The Diagonalization Matrix of the $\overset{n}{\otimes}(\equiv^* Z_p)$, *Where p is an Odd Prime*

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Abstract

In this paper we give some concepts that we shall use to determinate the diagonalization of the $\overset{n}{\otimes}(\equiv^* Z_p)$, where $\overset{n}{\otimes}(\equiv^* Z_p)$ is the tensor product of the matrix of the rational valued character table Z_p by itself n-times, p is an odd prime.

1. Introduction

The tensor product of two matrices and the rational character table of Z_p has been given in [2], [5] and [6] respectively.

Many studies present new results for finding the rational valued character of $Z_p^{(n)}$ and determination the cyclic decomposition of the factor group K(G), when $G = Z_p^{(n)}$ for p = 3, 5, 7, 11, 13 in [6], [8], [7], [9] and [1] respectively.

But in this work we found two matrices P and Q and using some concepts to determine the diagonalization of the $\overset{n}{\otimes}(\equiv^* Z_p)$ where $\equiv^* Z_p$ is the matrix of the rational character table of Z_p, p is an odd prime.

2. Preliminaries

In this section some definitions and basic concepts of tensor product, character theory and the characters table of finite abelian group Z_p are introduced.

Can found these concepts in [2], [3], [5] and [6].

Definition (2-1) [2]

Let $A \in M_n(K)$, $B \in M_m(K)$ we define a matrix $A \otimes B \in M_{nm}(K)$ put:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}_{nm \times mn}$$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}_{m \times m}$$

Thus

$$A \otimes B = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{kk} \end{bmatrix}$$

Where

$$\alpha_{11} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1m} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{11}b_{m1} & a_{11}b_{m2} & \cdots & a_{11}b_{mm} \end{bmatrix}_{m \times m}$$

$$\alpha_{1k} = \begin{bmatrix} a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1m} \\ a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n}b_{m1} & a_{1n}b_{m2} & \cdots & a_{1n}b_{mm} \end{bmatrix}_{m \times m}$$

$$\alpha_{kk} = \begin{bmatrix} a_{nn}b_{11} & a_{nn}b_{12} & \cdots & a_{nn}b_{1m} \\ a_{nn}b_{21} & a_{nn}b_{22} & \cdots & a_{nn}b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{nn}b_{m1} & a_{nn}b_{m2} & \cdots & a_{nn}b_{mm} \end{bmatrix}_{m\times m}$$

and k = nm

Example

Consider $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & 3 \\ 1 & 2 & 4 \end{bmatrix}_{3\times 3}$ then:

$$A \otimes B = \left[\begin{array}{ccc|ccc} -2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 & 0 & 0 \\ -1 & -2 & -4 & 0 & 0 & 0 \\ \hline 2 & 1 & 0 & 4 & 2 & 0 \\ -1 & -2 & 3 & -2 & -4 & 6 \\ 1 & 2 & 4 & 2 & 4 & 8 \end{array}\right]_{6\times 6}$$

Proposition (2-2)

Let $A, A^{\backslash}$ be two different matrices in $M_n(k)$ and $B, B^{\backslash}$ be two different matrices in $M_m(k)$ then:

1- $(A + A^{\backslash}) \otimes B = (A \otimes B) + (A^{\backslash} \otimes B)$

2- $(A \otimes B)\,(A^{\backslash} \otimes B^{\backslash}) = AA^{\backslash} \otimes BB^{\backslash}$

Proof

(1)

Let $A = (a_{ij})_{n\times n}$, $A^{\backslash} = (a^{\backslash}_{ij})_{n\times n}$ and $B = (b_{ij})_{m\times m}$

Then $(A + A^{\backslash}) = (a_{ij} + a^{\backslash}_{ij})_{n\times n}$

$\Rightarrow (A + A^{\backslash}) \otimes B = ((a_{ij} + a^{\backslash}_{ij})B)_{nm\times nm}$

