Approximate Solution to Calculus of Variational Problem Using Orthogonal Polynomials

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Abstract

In this paper, some properties concerning orthonormal Bernstein polynomials are derived. An explicit formula for the derivative orthonormal Bernstein polynomials of degree five is obtained in terms of Bernstein polynomials themselves to construct the operational matrix of derivatives of orthonormal Bernstein polynomials. Then we apply it for solving calculus of variational approximately using direct method. Illustrative example is given to demonstrate the efficiency of the method.

1. Introduction

Orthogonal functions have received considerable attention in dealing with analysis, optimal control and venous problems of dynamic systems [4,6,7,8]. Some Orthogonal Polynomials are applied on variational problems to find continuous solutions for these problems [1,2,5]. In this paper, a new operational matrix of derivative for orthonormal Bernstein polynomials of degree five is derived. Orthonormal Bernstein polynomials and their properties are employed for deriving formula of this matrix. Then orthonormal Bernstein polynomials of degree five with the operational matrix of derivative are applied for solving variational problems.

The paper is organized as follows: we begin by introducing some necessary definitions of Bernstein polynomials which are required for establishing our results. In section 3, the operational orthonormal Bernstein polynomials matrix of derivative of degree five is obtained. Section 4, is devoted to apply the orthonormal Bernstein matrix of derivative for solving variational problems. Also a conclusion is given in section 4.

2. Bernstein Polynomials and Their Properties

In this section, some necessary definitions and properties of Bernstein Polynomials are given which are used further in this paper.

The Bernstein Polynomials of nth degree form a complete basis over $[0,1]$, and they are defined by $B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}$, $0 < i < n$ ...(1)

where the binomial coefficients are given by $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

The derivatives of the n^{th} degree Bernstein Polynomials are Polynomials of degree $n-1$ and are given by [3]

$$\dot{B}_{i,n}(x) = n(B_{i-1,n-1}(x) - B_{i,n-1}(x))$$

…(2)

The first derivative of Bernstein Polynomials of degree n in terms of Bernstein of same degree n is given by the following formula

$$\dot{B}_{k,n} = (n-k+1)B_{k-1,n}(t) - (n-2k)B_{k,n}(t) - (k+1)B_{k+1,n}(t) \qquad \ldots(3)$$

3. The Orthogonal Bernstein Polynomials of Degree Five $b_{k,5}$

The orthogonal Bernstein polynomials are not orthogonals using Gram-Schmidt orthogonalization process on $B_{k,5}$ to obtain a class of orthogonal polynomials from Bernstein polynomials.

The five orthogonal polynomials are given by [7]

$$b_{05}(t) = \sqrt{11}\,(1-t)^5$$

$$b_{15}(t) = 6[5(1-t)^4 t - \frac{1}{2}(1-t)^5]$$

$$b_{25}(t) = \frac{18\sqrt{7}}{5}[10(1-t)^3 t^2 - 5(1-t)^4 t + \frac{5}{18}(1-t)^5]$$

$$b_{35}(t) = \frac{28}{\sqrt{5}}[10(1-t)^2 t^3 - 15(1-t)^3 t^2 + \frac{30}{7}(1-t)^4 t - \frac{5}{28}(1-t)^5]$$

$$b_{45}(t) = 7\sqrt{3}[5(1-t)t^4 - 20(1-t)^2 t^3 + 18(1-t)^3 t^2 - 4(1-t)^4 t + \frac{1}{7}(1-t)^5]$$

$$b_{55}(t) = 6[t^5 - \frac{25}{2}(1-t)t^4 + \frac{100}{3}(1-t)^2 t^3 - 25(1-t)^3 t^2 + 5(1-t)^4 t - \frac{1}{6}(1-t)^5]$$

