Solving the Eigen Value Problem For Euler Equation With Nonlocal Integral Condition

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Abstract

The aim of this paper is to find the solution and the eigen values for the eigen value problem which its differential operator is Euler equation with nonlocal integral condition, also the existence of non-positive eigen value with more general case are obtained.

1. Introduction

During many years and especially in the last three decades a lot of attention has been paid to problems of ordinary differential equations with different type of boundary conditions [1], [2], [3] and [4]. However the eigen value problems for one-dimensional differential operator with nonlocal condition are met quite rare and is considerably less investigated. There are only a few papers [5], [6], [7] and [8] devoted to this problem.

This paper dealing with similar eigen value problem but with differential operator not all coefficients are constant with nonlocal integral condition. It is work while to note, the solution technique for eigen value problems with nonlocal condition is closely related to the method of solution of nonlocal boundary value problems [9], [10] and [11]

This paper dealing with the eigen value problem subject to nonlocal integral condition in two cases the simple one and the general.

2. The Main Statement

First of all, we consider the eigen value problem for Euler equation with given integral condition

$$x^2y'' + 3xy' + \lambda y = 0 \qquad \dots(1)$$

where $x \in [1,2]$

$$y(1) = 0 \qquad \dots(2)$$

$$y(2) = a\int_1^2 y(x)dx \qquad \dots(3)$$

We find the value of λ such that the problem has the solution y(x) identically not equal to zero. Therefore we formulate three different cases.

Case 1: $\lambda = 1$, then the equation (1) will be

$x^2y'' + 3xy' + y = 0$

we can solve this equation by assumed that $x = e^u$ and with a simple operations we get the following solution

$$y(x) = \frac{1}{x}[c_1 + c_2 \ln(x)] \qquad \ldots(4)$$

Substituting the condition (2) in equation (4) implies

$$y(x) = c_2 \frac{1}{x} \ln(x)$$

Putting it into equation (3), we get

$$\frac{c_2}{2} \ln(2) = a \int_1^2 \frac{c_2}{x} \ln(x) dx$$

$$\frac{c_2}{2} \ln(2) = a c_2 (\ln(x))^2 \Big|_1^2$$

$$\frac{c_2}{2} \ln(2) = a c_2 (\ln(2))^2$$

$$c_2 \left(\frac{1}{2} - a \ln(2) \right) = 0$$

Now we conclude the following. If $a \neq \frac{1}{2\ln(2)}$ then for $\lambda = 1$ there exists only a trivial solution $y(x) = 0$ of the problem. If $a = \frac{1}{2\ln(2)}$ then $= 1$ is the eigen value of the problem (1) – (3), along with the corresponding eigen function $y(x) = \frac{c}{x} \ln(x)$, where c is any number.

Case 2: $\lambda < 1$.

The solution of Eular equation is

$$y(x) = c_1 x^{-1} \cos\left(\ln(x)\sqrt{\lambda - 1}\right) + c_2 x^{-1} \sin\left(\ln(x)\sqrt{\lambda - 1}\right)$$

From this equation and condition (2), we have

$$y(x) = c_2 x^{-1} \sin\left(\ln(x)\sqrt{\lambda - 1}\right)$$

Assume that $\alpha = \sqrt{1 - \lambda} > 0$ then $\alpha i = \sqrt{\lambda - 1} > 0$.