$= (a_{ij}B + a^{\backslash}_{ij}B)_{nm\times nm}$

And

$(A \otimes B) = (a_{ij}B)_{nm\times nm}$

$(A^{\backslash} \otimes B^{\backslash}) = (a^{\backslash}_{ij}B)_{nm\times nm}$

Thus,

$(A \otimes B) + (A^{\backslash} \otimes B^{\backslash}) = (a_{ij}B)_{nm\times nm} + (a^{\backslash}_{ij}B)_{nm\times nm}$

$$= (a_{ij}B + a^{\backslash}_{ij}B)_{nm\times nm}$$

Then

$(A + A^{\backslash}) \otimes B = (A \otimes B) + (A^{\backslash} \otimes B)$

(2)

Let $A = (a_{ij})_{n\times n},\ A^{\backslash} = (a^{\backslash}_{ij})_{n\times n}$

$B = (b_{ij})_{m\times m},\ B^{\backslash} = (b^{\backslash}_{ij})_{m\times m}$

Then, $AA^{\backslash} \otimes BB^{\backslash} = (a_{ij})(a^{\backslash}_{ij}) \otimes (b_{ij})(b^{\backslash}_{ij})$

Let $C_{ij} = \sum_{k=1}^{n} a_{ik}\ a_{kj}$

Where

$$AA^{\backslash} \otimes BB^{\backslash} = (C_{ij}) \otimes BB^{\backslash} = (C_{ij}BB^{\backslash})$$

$$= (\sum_{k=1}^{n} a_{ik}\ a^{\backslash}_{kj}\ BB^{\backslash})$$

$$= [a_{i1}a^{\backslash}_{1j}B + a_{i2}a^{\backslash}_{2j}B + \cdots + a_{in}a^{\backslash}_{nj}B]\ B^{\backslash}$$

$$= [a_{i1}Ba^{\backslash}_{1j}B^{\backslash} + a_{i2}Ba^{\backslash}_{2j}B^{\backslash} + \cdots + a_{in}Ba^{\backslash}_{nj}B^{\backslash}]$$

$$= \left[\sum_{k=1}^{n} a_{ik}B\ a^{\backslash}_{kj}B^{\backslash}\right]_{nm\times nm}$$

$$= \sum_{k=1}^{n.m} a^{*}_{ik}\ a^{\backslash}_{kj}$$

And

$$(A \otimes B) + (A^{\backslash} \otimes B) = (a_{ij}B)\ (a^{\backslash}_{ij}B^{\backslash})$$

$$= (a^{*}_{ik})\ (a^{\backslash *}_{kj})$$

$$= \sum_{k=1}^{n.m} a^{*}_{ik}\ a^{\backslash *}_{kj}$$

Where $a^{*}_{ik} = (a_{ik}B)$

<u>Definition (2-3)</u>

Let T be a matrix representation of finite group G over the field F.

The character χ of T is the mapping χ: G → F defined by χ(g) = Tr(T(g)), $\forall g \in G$, where Tr(T(g)) refers to the trace of the matrix T(g).

Clearly χ(1) = n, which is called the **degree of χ**. Also characters of degree 1 are called **linear characters**.

Example

In symmetric group S_3 = <x, y: $x^2 = y^3 = 1$, $xy = y^2x$>, define the representation T: $S_3 \to GL(2, \mathbb{C})$ such that:

$T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $T(y) = \begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix}$, where $w = e^{2\pi i/3}$

The character χ of T is:

$\chi(T(x)) = 0 + 0 = 0$, $\chi(T(y)) = w + w^2 = -1$.

Definition (2-4)

The character afforded by irreducible representation is called **irreducible character**, otherwise it is called **compound character**.

Example

Linear characters are irreducible character.

Definition (2-5)

A **class function** on a group G is a function f: G → ℂ which is constant on conjugacy classes, that is $f(x^{-1}yx) = f(y)$, $\forall x, y \in G$. If al values of f are in Z, then it is called **Z-valued class function**.

Lemma (2-6)

Characters are class function.

Proof

Let T be matrix representation and χ character of T,

Then,

$$\begin{aligned}\chi(x^{-1}yx) &= Tr(T(x^{-1}yx)) \\ &= Tr(T(x^{-1})T(y)T(x)) \\ &= Tr(T(x^{-1})T(x)T(y)) \\ &= Tr(T(y)) = \chi(y)\end{aligned}$$

Theorem (2-7) [3]

A finite abelian group G of order n has exactly n distinct characters.