3.1 The Relation Between b_{k5} and B_{k5}

The transformation matrix between Bernstein polynomials of order five and orthogonal polynomials of order five can be derived as follows

$$b_{0,5} = \sqrt{11}\, B_{0,5}$$

$$b_{1,5} = 6[\,B_{1,5} - \frac{1}{2}\,B_{0,5}]$$

$$b_{2,5} = \frac{18\sqrt{7}}{5}[\,B_{2,5} - B_{1,5} + \frac{5}{18}\,B_{0,5}]$$

$$b_{3,5} = \frac{28}{\sqrt{5}}[\,B_{3,5} - \frac{3}{2}\,B_{2,5} + \frac{6}{7}\,B_{1,5} - \frac{5}{28}\,B_{0,5}]$$

$$b_{4,5} = 7\sqrt{3}\,[\,B_{4,5} - 2\,B_{3,5} + \frac{9}{5}\,B_{2,5} - \frac{4}{5}\,B_{1,5} + \frac{1}{7}\,B_{0,5}]$$

$$b_{5,5} = 6[\,B_{5,5} - \frac{5}{2}\,B_{4,5} + \frac{10}{3}\,B_{3,5} - \frac{5}{2}\,B_{2,5} + B_{1,5} - \frac{1}{6}\,B_{0,5}]$$

In matrix form, the relation between b_{k5} and B_{k5} , $k = 0,1,2,3,4,5$ can be written as follows

$$\begin{bmatrix} b_{0,5} \\ b_{1,5} \\ b_{2,5} \\ b_{3,5} \\ b_{4,5} \\ b_{5,5} \end{bmatrix} = \begin{bmatrix} \sqrt{11} & 0 & 0 & 0 & 0 & 0 \\ -3 & 6 & 0 & 0 & 0 & 0 \\ \sqrt{7} & -\frac{18\sqrt{7}}{5} & \frac{18\sqrt{7}}{5} & 0 & 0 & 0 \\ -\sqrt{5} & \frac{288}{7\sqrt{5}} & -\frac{284}{2\sqrt{5}} & \frac{28}{\sqrt{5}} & 0 & 0 \\ \sqrt{3} & -\frac{28\sqrt{3}}{5} & \frac{63\sqrt{3}}{5} & -14\sqrt{3} & 7\sqrt{3} & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 \end{bmatrix} \begin{bmatrix} B_{0,5} \\ B_{1,5} \\ B_{2,5} \\ B_{3,5} \\ B_{4,5} \\ B_{5,5} \end{bmatrix} \qquad \ldots(4)$$

3.2 The Derivative of b_{k5}

The derivative of the orthonormal Bernstein polynomials of degree five in terms of Bernstein polynomials themselves is derived and given by

$$\dot{b}_{0,5} = \sqrt{11}[-5B_{0,5} - B_{1,5}]$$

$$\dot{b}_{1,5} = 6[\frac{15}{2}B_{0,5} - \frac{5}{2}B_{1,5} - 2B_{2,5}]$$

$$\dot{b}_{2,5} = \frac{18\sqrt{7}}{2}[-\frac{115}{18}B_{0,5} + \frac{121}{18}B_{1,5} + B_{2,5} - 3B_{3,5}]$$

$$\dot{b}_{3,5} = \frac{28}{\sqrt{5}}[\frac{145}{28}B_{0,5} - \frac{235}{28}B_{1,5} + \frac{39}{14}B_{2,5} + \frac{11}{2}B_{3,5} - 4B_{4,5}]$$

$$\dot{b}_{4,5} = 7\sqrt{3}[-\frac{33}{7}B_{0,5} + \frac{331}{35}B_{1,5} - \frac{31}{5}B_{2,5} - \frac{18}{5}B_{3,5} + 11B_{4,5} - 5B_{5,5}]$$

$$\dot{b}_{5,5} = 6[\frac{35}{6}B_{0,5} - \frac{77}{6}B_{1,5} + \frac{21}{2}B_{2,5} + \frac{35}{6}B_{3,5} - \frac{119}{6}B_{4,5} + \frac{35}{2}B_{5,5}]$$

3.3 The Orthonormal Matrix of Derivative of b_{k5}

The operational matrix of derivatives D is given by $\frac{d\Psi(t)}{dt} = D\,\Psi(t)$ where $\Psi(t) = [\Psi_1, \Psi_2, \ldots, \Psi_N]$ and Ψ_i , $i = 1,2,3,\ldots,N$ are orthogonal basis functions. Several papers have appeared in the literature concerned with the application of operational matrix of derivatives [4]. In the following section, we introduce a new method for deriving operational matrix of derivative for orthonormal Bernstein polynomials of degree five.

