Modified Adomian Decomposition Method for Solving the Fractional Heat Equation in the Caputo form

Eman. I. Gorial
Reem Monther Kubba
Department of Mathematics, Ibn Al–Haitham College Education, Baghdad University

Abstract

In this paper, analytical solution of the fractional heat equation has been presented. The algorithm for the analytical solution for this equation is based on modified Adomian's decomposition method. The fractional derivative is described in Caputo's sense. The analytical method has been applied to solve a practical example and the results have been compared with exact solution.

Introduction

The fractional calculus is used in many fields of science and engineering [1, 2, 3, 4]. The solution of differential equation containing fractional derivatives is much involved and its classic a analytic methods are mainly integral transforms, such as Laplace transform, Fourier transform, Mellin transform, etc.[1,2,5]

In recent years Adomiande composition method is applied to solving fractional differential equations. This method efficiently works for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations, and even for stochastic systems [6] as well. By using this method Saha Ray et al [7, 8, 9, 10] solved linear differential equations containing fractional derivative of order 1/2 or 3/2, and nonlinear differential equation containing fractional derivative of order1/2.

In this paper, we consider the fractional heat equations of the form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + q(x,t) \qquad (1)$$

on a finite domain $x_L < x < x_R$ with $1 < \alpha < 2$. We also assume an initial condition $u(x,0) = f(x)$ for $x_L < x < x_R$ and boundary conditions of the form $u(x_L,t) = 0$ and $u(x_R,t) = g_R(t)$ eq.(1) uses a caputo fractional derivative of order α.

In the present work, we apply the modified Adomian's Decomposition method for solving eq.(1) and compare the results with exact solution. The paper is organized as follows. In section 2, mathematical aspects. In section 3, basic idea of modified Adomian's decomposition method. In section 4, the fractional heat equation and its solution by modified Adomian's decomposition method. In section 5, numerical example is solved using the modified Adomian's decomposition method. Finally, we present conclusion about solution the fractional heat equation in section 6.

2. Mathematical Aspects

The mathematical definition of fractional calculus has been the subject of several different approaches [12, 13]. The Caputo fractional derivative operator D^{α} of order α is defined in the following form:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \qquad \alpha > 0,$$

where $m-1 < \alpha < m, m \in \mathrm{N}, x > 0$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x),$$

where λ and μ are constants.

For the Caputo's derivative we have:

$$D^{\alpha}_{L+}(x-L)^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(x-L)^{n-\alpha}$$

and

$$D^{\alpha}_{R-}(R-x)^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(R-x)^{n-\alpha}$$

3. Basic Idea of Modified Adomian's Decomposition Method

Consider a general nonlinear equation, [14]

$$Lu + R(u) + F(u) = g(t) \qquad (2)$$

where L is the operator of the highest-ordered derivatives with respect to t and R is the remainder of the linear operator. The nonlinear term is represented by F(u). Thus we get

$$Lu = g(t) - R(u) - F(u) \qquad (3)$$

The inverse L^{-1} is assumed an integral operator given by

$$L^{-1} = \int_0^t (\cdot) dt \, ,$$

The operating with the operator L^{-1} on both sides of Equation (3) we have

$$u = f + L^{-1}(g(t) - R(u) - F(u)) \qquad (4)$$

where f is the solution of homogeneous equation

$$Lu = 0 \qquad (5)$$

involving the constants of integration. The integration constants involved in the solution of homogeneous Equation (5) are to be determined by the initial or boundary condition according as the problem is initial-value problem or boundary-value problem.

The ADM assumes that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

and the nonlinear operator F(u) can be decomposed by an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n$$

where $u_n(x,t)$ will be determined recurrently, and A_n are the so-called polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2,\ldots,$$

But the modified decomposition method was introduced by Wazwaz [15]. This method is based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x)$$

We apply this decomposition when the function f consists of several parts and can be decomposed into two different parts. In this case, f is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part f_1 is important.

For the method to be more efficient, we select f_1 as one term of f or at least a number of terms if possible and f_2 consists of the remaining terms of f.

4. The Modified Adomian's Decomposition Method for Solving the Fractional Heat Equations

We adopt modified decomposition method for solving Equation (1). In the light of this method we assume that

$$u = \sum_{n=0}^{\infty} u_n$$

Now, Equation (1) can be rewritten as

$$Lu(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + q(x,t)$$

where $L_t = \frac{\partial}{\partial t}$ which is an easily invertible linear operator, $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$ is the Caputo derivative of order α.