Hence

$y(x) = c_2 x^{-1} \sin(\ln(x)\, \alpha i)$

$y(x) = c_2 x^{-1} i \sinh(\ln(x)\, \alpha) \qquad \ldots(5)$

putting (5) into condition (3), we have

$$\frac{c_2}{2} i \sinh(\ln(2)\alpha) = a\int_1^2 c_2 x^{-1} \sinh(\ln(x)\alpha)\,dx$$

$$= \frac{a c_2 i}{\alpha} \cosh(\ln(x)\alpha)\Big|_1^2$$

$$= \frac{a c_2 i}{\alpha}\left[\cosh(\ln(2)\alpha) - 1\right]$$

$$c_2 \sinh\left(\frac{\ln(2)\alpha}{2}\right)\cosh\left(\frac{\ln(2)\alpha}{2}\right) = \frac{a c_2}{\alpha}\cdot 2\sinh^2\left(\frac{\ln(2)\alpha}{2}\right)$$

$$c_2 \cosh\left(\frac{\ln(2)\alpha}{2}\right) - \frac{2a}{\alpha}\sinh\left(\frac{\ln(2)\alpha}{2}\right) = 0$$

Hence, for $\lambda < 1$, solution (5) of the problem (1)-(3) exists for the value of α being a root of the equation

$$\cosh\left(\frac{\ln(2)\alpha}{2}\right) = \frac{2a}{\alpha}\sinh\left(\frac{\ln(2)\alpha}{2}\right) \qquad \ldots(6)$$

Let us find out dependency of number of roots of the equation (6) on the value of a. Firstly, we put equation (6) into the form

$$\tanh\left(\frac{\ln(2)\alpha}{2}\right) = \frac{\alpha}{2a} \qquad \ldots(7)$$

Taking into account the properties of the function $\tanh\frac{\alpha}{2}$ we conclude, that equation (7) has a single root $\alpha = 0$ as $-\infty < a \le \frac{1}{\ln(2)}$. For $a > \frac{1}{\ln(2)}$ there exist three roots of equation (7): $\alpha = 0$, $\bar{\alpha} > 0$ and $-\bar{\alpha} < 0$. For the root $\alpha = 0$ it implies that $\lambda = 1$, while for the roots $\pm\bar{\alpha}$ we have $\bar{\lambda} = -\left(\pm\bar{\alpha}\right)^2 + 1 < 1$. Thus for $a > 2$, the eigen value $\bar{\lambda} < 1$ of the problem (1)-(3) exists , such that $\sqrt{1-\bar{\lambda}} = \bar{\alpha}$ is the only positive root of the equation (7). The corresponding eigen vector is defined by the formula (5), where $\alpha = \sqrt{1-\bar{\lambda}}$.

Case 3: $\lambda > 1$.

By the same way in case 2, putting condition (2) in the solution of Eular equation, then we have

$$y(x) = c_2\, x^{-1} \sin(\ln(x)\alpha) \qquad \ldots(8)$$

where $\alpha = \sqrt{\lambda - 1} > 0$.

Now putting condition (3) in equation (8) after simple rearrangement, we obtain

$$c_2\left(\cosh\frac{\ln(2)\alpha}{2}-\frac{2a}{\alpha}\sinh\frac{\ln(2)\alpha}{2}\right)=0$$

Then

$$\tan\frac{\ln(2)\alpha}{2}=\frac{\alpha}{2a} \qquad ...(9)$$

Hence, for any value of a there exist infinitely many positive eigen values $\lambda_k=\alpha_k^2+1>0$ along with the corresponding eigen vectors of the form (8).

The results of the three cases can be joined into the following statement:

For any value of a there exist infinitely many positive eigen value λ_k of the problem (1)-(3). These eigen values are the roots of the equation

$$\tan\frac{\ln(2)\sqrt{\lambda-1}}{2}=\frac{\sqrt{\lambda-1}}{2a}$$

These correspond to eigen functions of the form

$$y_k(x)=x^{-1}\sin\left(\ln(x)\sqrt{\lambda_k-1}\right)\ ;\ k=1,2,...$$

Moreover, the following statement is true

1- If $-\infty<a\le\frac{1}{\ln(2)}$, then there are no other eigen values;

2- If $a=\frac{1}{\ln(2)}$ then $\lambda_0=1$ is the eigen value of the problem (1)-(3) with the corresponding eigen function $y_0(x)=\frac{1}{x}\ln(x)$;

3- If $\frac{1}{\ln(2)}<a<\infty$, then one more negative eigen value $\bar{\lambda}=-(\bar{\alpha})^2-1$ exists.

