On the Parameters of Teichmüller Space

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Abstract :
The main purpose of this paper is to establish a connection between the length and angular parameters with the parameters of Maskit for Teichmüller space of the Riemann's surfaces of topological type $(1,1)$, i.e, for tori with one boundary component.

Keywords: Teichmüller space, Fuchsian group, Jacobian of the mapping, Maskit's parameters.

1- Introduction:

At different times, some authors have introduced a different parameters in the Teichmüller space, adapted to respective problems. Maskit B.[2] defined, studied and developed the parameters of the Teichmüller space, within a series of his works, then they have been called as the parameters of Maskit .

Maskit ([2]) showed that the Teichmüller space $T(1,1)$ can be parameterized by three real variables in the parameters space

$$P = \{(x,s,\lambda) \in R^3 : x > 1, \quad s > 0, \lambda \geq 1\}$$

Each element in the parameters space correspond to the Fuchsian group – be a representative of the element of the Teichmüller space. The generators {A, B, $[A^{-1}, B^{-1}]$} of this group determined by the matrices

$$A = \begin{pmatrix} \lambda xs & x \\ \lambda s & x \end{pmatrix}, \quad B = \begin{pmatrix} 0 & x \\ -s & x(1+s) \end{pmatrix},$$

which act in the complex upper half plane . Note that when $\lambda = 1$ the corresponding Riemann surface represents topologically a torus with one puncture. When $\lambda > 1$ we will have a torus with deleted disk.

If we denote by r_A, a_A, r_B, a_B repelling and attracting fixed point of A and B respectively, the direct calculation gives the following formula:

$$r_A = \frac{x(s-1) - \sqrt{x^2(s-1)^2 + 4sx}}{2s} \qquad a_A = \frac{x(s-1) + \sqrt{x^2(s-1)^2 + 4sx}}{2s}$$

$$r_B = \frac{x(s+1) + \sqrt{x^2(s+1)^2 - 4sx}}{2s} \qquad a_B = \frac{x(s+1) - \sqrt{x^2(s+1)^2 - 4sx}}{2s}$$

For simplicity, calculations were realized for $\lambda = 1$.

For the Teichmüller space $T(1,1)$, Okumura I. ([4]) considered another parameterization: parameters are the cosines of the interior angles of a triangle formed by the axes of hyperbolic transformations that generate the corresponding Fuchsian group acting on the unit disk. Of course, between these parameters and Maskit parameters the relationship exists. Explicit formulas are very cumbersome, with one exception. This case we will consider in theorem 3.1.

The space $T(1,1)$ recently considered Komori Y. ([1]).

2- Preliminaries:

From the topological point of view any connected oriented two-dimensional surface is homeomorphic to the sphere with g handles and m holes, i.e. it has topological type (g, m). Two surfaces of common topological type are homeomorphic among themselves, i.e. they are topological equivalent. Riemann surface be the topological surface equipped by conformal structure. Two Riemann surfaces are not necessarily conformal equivalent. The elements of the set R (g, m) are classes of conformal equivalent Riemann surfaces of topological type (g, m). This set depends on the parameters named as modules their number depends on numbers g and m. The problem of modules consisted in studying the structure of this set.

O.Teichmüller found the solution of a problem of modules by the theory of quasiconfomal mappings. He has considered the space T (g, m), and he has shown that T(g, m) be the covering of R (g, m). He has determined the metrics in T (g, m) and by this metrics he has introduced a complex structure in T (g, m). The parameters of Teichmüller are difficult computable, therefore some authors determined another parameters.

3- The main results:

Theorem 3.1 If we denote the angle between the axes for the transformation A and B through γ, we obtain

$$\cos\gamma = \frac{x(1-s)^2}{\sqrt{x^2(s^2-1)^2 + 16x^2(x-1)}} .$$

Proof: In fact the proof is a direct calculation. To do this, you need to determine the fixed points of the commutator $[A^{-1}, B^{-1}]$, to construct a corresponding triangle and calculate $\cos\gamma$.

Now we find the relation between the parameters of Maskit and the parameters of Okai T. ([3]) for $T(1,1)$.

In this case, we prefer to take a different set of parameters of Maskit. For a Fuchsian group Γ of topological type $(1,1)$, acting in the upper half of plane $H = \{z; \operatorname{Im} > 0\}$, Maskit considered the parameter space of the form

$$P = \{(x, s, t) \in R^3 ; \quad x > 1, \quad s > 0, \quad t \geq s\}.$$

Then the standard normalized generators A and B of a Fuchsian group Γ are as follows:

$A = \begin{pmatrix} tx & x \\ t & x \end{pmatrix}$ and $B = \begin{pmatrix} 0 & x \\ -s & x(1+s) \end{pmatrix}$.

If $t = s$, then Γ corresponds to the torus S with one puncture.

If $t > s$, then Γ corresponds to a torus S with deleted one disk.