Theorem 1: let $\Psi(t)$ and $\phi(t)$ be the orthogonal Bernstein polynomials and Bernstein polynomials of degree five respectively defined by $\Psi(t) = [b_{05}\ \ b_{15}\ \ b_{25}\ \ b_{35}\ \ b_{45}\ \ b_{55}]^T$ and $\phi(t) = [B_{05}\ \ B_{15}\ \ B_{25}\ \ B_{35}\ \ B_{45}\ \ B_{55}]^T$ the derivative of the vector $\Psi(t)$ can be expressed by $\frac{d\Psi(t)}{dt} = D\,\phi(t)$ where D is the $6x6$ operational matrix of derivative defined as follow

$$\begin{bmatrix} -16.5831 & -3.3166 & 0 & 0 & 0 & 0 \\ 45 & -15 & -12 & 0 & 0 & 0 \\ -152.1307 & 160.0680 & 23.8118 & -71.4353 & 0 & 0 \\ 64.8460 & -105.0952 & 34.8827 & 68.8709 & -50.0879 & 0 \\ -57.1577 & 114.6618 & -75.1710 & -43.6477 & 133.3679 & -60.6218 \\ 35 & -77 & 63 & 35 & -119 & 105 \end{bmatrix} \quad \ldots(5)$$

4. Solution of Variational Problem Using the Operational Matrix of Derivative

Consider the problem of finding the minimum of the time-varying function [5]

$$J(x)=\int_0^1[\dot{x}^2 + t\dot{x} + x^2]dt \qquad \ldots(6)$$

with boundary conditions $x(0) = 0$, $\quad x(1) = \frac{1}{4}$ …(7)

The exact solution is $x(t) = \frac{1}{2} + c_1 e^t + c_2 e^{-t}$ where $c_1 = \frac{2-e}{4(e^2-1)}$ and $c_2 = \frac{e-2e^2}{4(e^2-1)}$.

Let $\quad x(t) = a^T b$ …(8)

then, $\quad \dot{x}(t) = a^T D b$ …(9)

$\quad x^2(t) = a^T\, b\, b^T a$ …(10)

$$\dot{x}^2 = a^T Db\, Db^T a$$

where

$$a = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]^T$$

$$b = [b_{05} \quad b_{15} \quad b_{25} \quad b_{35} \quad b_{45} \quad b_{55}]$$

$$J(x) = \int_0^1 [a^T Db\, Db^T a + t\, a^T Db + a^T b\, b^T a]\, dt$$

$$J(x) = \int_0^1 [a^T (Db\, Db^T + b\, b^T) a]\, dt + \int_0^1 t a^T Db \; dt$$

$$J(x) = \frac{1}{2} a^T H a + c^T a$$

where

$H = 2\int_0^1 [Db\, Db^T + b\, b^T]\, dt$ and $c^T = \int_0^1 t\, Db^T \; dt$

subject to $F_1 a - b_1 = 0$

where

$$H = \begin{bmatrix} 5.7374 & -2.6869 & -1.5469 & -0.7785 & -0.3131 & -0.0786 \\ -2.6869 & 3.2756 & 0.8694 & -0.2742 & -0.5375 & -0.3131 \\ -1.5469 & 0.8694 & 1.3564 & 0.7071 & -0.2742 & -0.7785 \\ -0.7785 & -0.2742 & 0.7071 & 1.3564 & 0.8694 & -1.5469 \\ -0.3131 & -0.5375 & -0.2742 & 0.8694 & 3.2756 & -2.6869 \\ -0.0786 & -0.3131 & -0.7785 & -1.5469 & -2.6869 & 5.7374 \end{bmatrix}$$

…(11)

$$C^T = \begin{bmatrix} \frac{-1}{6} \\ \frac{-1}{6} \\ \frac{-1}{6} \\ \frac{-1}{6} \\ \frac{-1}{6} \\ \frac{5}{6} \end{bmatrix}, \qquad F_1 = \begin{pmatrix} b^T(0) \\ b^T(1) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} \qquad \ldots(12)$$

Table 1 shows the results using the orthonormal Bernstein polynomials of degree five against the exact solutions.

table (1): the orthonormal Bernstein polynomials of degree five against the exact solutions.

t	Exact solution	Bernstein Polynomials	Absolute Error $\lvert Exact - B_{k,n} \rvert$
0	0	0	0
0.1	0.0420	0.0420	0
0.2	0.0793	0.0793	0
0.3	0.1125	0.1125	0
0.4	0.1418	0.1418	0
0.5	0.1674	0.1675	0
0.6	0.1898	0.1898	0
0.7	0.2091	0.2091	0
0.8	0.2254	0.2254	0
0.9	0.2390	0.2390	0
1	0.2500	0.2500	0

The algorithm:

To approximate the solution of variational problems (1), (2) using the orthonormal B-spline of degree 5 in the interval $[0,1]$.