It is corresponds to the only positive root of equation (7). The corresponding eigen function is

$$\overline{y(x)}=x^{-1}\sin\left(\ln(x)\bar{\alpha}\right)$$

3. More General Problem

Let us consider now more general eigen value problem:

$$x^2y''+3xy'+\lambda y=0 \qquad ...(10)$$

$$y(1)=a_1\int_1^2 y(x)dx \qquad ...(11)$$

$$y(2)=a_2\int_1^2 y(x)dx \qquad ...(12)$$

The methodology of the solution of the problem let be the same as earlier.

Case 1: $\lambda = 1$, the general solution of equation (10) is the previous equation (4) putting it into expressions (11), (12) as the following

$$y(1) = a_1 \int_1^2 \frac{1}{x}[c_1 + c_2 \ln(x)]dx$$

$$c_1 = a_1[c_1 \ln(2) + \frac{c_2}{2}(\ln(2))^2]$$

$$c_1(1 - a_1 \ln(2)) - \frac{a_1 c_2}{2}(\ln(2))^2 = 0 \qquad \ldots(13)$$

$$y(2) = a_2 \int_1^2 \frac{1}{x}[c_1 + c_2 \ln(x)]dx$$

$$\frac{1}{2}[c_1 + c_2 \ln(2)] = a_2[c_1 \ln(2) + \frac{c_2}{2}(\ln(2))^2]$$

$$c_1(\frac{1}{2} - a_2 \ln(2)) + c_2(\frac{1}{2}\ln(2) - \frac{a_1}{2}(\ln(2))^2) = 0 \qquad \ldots(14)$$

Thus there exists the solution

$$y(x) = \frac{1}{x}[c_1 + c_2 \ln(x)]$$

Not identical zero, if the determinant of the coefficients of equations (13), (14) equal to zero,

$$D = \begin{vmatrix} 1 - a_1 \ln(2) & -\frac{a_1}{2}(\ln(2))^2 \\ \frac{1}{2} - a_2 \ln(2) & \frac{1}{2}\ln(2) - \frac{a_2}{2}(\ln(2))^2 \end{vmatrix} = 0$$

This implies that $a_2 - a_1 = \frac{1}{\ln(2)}$. Hence, if $a_2 - a_1 \neq \frac{1}{\ln(2)}$ then $\lambda = 1$ is not an eigen value of the problem, since y(x) = 0, if $a_2 - a_1 = \frac{1}{\ln(2)}$ then $\lambda = 1$ is an eigen value of the problem along with the corresponding eigen function:

$$y(x) = \frac{1}{x}\left[1 + \frac{2(1 - a_1 \ln(2))}{a_1(\ln(2))^2}\ln(x)\right]$$

Case 2: $\lambda < 1$, let $\alpha = \sqrt{1-\lambda} > 0$ in this case the general solution of equation (10) has the form

$y(x) = c_1 x^{-1}\cosh(\ln(x)\alpha) + c_2\, i \sinh(\ln(x)\alpha)$

putting expression for y(x) into both (11) and (12) we get

$$c_1 = a_1\left[\frac{c_1}{\alpha}\sinh(\ln(2)\alpha) + \frac{c_2 i}{\alpha}\cosh(\ln(2)\alpha) - \frac{c_2 i}{\alpha}\right]$$

$$c_1\left(1 - \frac{a_1}{\alpha}\sinh(\ln(2)\alpha)\right) - c_2\left(1 - \frac{a_1 i}{\alpha}\mathrm{conh}(\ln(2)\alpha) - \frac{a_1 i}{\alpha}\right) = 0 \qquad \ldots(15)$$

and

$$\frac{c_1}{2}\mathrm{conh}(\ln(2)\alpha) + \frac{c_2 i}{2}\sinh(\ln(2)\alpha) = a_2\left[\frac{c_1}{\alpha}\sinh(\ln(2)\alpha) + \frac{c_2 i}{\alpha}\mathrm{conh}(\ln(2)\alpha) - \frac{c_2 i}{\alpha}\right] \ldots(16)$$

