Programming for Compute Generalized Inverse Using Biorthogonalization Method

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Abstract:

The aim of this paper is to construct a program based on biorthogonalization method for computing generalized inverse of real matrices (Moore-Penrose inverse matrices) [1, 2]. The new program has been illustrated by solving different examples.

The accuracy of the results proved that the new program has very high efficiently.

1- Introduction:

The generalized inverse for every matrix A is denoted as A^+,where $X=A^+$ is unique solution to the Penrose defining relations[1]

$AXA=A$, $XAX=X$,$(AX)^T=AX$, $(XA)^T=XA$ (where T denotes the transpose of matrix).We will use relations $A^+A = I$ ($A A^+ = I$) if A has full column (row) rank so that $A^+=A^{-1}$ if A is non singular. The following some properties of generalized inverse [3, 4]

a) $(A^+)^+=A$
b) $(A^T)^+=(A^+)^T$ and $(AA^T)^+=A^+(A^+)^T$
c) $(\lambda A)^+= \lambda^{-1}A^+$ $(\lambda \neq 0)$
d) $|A| \neq 0 \rightarrow A^+=A^{-1}$

2- Biorthogonalization method :[5 , 6]

In this section we will describe the process to compute the generalized inverse by Biorthogonalization method.

Definition 2.1: Two sets of n-vectors $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ are said to form a biorthogonal system if and only if the inner products (v_i, u_j) satisfy the relation

$$(v_i, u_j) = \delta_{ij} \quad (i, j= 1, 2, \dots, n) \qquad \dots (1)$$

Where

$\delta_{ij}= 0$ when $i\neq j$ and $\delta_{ii}=1$.

To constitute a biorthogonal system each of the two sets must be linearly independent. To show this, suppose that there exist constants $k_1, k_2, \dots, k_n$ not all zero, such that

$$k_1u_1 + k_2u_2 + \dots + k_nu_n = 0$$

multiplying this equation by v_i^t we obtain

$$k_1 v_1^t u_1 + k_2 v_2^t u_2 + \dots + k_n v_i^t u_n = 0 \quad (i = 1, 2, \dots, n)$$

Since the sets $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ constitute a biorthogonal system, this system of equations reduces to

$$k_i = 0 \quad (i=1,2,\dots,n).$$

It follows that $\{u_1, u_2, \dots, u_n\}$ is linearly independent. A similar argument shows that $\{v_1, v_2, \dots, v_n\}$ must be linearly independent. It is also true that $n \le m$, for otherwise each of sets would be linearly independent.

Now we describe biorthogonal systems in term of matrices. The vectors $u_1, u_2, \dots, u_n$ can be considered to be the column vectors of a matrix U and $v_1, v_2, \dots, v_n$ the row vectors of a matrix V.

In [5,6] a process called "Biorthogonalization" for inverting matrices is described. This process can be extended and modified to compute the generalized inverse of rectangular matrices.

Given the matrix $U=[u_1, u_2, \dots, u_n]$ we wish to find a matrix $V=[v_1, v_2, \dots, v_n]^t$ such that the sets $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ constitute a biorthogonal system. The construction will be one in which we begin with an initial estimate $V^{(0)}=[v_1^{(0)}, v_2^{(0)}, \dots, v_n^{(0)}]$ of V and by series of n transformations on $V^{(0)}$ obtain the matrix V. The procedure for constructing the matrix V will be an iterative one in which we transform the vectors $v_1^{(0)}, v_2^{(0)}, \dots, v_n^{(0)}$ into vectors $v_1, v_2, \dots, v_n$ in such a manner that the sets $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ form a biorthogonal system. This process will require (n) iterations. In the first iteration we construct the vectors

$$v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}$$

so that

$$(v_1^{(1)}, u_1) = 1$$

and

$$(v_j^{(1)}, u_1) = 0 \qquad (j=2, 3, \dots, n),$$

To begin we set

$$c_{j1} = (v_j^{(0)}, u_1) \quad (j=1, 2, \dots, n),$$

$$c_1 = \frac{1}{c_{11}} \quad (\text{assuming that } c_{11} \neq 0)$$

$$v_1^{(1)} = c_1 v_1^{(0)}$$

$$v_j^{(1)} = v_j^{(0)} - c_{j1} v_1^{(1)}$$

Assuming that (k-1) iteration have been completed and the vectors $v_1^{(k-1)}, v_2^{(k-1)}, \dots, v_n^{(k-1)}$ have been obtained, we set

$$c_{jk} = (v_j^{(k-1)}, u_k) \quad (j=1, 2, \dots, n) \quad \dots(2)$$

$$c_k = \frac{1}{c_{kk}} \quad (\text{assuming that } c_{kk} \neq 0) \quad \ldots(3)$$

Assuming that $c_{kk} \neq 0$, we construct the set $\{v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)}\}$ using the formulas

$$v_k^{(k)} = c_k\, v_k^{(k-1)} \quad \ldots(4)$$
$$v_j^{(k)} = v_j^{(k-1)} - c_{jk}\, v_k^{(k)} , \; (j\neq k, j=1, 2, \ldots, n) \quad \ldots(5)$$

We note that when k=1 these formulas agree with those used in constructing $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$. After n iterations we will have constructed the n-vectors $v_1^{(n)}, v_2^{(n)}, \ldots, v_n^{(n)}$.

