_2) = 1$, we have

$\equiv^* (H \times L) = \equiv^* (H) \otimes \equiv^* (L)$

Example

The rational valued character $Z_2$ and $Z_3$ are

$\equiv^* (Z_2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\equiv^* (Z_3) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

Let

$P_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

Then

$P_1 \circ \equiv^* (Z_2) \circ Q_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$

$P_2 \circ \equiv^* (Z_3) \circ Q_2 = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$

by lemma (3-5), we have

$(P_1 \otimes P_2) \circ (\equiv^* (Z_2) \otimes \equiv^* (Z_3)) \circ (Q_1 \otimes Q_2) = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The Diagonal Matrix for $\otimes(\equiv^* Z_p)$ (3-6)

We denote for the tensor product of the matrix of the rational character table of $Z_p$ of $n$-times of itself by $\otimes(\equiv^* Z_p)$.

We can apply lemma (3-5) to determine diagonal of $\otimes(\equiv^* Z_p)$, where $p$ is an odd prime.
The Diagonalization Matrix of the $\otimes(\equiv^* Z_p)$, Where $p$ is an Odd Prime.....

Dunya M. Hamed

Let $P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ be two matrices which is the invariant factor matrix for $\equiv^* Z_p$.

where $\equiv^* Z_p = \begin{bmatrix} 1 & 1 \\ p-1 & -1 \end{bmatrix}$.

Hence, by lemma (3-5)

$P \cdot \equiv^* (Z_p) \cdot Q = \begin{bmatrix} -p & 0 \\ 0 & -1 \end{bmatrix}$

Now, we consider explicitly the case $n = 2$, then

$P \otimes P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4 \times 4}$, $Q \otimes Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4 \times 4}$

And

$\equiv^* (Z_p) \otimes \equiv^* (Z_p) = \otimes(\equiv^* Z_p) = \begin{bmatrix} 1 & 1 \\ p-1 & -1 \\ p-1 & p-1 \\ (p-1)^2 & -(p-1) & -(p-1) & 1 \end{bmatrix}_{4 \times 4}$

We obtain

$(P \otimes P) \cdot (\otimes(\equiv^* Z_p)) \cdot (Q \otimes Q) = \begin{bmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Hence, by lemma (3-5)

$D(\otimes(\equiv^* Z_p)) = \text{diag} \{p^2; p; p; 1\}$

We consider explicitly the case $n = 3$, then we have
The Diagonalization Matrix of the \( \otimes(\equiv^* Z_p) \), Where \( p \) is an Odd Prime.....

Dunya M. Hamed

\[
P \otimes P \otimes P = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{8 \times 8}
\]

\[
Q \otimes Q \otimes Q = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}_{8 \times 8}
\]

And \( \equiv^* (Z_p) \otimes \equiv^* (Z_p) \otimes \equiv^* (Z_p) = \otimes(\equiv^* Z_p) = \]

\[
\begin{bmatrix}
p - 1 & -1 & p - 1 & -1 & p - 1 & -1 & 1 & 1 \\
p - 1 & p - 1 & p - 1 & -1 & p - 1 & p - 1 & -1 & -1 \\
p - 1 & (p - 1)^2 & - (p - 1) & (p - 1)^2 & - (p - 1) & 1 & (p - 1)^2 & - (p - 1) \\
p - 1 & (p - 1)^2 & - (p - 1) & p - 1 & - (p - 1) & - (p - 1) & 1 & - (p - 1) \\
p - 1 & (p - 1)^3 & - (p - 1)^2 & (p - 1)^3 & - (p - 1)^2 & p - 1 & (p - 1)^3 & p - 1 \\
\end{bmatrix}_{8 \times 8}
\]

Now, we obtain

\[
(P \otimes P \otimes P) \cdot (\otimes(\equiv^* Z_p)) \cdot (Q \otimes Q \otimes Q) =
\]

\[
\begin{bmatrix}
-p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p & - p^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - p^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & - p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & - p^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & - p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]
The Diagonalization Matrix of the $\otimes(\equiv^* \mathbb{Z}_p)$, Where $p$ is an Odd Prime……