We shall consider only the case when $t > s$. Using the transformation

$$\varphi(z) = \frac{z-i}{z+i}$$

we get a new Fuchsian group $\tilde{\Gamma}$ acting on the unit circle

$$\Delta = \{z: |z| < 1\}.$$

Then the standard normalized form $\tilde{A}$ and $\tilde{B}$ to the group $\tilde{\Gamma}$ are as follows:

$$\tilde{A} = \varphi . A \circ \varphi^{-1} = \left[\frac{r_1}{\sqrt{r_1^2 - r_2^2}} e^{i\gamma_1} ; \frac{r_2}{\sqrt{r_1^2 - r_2^2}} e^{-i\gamma_2}\right]$$

and $$\tilde{B} = \varphi \circ B . \varphi^{-1} = \left[\frac{r_3}{\sqrt{r_3^2 - r_4^2}} e^{i\gamma_3} ; \frac{r_4}{\sqrt{r_3^2 - r_4^2}} e^{i\gamma_4}\right],$$

where

$$r_1 = \sqrt{x^2(t+1)^2 - (x-t)^2} ,$$

$$r_2 = \sqrt{x^2(t-1)^2 + (x+t)^2} ,$$

$$r_3 = \sqrt{x^2(s+1)^2 + (x+s)^2} ,$$

$$r_4 = \sqrt{x^2(s+1)^2 + (x-s)^2} ,$$

$$\tan\gamma_1 = \frac{x-t}{x(t+1)} , \qquad \tan\gamma_2 = \frac{x+t}{x(t-1)} ,$$

$$\tan\gamma_3 = \frac{x+s}{x(s+1)^2} , \qquad \tan\gamma_4 = \frac{x-s}{x(s+1)} .$$

Riemann surface $S = \Delta/\tilde{\Gamma}$ be a torus with one hole. For any element of the fundamental group $\psi \in \pi_1(S,p)$ there exists a unique (not necessarily simple) geodesic arc from p to a point p , which represents the element ψ . So $\tilde{\Gamma}$ isomorphic $\pi_1(S,p)$ then the generators $\tilde{A}$ and $\tilde{B}$ correspond the geodesic arcs α and β , respectively.

Theorem 3.2 Let the length of the geodesic α is equal to ρ , the length of the geodesic β is equal to ρ and the angle between geodesics α and β at the intersection point p is equal to θ .

Then we have the following formula:

(a) $$2\cosh\frac{\rho}{2} = (t+1)\sqrt{\frac{x}{t(x-1)}} ;$$

(b) $2\cosh\frac{\bar{\rho}}{2} = (s+1)\sqrt{\frac{x}{s}}$;

(c) $\cos\theta = \frac{x(1-ts)-(x-2)(t-s)}{\sqrt{x(t-1)^2+4t}\,.\sqrt{x(s+1)^2-4s}}$.

Proof: It is sufficient to perform all calculations in the complex upper half - plane H, so $\rho, \bar{\rho}, \theta$ are invariant under the transformation φ, and we use the following known properties of the hyperbolic metric.

let S be Riemann surface with universal covering H and Γ is a Fuchsian model of the surface S, acting in H. Suppose

$$T(z) = \frac{az+b}{cz+d} \quad , \quad a, b, c, d \in R, \quad ad - bc = 1 \; ,$$

be a hyperbolic element of group Γ and let L_T be a closed geodesic on S, representing the element T. Then the hyperbolic length $\rho(L_T)$ satisfies the equation

$$2\cosh\frac{\rho(L_T)}{2} = |a+d|.$$

We now consider the mapping of the parameters

$$f : (x, s, t) \to (\rho, \bar{\rho}, \theta),$$

defined by the formulas (a), (b) and (c) of Theorem 3.2.

Proposition 3.3 The mapping f is locally invertible mapping at almost everywhere in the space of parameters ($\rho, \bar{\rho}, \theta$).

Proof: We know that the Jacobian matrix of mapping f vanish on some algebraic curve K, which is defined by the equation:

$$x^2(s-1)(t^2-s^2)(t-1)+x(x-2)(s-1)(s+1)(1-st)(t-1)+$$
$$+2x(s-1)(s+1)^2t(t+1)-8s(s-1)t(t+1)2x(x-1)s(s+1)(t-1)^2-8(x-1)s(s+1)t(t-1)=0$$

Then by Sard's theorem $f(K)$ is a set of measure zero. Outside the set $f(K)$ the mapping f is locally invertible by the theorem of the inverse mapping ∎

References

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البحث في بارمترات فضاء تيخميلر"

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ملخص البحث : أن الهدف الرئيسي من هذا البحث هو ايجاد اتصال بين بارمترات الطول والزاوية من جهه و سطوح ريمان في فضاء تيخميلر التبولوجي من النوع (1,1) من جهه أخرى .