The character table of finite abelian group (2-8)

For a finite abelian group G of order n a complete information about the irreducible characters of G is displayed in a table called **the character table** of G.

We list the elements of G in the 1st row, we put $\chi_i(x^j) = \chi_i^j, 1 \le i \le n-1$

$\equiv G =$

C_g	1	x	x^2	…	x^{n-1}
$\lvert C_g \rvert$	1	1	1	…	1
$\lvert C_G(C_g) \rvert$	n	n	n	…	n
χ_1	1	1	1	…	1
χ_2	1	$(\chi_2)^1$	$(\chi_2)^2$	…	$(\chi_2)^{n-1}$
⋮	⋮				⋮
χ_n	1	$(\chi_n)^1$	$(\chi_n)^2$	…	$(\chi_n)^{n-1}$

Table 1

Where $\lvert C_g \rvert$ = order of conjugacy class of g and $\lvert C_{G(g)} \rvert$ = order of centralizer of g in G.

If G = Z_n, the cyclic group of order n, and let $w = e^{2\pi i/n}$ be a primitive n-th root of unity then the general formula of the character table of Z_n is:

$\equiv G =$

C_g	1	Z	Z^2	…	Z^{n-1}
$\lvert C_g \rvert$	1	1	1	…	1
$\lvert C_G(C_g) \rvert$	n	n	n	…	n
χ_1	1	1	1	…	1
χ_2	1	w^1	w^2	…	w^{n-1}
χ_3	1	w^2	w^3	…	w^{n-2}
⋮	⋮				⋮
χ_n	1	w^{n-1}	w^{n-2}	…	w

Table 2

Example

The group Z_5 consists the elements 1, z, z^2, z^3, z^4, (z^5 = 1).

Let $w = e^{2\pi i/5}$, then

The character table of Z_5 is:

$\equiv Z_5 =$

C_g	1	z	z^2	z^3	z^5
$\lvert C_g \rvert$	1	1	1	1	1
$\lvert C_G(C_g) \rvert$	5	5	5	5	5
χ_1	1	1	1	1	1
χ_2	1	w	w^2	w^3	w^4
χ_3	1	w^2	w^4	w	w^3
χ_4	1	w^3	w	w^4	w^2
χ_5	1	w^4	w^3	w^2	w

The Rational Valued Character of Z_p (2-9)

The general formula of the rational valued character table of Z_p is:

$\equiv^* Z_p =$

	1	x
	1	p-1
ϕ_1	1	1
ϕ_2	p-1	-1

Table 3

3. The Diagonal Matrix of the Tensor Product for ($\equiv^* Z_p$)

In this section we give some concepts that we shall use to determinate $D(\overset{n}{\otimes}(\equiv^* Z_p))$, where p is an odd prime.

Definition (3-1) [3]

A **rational valued** character θ of G is a character whose values are in Z, that is $\theta(x) \in Z, \forall\ x \in G$.

Theorem (3-2) [4]

Let M be an m × n matrix with entries in a principal domain R. then there exist matrices P, Q, D such that:

1. P and Q are invertible.
2. $Q\ M\ P^{-1} = D$.
3. D is diagonal matrix.
4. If we denote D_{ii} by d_i then there exists a natural number r, $0 \le r \le \min(m, n)$ such that j > r implies $d_j = 0$ and $j \le r$ implies $d_j \neq 0$ and $1 \le j \le r$ implies d_j divides d_{j+1}.

Definition (3-3) [4]

Let M be a matrix with entries in a principal domain R, be equivalent to a matrix D = diag $\{d_1, d_2, \ldots, d_r, 0, \ldots, 0\}$ such that d_j/d_{j+1} for $1 \le j \le r$, we call **D the invariant factor matrix of M** and **d_1, d_2, …, d_r the invariant factors of M**.

Theorem (3-4) [4]

Let M be a matrix with entries in a principal domain R, then the invariant factors are unique.