Therefore, we can write,

$$u(x,t) = u(x,0) + L_t^{-1}\left(\frac{\partial^{\alpha}\left(\sum_{n=0}^{\infty} u_n\right)}{\partial x^{\alpha}} \right) + L^{-1}\left(q(x,t)\right) \qquad (6)$$

Then the modified decomposition method (MDM) recursive scheme is as follows

$$u_0 = f_1$$

$$u_1 = f_2 + L_t^{-1}\left(\frac{\partial^{\alpha} u_0}{\partial x^{\alpha}} \right)$$

$$u_{n+1} = L_t^{-1}\left(\frac{\partial^{\alpha}\left(\sum_{n=0}^{\infty} u_n\right)}{\partial x^{\alpha}} \right), n \geq 0$$

5. Numerical Application:

In this section, we apply modified decomposition method for finding the analytical solution of fractional heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t)$$

with the source function $q(x,t) = 5.445622 e^t x^{6/5} + e^t x^3$, subject to the initial condition u (x,0) = x^3, $0 < x < 1$, and the boundary conditions u (0,t) = 0, t >0, u (1, t) = e^t, t > 0 . Note that the exact solution to this problem is: u(x,t) = $x^3 e^t$. Table 1 shows the analytical solutions for fractional heat equation obtained for different values and comparison between exact solution and analytical solution.

Table1. Comparison between exact solution and analytical solution when $\alpha = 1.8$ for fractional heat equation

x	t	Exact Solution	Modified Adomian Decomposition Method	|uex-uMADM|
0	1	0.000000	0.000000	0.0000000
0.1	1	0.002718	0.002718	0.0000000
0.2	1	0.022000	0.022000	0.0000000
0.3	1	0.073000	0.073000	0.0000000
0.4	1	0.174000	0.174000	0.0000000
0.5	1	0.340000	0.340000	0.0000000
0.6	1	0.587000	0.587000	0.0000000
0.7	1	0.932000	0.932000	0.0000000
0.8	1	1.392000	1.392000	0.0000000
0.9	1	1.982000	1.982000	0.0000000
1	1	2.718000	2.718000	0.0000000
0	2	0.000000	0.000000	0.0000000
0.1	2	0.007389	0.007389	0.0000000
0.2	2	0.059000	0.059000	0.0000000
0.3	2	0.200000	0.200000	0.0000000
0.4	2	0.473000	0.473000	0.0000000
0.5	2	0.924000	0.924000	0.0000000
0.6	2	1.596000	1.596000	0.0000000
0.7	2	2.534000	2.534000	0.0000000
0.8	2	3.783000	3.783000	0.0000000
0.9	2	5.387000	5.387000	0.0000000
1	2	7.389000	7.389000	0.0000000
0	3	0.000000	0.000000	0.0000000
0.1	3	0.020000	0.020000	0.0000000
0.2	3	0.161000	0.161000	0.0000000
0.3	3	0.542000	0.542000	0.0000000
0.4	3	1.285000	1.285000	0.0000000
0.5	3	2.511000	2.511000	0.0000000
0.6	3	4.338000	4.338000	0.0000000
0.7	3	6.889000	6.889000	0.0000000
0.8	3	10.28400	10.28400	0.0000000
0.9	3	14.64200	14.64200	0.0000000
1	3	20.08600	20.08600	0.0000000

6. Conclusion

1- Analytical solutions for fractional heat equation obtained for different values of a using the modified decomposition method has been described and demonstrated.
2- It is clear that the modified decomposition method is in high agreement with the exact solutions.

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طريقة تحليل ادوميان المعدلة لحلّ معادلة الحرارة الكسرية بصيغة كابوتو

ايمان ايشو كوريال و ريم منذر كبة

قسم الرياضيات -كلية التربية-ابن الهيثم-جامعة بغداد

الخلاصة

في هذا البحث ، تم تقديم الحل التحليلي للمعادلة الحرارية الكسرية . وان خوارزمية الحل التحليلي قائمة على اساس طريقة التحلل ادوميان المعدلة .المشتقة الكسرية هو موصوف بتعريف كابوتو وقد تم تطبيق الطريقة التحليلية في حل مثال عملي وتمت مقارنة النتائج مع الحل المضبوط.