Input

- The vector a^* as syms.
- The vector b.
- The boundary condition.

Output:

- $a_i\ ;\ i = 0,1,2,\dots\dots,5$ the unknown parameters.

Step 1 for $k = 0:5$ do step 2

Step 2 $\binom{5}{k} t^k (1-t)^{5-k}$

Step 3 set B(k,5,t)=b

% output ‘the Bernstein polynomial B_{kn}’

Step 4 for $k = 0:5$ do step 5

Step 5 diff (B(k,5,t),t)

Step 6 set Db (k,5,t) = Db

% output ‘the derivative of B-spline polynomials $\dot{B}_{kn}(t)$’

Step 7 set $a^* = [a_0; a_1; a_2; a_3; a_4; a_5]^T$

Step 8 for k=0: 5 do step 9

Step 9 set $b = [B(0,5,t); B(1,5,t); B(2,5,t); B(3,5,t); B(4,5,t); B(5,5,t)]$

Step 10 set Db = [diff (spline (0, 5, t), t); diff (spline (1, 5, t), t); diff (spline (2, 5, t), t); diff (spline(3, 5, t), t);diff (spline (4, 5, t), t);diff (spline (5, 5, t), t)]

Step 11 for k= 0: 5 do step 12

Step 12 set $l_1 = t\ Db^T$

$C^T = int(l_1,0,1)$

% 'C^T'

Step 13 for k= 0: 5 do step 14

Step 14 set $l_2 = bb^T$

$l_3 = DbDb^T$

$l_4 = 2(l_3 + l_4)$

$H = int(l_4,0,1)$

% output' the matrix H '

Step 15 for k = 0:5 do step 16

Step 16 $x_1 = [B(0,5,0); B(1,5,0); B(2,5,0); B(3,5,0); B(4,5,0); B(5,5,0)]$

Step 17 $x_2 = [B(0,5,1); B(1,5,1); B(2,5,1); B(3,5,1); B(4,5,1); B(5,5,1)]$

Step 18 $F_1 = [x_1^T; x_2^T]$

Step 19 $b_1 = [0; \frac{1}{4}]$

% output 'F_1, b_1'

Step 20 set $x_3 = (F_1 H^{-1} F_1^T)^{-1}$

$x_4 = H^{-1} F_1^T$

$x_5 = x_4 x_3$

$a^* = -H^{-1}c + x_5(F_1 H^{-1}c + b_1)$

Step 21 Stop

$a_0 = 0, \quad a_1 = 0.0888, \quad a_2 = 0.1525, \quad a_3 = 0.1986, \quad a_4 = 0.2306, \quad a_5 = 0.2500$

5. Conclusion

The relation between the orthonormal Bernstein polynomials of degree five $b_{k5}$and the Bernstein polynomials B_{kn} is demonstrated. By using the relation, the operational matrix of derivative of orthonormal Bernstein polynomials id derived. They are applied to solve variational problems. The present method reduces the variational problem into quadratic programming problem. Applications of the presented method are demonstrated through illustrative example. The results reveal that the present method is very effective and give an exact solution.

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حل تقريبي لمسائل التغاير باستخدام متعددات حدود متعامدة.

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المستخلص

في هذا البحث، تم اشتقاق بعض الصيغ الخاصة لمتعددات حدود برنشتن المتعامدة. تم الحصول على صيغة صريحة لمشتقة متعدادات حدود برنشتن المتعامدة من الرتبة الخامسة بدلالة متعددات حدود برنشتن نفسها لتكوين مصفوفة العمليات لمشتقة متعددات حدود برنشتن. تم تطبيق هذه المصفوفة لحل مسائل التغاير بصورة تقريبية باستخدام طريقة مباشرة. أعطي مثال توضيحي لبيان كفاءة الطريقة.