Lemma (3-5) [5]

Let A and B are two non-singular matrices of degree n and m respectively over a principal domain R, and let

$P_1\ A\ Q_1 = D(A) = \text{diag}\{d_1(A), d_2(A), \ldots, d_n(A)\}$,

$P_2 B Q_2 = D(B) = \text{diag}\{d_1(B), d_2(B), \ldots, d_n(B)\}$,

be the invariant factor matrices of A and B then,

$(P_1 \otimes P_2).(A \otimes B).(Q_1 \otimes Q_2) = D(A) \otimes D(B)$

and from this the invariant factor matrix of $A \otimes B$ can be written down.

Let H and L be P_1 and P_2 – groups respectively, where P_1 and P_2 are distinct prime. We know that

$\equiv (H \times L) = \equiv (H) \otimes \equiv (L)$.

Since g.c.d $(P_1, P_2) = 1$, we have

$\equiv^* (H \times L) = \equiv^* (H) \otimes \equiv^* (L)$

Example

The rational valued character Z_2 and Z_3 are

$\equiv^* (Z_2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\equiv^* (Z_3) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

Let

$P_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

Then

$P_1 \circ \equiv^* (Z_2) \circ Q_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$,

$P_2 \circ \equiv^* (Z_3) \circ Q_2 = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$

by lemma (3-5), we have

$$(P_1 \otimes P_2) \circ (\equiv^* (Z_2) \otimes \equiv^* (Z_3)) \circ (Q_1 \otimes Q_2) = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Diagonal Matrix for $\overset{n}{\otimes}(\equiv^* Z_p)$ (3-6)

We denote for the tensor product of the matrix of the rational character table of Z_p of n-times of itself by $\overset{n}{\otimes}(\equiv^* Z_p)$.

We can apply lemma (3-5) to determine diagonal of $\overset{n}{\otimes}(\equiv^* Z_p)$, where p is an odd prime.

Let $P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, Q = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ be two matrices which is the invariant factor matrix for $\equiv^* Z_p$.

where $\equiv^* Z_p = \begin{bmatrix} 1 & 1 \\ p-1 & -1 \end{bmatrix}$.

Hence, by lemma (3-5)

$$P \cdot \equiv^* (Z_p) \cdot Q = \begin{bmatrix} -p & 0 \\ 0 & -1 \end{bmatrix}$$

Now, we consider explicitly the case n = 2, then

$$P \otimes P = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ \hline -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]_{4\times4} , Q \otimes Q = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ \hline -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]_{4\times4}$$

And

$$\equiv^* (Z_p) \otimes \equiv^* (Z_p) = \overset{2}{\otimes}(\equiv^* Z_p) = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ p-1 & -1 & p-1 & -1 \\ \hline p-1 & p-1 & -1 & -1 \\ (p-1)^2 & -(p-1) & -(p-1) & 1 \end{array}\right]_{4\times4}$$

We obtain

$$(P \otimes P) \cdot (\overset{2}{\otimes}(\equiv^* Z_p)) \cdot (Q \otimes Q) = \begin{bmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, by lemma (3-5)

$$D(\overset{2}{\otimes}(\equiv^* Z_p)) = \text{diag } \{p^2; p, p; 1\}$$

We consider explicitly the case n = 3, then we have

$$P\otimes P\otimes P=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{8\times 8}$$

$$Q\otimes Q\otimes Q=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{8\times 8}$$

And $\equiv^{*}(Z_p)\otimes\equiv^{*}(Z_p)\otimes\equiv^{*}(Z_p)=\overset{3}{\otimes}(\equiv^{*} Z_p)=$

$$=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ p-1 & -1 & p-1 & -1 & p-1 & -1 & p-1 & -1 \\ p-1 & p-1 & -1 & -1 & p-1 & p-1 & -1 & -1 \\ (p-1)^2 & -(p-1) & -(p-1) & 1 & (p-1)^2 & -(p-1) & -(p-1) & 1 \\ p-1 & p-1 & p-1 & p-1 & -1 & -1 & -1 & -1 \\ (p-1)^2 & -(p-1) & (p-1)^2 & -(p-1) & -(p-1) & 1 & -(p-1) & 1 \\ (p-1)^2 & (p-1)^2 & -(p-1) & -(p-1) & -(p-1) & -(p-1) & 1 & 1 \\ (p-1)^3 & -(p-1)^2 & -(p-1)^2 & p-1 & -(p-1)^3 & p-1 & p-1 & -1 \end{bmatrix}_{8\times 8}$$