Equating to zero the determinant of the system of equation (15) and (16)

$$D = \begin{vmatrix} 1 - \frac{a_1}{\alpha}\sinh(\ln(2)\alpha) & -\frac{a_1 i}{\alpha}\mathrm{conh}(\ln(2)\alpha) + \frac{a_1 i}{\alpha} \\ \frac{1}{2}\cosh(\ln(2)\alpha) - \frac{a_2}{\alpha}\sinh(\ln(2)\alpha) & \frac{i}{2}\sinh(\ln(2)\alpha) - \frac{a_2 i}{\alpha}\mathrm{conh}(\ln(2)\alpha) + \frac{a_2 i}{\alpha} \end{vmatrix} = 0$$

We get the following condition of existence of the negative eigen value

$$\tanh\frac{\ln(2)\alpha}{2} = \frac{\alpha}{-a_1 - 2a_2} \qquad \ldots(17)$$

Equation (17) has a single positive root $\bar{\alpha}$, provided $-a_1 - 2a_2 > 2$. In this case, problems (10)-(12) has the negative eigen value $\bar{\lambda} = -(\bar{\alpha})^2 + 1$.

Case 3: $\lambda > 1$, in this case the general solution of equation (1) is of the form

$$y(x) = c_1 x^{-1}\cos(\ln(x)\alpha) + c_2\, x^{-1}\sin(\ln(x)\alpha) \qquad \ldots(18)$$

where $\alpha = \sqrt{\lambda - 1} > 0$.

Putting this expression into (11) and (12) by the same way in case (2) we obtain the system of the equations with respect to unknown constants c_1 and c_2.

$$c_1\left(1 - \frac{a_1}{\alpha}\sin(\ln(2)\alpha)\right) + c_2\left(1 - \frac{a_1}{\alpha}\mathrm{con}(\ln(2)\alpha) - \frac{a_1}{\alpha}\right) = 0 \qquad \ldots(19)$$

$$c_1\left(\frac{1}{2}\mathrm{con}(\ln(2)\alpha) - \frac{a_2}{\alpha}\sin(\ln(2)\alpha)\right) + c_2\left(\frac{1}{2}\sin(\ln(2)\alpha) + \frac{a_2}{\alpha}\mathrm{con}(\ln(2)\alpha) + \frac{a_2}{\alpha}\right) = 0 \ldots(20)$$

Again, there exists a solution of a form (18) by equating to zero the determinant of this system

$$D = \begin{vmatrix} 1 - \frac{a_1}{\alpha}\sin(\ln(2)\alpha) & -\frac{a_1}{\alpha}\mathrm{con}(\ln(2)\alpha) + \frac{a_1}{\alpha} \\ \frac{1}{2}\cos(\ln(2)\alpha) - \frac{a_2}{\alpha}\sin(\ln(2)\alpha) & \frac{1}{2}\sin(\ln(2)\alpha) - \frac{a_2}{\alpha}\mathrm{con}(\ln(2)\alpha) + \frac{a_2}{\alpha} \end{vmatrix} = 0$$

By solving this determinant, we obtain a condition of existence of the positive eigen value

$$\tan\frac{\ln(2)\alpha}{2}=\frac{\alpha}{a_1+2a_2} \quad \ldots(21)$$

This equation has infinitely many positive roots α_k, and $\lambda_k=\alpha_k^2+1>0$. The corresponding eigen function are definite by (21).

Conclusion:

We conclude that we can solve the eigen value problem where the coefficients are variables with nonlocal integral condition where similar problems are solved previously but with constant coefficients.

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