Hence, by lemma (3-5)

$$D(\otimes(\equiv^* \mathbb{Z}_p)) = \text{diag}\left\{ -p^3; -p^2, -p^2; -p, -p, -p; -1 \right\}$$

We consider explicitly the case $n = 4$

Then, we obtain

$$(P \otimes P \otimes P \otimes P) \cdot (\otimes(\equiv^* \mathbb{Z}_p)) \cdot (Q \otimes Q \otimes Q \otimes Q) =$$

$$\begin{bmatrix}
    p^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

Hence, by lemma (3-5)

$$D(\otimes(\equiv^* \mathbb{Z}_p)) = \text{diag}\left\{ p^4; p^3, p^3; p^3; p^2, p^2, p^2, p^2, p^2; p, p, p, p; 1 \right\}$$

When $n = 5$, we have

$$(P \otimes P \otimes P \otimes P \otimes P) \cdot (\otimes(\equiv^* \mathbb{Z}_p)) \cdot (Q \otimes Q \otimes Q \otimes Q \otimes Q) =$$
Hence, by lemma (3-5)

\[ D(\otimes(\equiv^* Z_p)) = \text{diag} \left\{ -p^5; -p^4, \ldots, -p^4; -p^3, \ldots, -p^3; \right\} \]

\[ \left\{ -p^2, \ldots, -p^2; -p, \ldots, -p; -1 \right\}. \]

The general case for P is an odd prime and \( n \in Z \) give by the following proposition.
**Proposition**

If \( P \) is an odd prime and \( n \in \mathbb{Z}^+ \) then

\[
(\otimes P) \cdot (\otimes (\equiv^* Z_p)) \cdot (\otimes Q) = 
\]

\[
D(\otimes (\equiv^* Z_p)) = \text{diag} \left\{ \pm p^n; \pm p^{n-1}, \ldots, \pm p^{n-2}, \ldots, \pm p^n; \right. \\
\left. \begin{pmatrix} n \\ n-1 \end{pmatrix}, \begin{pmatrix} n \\ n-2 \end{pmatrix}, \ldots, \begin{pmatrix} n \\ 2 \end{pmatrix}, \begin{pmatrix} n \\ 1 \end{pmatrix} \right\}
\]

If \( 1 \leq i \leq n \) then \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \)

(i.e) \( \binom{n}{i} \) is the number of combinations of objects taken \( i \).

**Proof**

By an inductive argument, the statement is certainly true for \( k = 1 \). Assuming it holds for an arbitrary \( k \), then

\[
D(\otimes (\equiv^* Z_p)) = \text{diag} \left\{ \pm p^k; \pm p^{k-1}, \ldots, \pm p^{k-2}, \ldots, \pm p^k; \right. \\
\left. \begin{pmatrix} k \\ k-1 \end{pmatrix}, \begin{pmatrix} k \\ k-2 \end{pmatrix}, \ldots, \begin{pmatrix} k \\ 2 \end{pmatrix}, \begin{pmatrix} k \\ 1 \end{pmatrix} \right\}
\]

Where \( \pm \) appearing at the end is forced by the fact that \( \equiv^* (Z_p^k) \). We must show that it also holds for \( k + 1 \). But this is immediate from lemma (3-5), we obtain

\[
(\otimes \equiv^* Z_p) = (\otimes (\equiv^* Z_p)) \otimes \equiv^* (Z_p).
\]
The Diagonalization Matrix of the $\otimes(\equiv^* \mathbb{Z}_p)$, Where $p$ is an Odd Prime.....

Dunya M. Hamed

Hence

$$D(\otimes(\equiv^* \mathbb{Z}_p)) = D(\otimes(\equiv^* \mathbb{Z}_p)) \otimes D(\equiv^*(\mathbb{Z}_p)).$$

References