Now, we obtain

$(P\otimes P\otimes P)\cdot(\overset{3}{\otimes}(\equiv^{*} Z_p))\cdot(Q\otimes Q\otimes Q)=$

$$\begin{bmatrix} -p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Hence, by lemma (3-5)

$$D(\overset{3}{\otimes}(\equiv^* Z_p)) = \text{diag}\left\{-p^3; \underbrace{-p^2, -p^2, -p^2}; \underbrace{-p, -p, -p}; -1\right\}$$

We consider explicity the case n = 4

Then, we obtain

$$(P\otimes P\otimes P\otimes P)\cdot(\overset{4}{\otimes}(\equiv^* Z_p))\cdot(Q\otimes Q\otimes Q\otimes Q) =$$

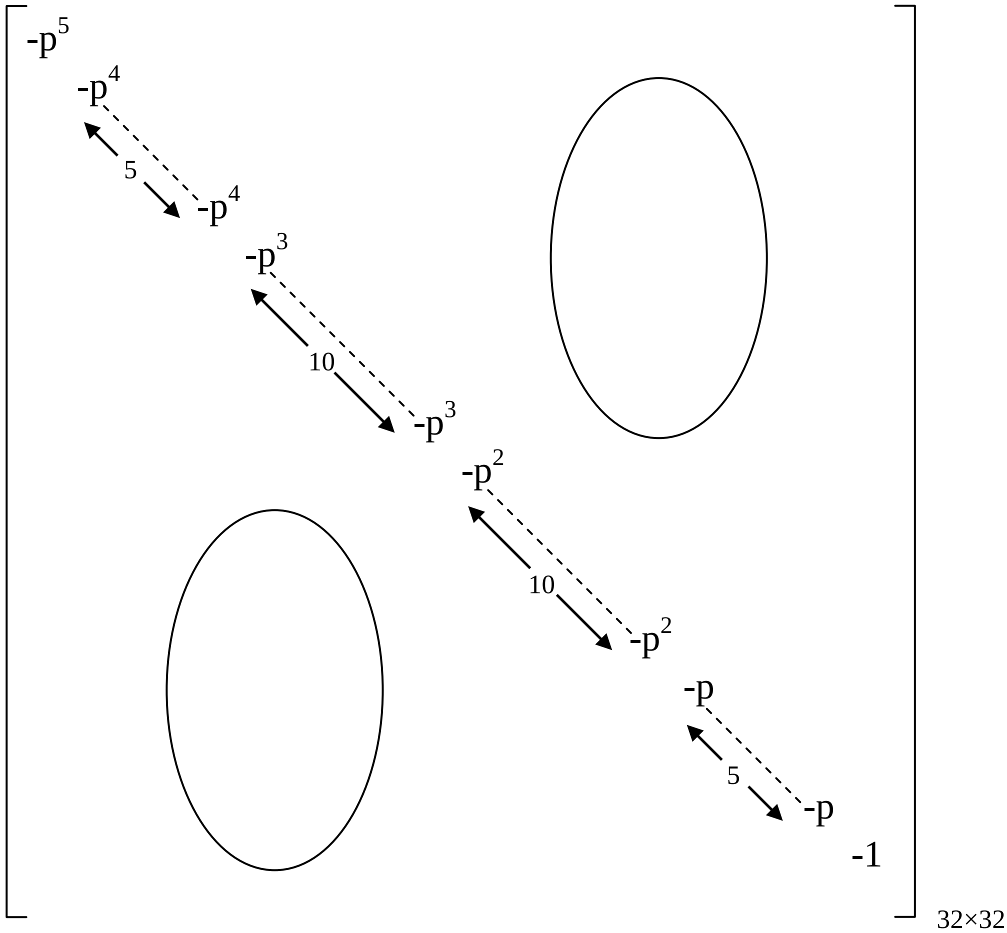
$$\begin{bmatrix}
p^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}_{16\times 16}$$

Hence, by lemma (3-5)

$$D(\overset{4}{\otimes}(\equiv^* Z_p)) = \text{diag}\left\{p^4; \underbrace{p^3, p^3, p^3, p^3}; \underbrace{p^2, p^2, p^2, p^2, p^2, p^2}; \underbrace{p, p, p, p}; 1\right\}$$

When n = 5, we have

$$(P\otimes P\otimes P\otimes P\otimes P)\cdot(\overset{5}{\otimes}(\equiv^* Z_p))\cdot(Q\otimes Q\otimes Q\otimes Q\otimes Q) =$$



Hence, by lemma (3-5)

$$D(\overset{5}{\otimes}(\equiv^{*} Z_p)) = \operatorname{diag}\Big\{-p^5; \underbrace{-p^4, \cdots, -p^4}_{5}; \underbrace{-p^3, \cdots, -p^3}_{10}; \underbrace{-p^2, \cdots, -p^2}_{10}; \underbrace{-p, \cdots, -p}_{5}; -1\Big\}.$$

The general case for P is an odd prime and $n \in \overset{+}{Z}$ give by the following proposition.

Proposition

If P is an odd prime and $n \in \overset{+}{Z}$ then

$$(\overset{n}{\otimes}P)\cdot(\overset{n}{\otimes}(\equiv^{*} Z_p))\cdot(\overset{n}{\otimes}Q) =$$

$$D(\overset{n}{\otimes}(\equiv^{*} Z_p)) = \text{diag}\left\{\pm p^n;\underbrace{\pm p^{n-1},\cdots,\pm p^{n-1}}_{\binom{n}{n-1}};\underbrace{\pm p^{n-2},\cdots,\pm p^{n-2}}_{\binom{n}{n-2}};\right.$$

$$\left.\cdots;\underbrace{\pm p^{2},\cdots,\pm p^{2}}_{\binom{n}{2}};\underbrace{\pm p,\cdots,\pm p}_{\binom{n}{1}};\pm 1\right\}$$

If $1 \le i \le n$ then $\binom{n}{i} = \dfrac{n!}{i!(n-1)!}$

(i.e) $\binom{n}{i}$ is the number of combinations of objects taken i.

Proof

By an inductive argument, the statement is certainly true for k = 1. assuming it holds for an arbitrary k, then

$$D(\overset{k}{\otimes}(\equiv^{*} Z_p)) = \text{diag}\left\{\pm p^k;\underbrace{\pm p^{k-1},\cdots,\pm p^{k-1}}_{\binom{k}{k-1}};\underbrace{\pm p^{k-2},\cdots,\pm p^{k-2}}_{\binom{k}{k-2}};\right.$$

$$\left.\cdots;\underbrace{\pm p^{2},\cdots,\pm p^{2}}_{\binom{k}{2}};\underbrace{\pm p,\cdots,\pm p}_{\binom{k}{1}};\pm 1\right\}$$

Where ± appearing at the end is forced by the fact that $\equiv^{*}(Z_p^k)$. We must show that it also holds for k + 1. But this is immediate from lemma (3-5), we obtain $(\overset{k+1}{\otimes}\equiv^{*} Z_p) = (\overset{k}{\otimes}(\equiv^{*} Z_p))\otimes \equiv^{*}(Z_p)$.

Hence

$$D(\overset{k+1}{\otimes}(\equiv^* Z_p)) = D(\overset{k}{\otimes}(\equiv^* Z_p)) \otimes D(\equiv^* (Z_p)).$$

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المستخلص

فـي هـذا العمـل سـنقدم بعـض المفـاهيم التـي سـوف تسـتخدم لتحديـد المصـفوفة القطرية للضرب الممتد (tensor) لمصفوفة جدول الشواخص النسبية Z_p، عندما p عدد أولـي فردي.