The following theorem is an obvious consequence of the above discussion.

Theorem 2.2 [5]: Having completed the first (k-1) iterations, the k-th iteration can be completed if and only if $c_{kk} \neq 0$.

Theorem 2.3 [5]: Assume that vectors $v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)}$ have been completed by the process described above. Then

$$(v_k^{(k)}, u_k) = 1 \quad \ldots(6)$$
$$(v_j^{(k)}, u_k) = 0 \; , (j\neq k, j=1, 2, \ldots, n) \quad \ldots(7)$$

Proof:

Since

$$v_k^{(k)} = c_k\, v_k^{(k-1)}$$

We have

$$(v_k^{(k)}, u_k) = c_k\, (v_k^{(k-1)}, u_k)$$

And hence from (2) and (3) we have

$$(v_k^{(k)}, u_k) = \frac{1}{c_{kk}} c_{kk} = 1$$

For $j\neq k$,

Since $v_j^{(k)} = v_j^{(k-1)} - c_{jk}\, v_k^{(k)}$

From (2) and (6) we have

$$(v_j^{(k)}, u_k) = c_{jk} - c_{jk}.1=0$$

To compute the generalized inverse of a matrix, the above method is modified by adding rows to the original matrix which are orthogonal to original rows of the matrix and which raise the rank of the matrix to its column dimension. This can be done depends on the rows of the matrix. It may be possible to determined these rows by inspection ,especially if most of the components of the original matrix are zero (sparse matrix). If it is not possible to determine these rows by inspection, then the Gram-Schmidt process can be used to determine the additional rows. Let k be the number of the rows that are added. Then the above method is now applied to the resulting matrix. The generalized inverse of the original matrix is obtained by deleting the last k columns from $V^{(n)}$.

3- Examples

In the following numerical example we will illustrate the method of biorthogonalization.

Example (3.1): Consider the matrix A (rectangular matrix)

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

The matrix is of rank 1. We add two rows to A which are orthogonal to all other rows so as to obtain the matrix

$$U = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

of rank 3 to which the method is applicable with $V^{(0)} = U^T$

$$V^{(0)} = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \end{pmatrix}$$

In the first iteration we put k=1 into formulas (2) to (5). we begin with k=1 and j=1. Then

$$C_{11} = (v_1^{(0)}, u_1)$$

$$C_{11} = (1, 2, -2, 1)\begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} = 10 , \quad c_1 = \frac{1}{C_{11}} = \frac{1}{10}$$

$$V_1^{(1)} = c_1 v_1^{(0)} = \frac{1}{10} (1, 2 , -2 , 1) = (1/10, \ 1/5 , (-1)/5, \ 1/10) .$$

When j=2 we have

$$C_{21} = (v_2^{(0)}, u_1)$$

$$C_{21} = (1, 2, 0, 1)\begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} = 6$$

$$V_2^{(1)} = v_2^{(0)} - c_{21} v_k^{(1)}$$

$$= (1, 2 , 0 , 1) - 6 (1/10, \ 1/5 , (-1)/5, \ 1/10)$$

$$= (2/5, \ 4/5 , 6/5, \ 2/5) .$$

When j=3 we have

$$C_{31} = (v_3^{(0)}, u_1)$$

$$C_{31} = (2, 4 , 1 , -1)\begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} = 7$$

$$V_3^{(1)} = v_3^{(0)} - c_{31} v_1^{(1)}$$

$$= (2, 4, 1, -1) - 7 (1/10. \ 1/5 . (-1)/5, \ 1/10)$$

$$V_3^{(1)} = (13/10. \ 13/5 . 12/5, \ (-17)/10).$$

Hence

$$V^{(1)} = \begin{pmatrix} \frac{1}{10} & \frac{1}{5} & \frac{-1}{5} & \frac{1}{10} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{2}{5} \\ \frac{13}{10} & \frac{13}{5} & \frac{12}{5} & \frac{-17}{10} \end{pmatrix}$$

When k=2 and j=2 we have

$C_{22}= (v_2^{(1)}, u_2) = \frac{12}{5}$, $C_2= \frac{5}{12}$

$V_2^{(2)} = c_2\, v_2^{(1)} = (1/6, \quad 1/3 , 1/2, \quad 1/6)$.

When j=1 we have

$C_{12}= (v_1^{(1)}, u_2) = \frac{3}{5}$

$V_1^{(2)} = v_1^{(1)} - c_{12}\, v_2^{(2)} = (0, 0 , (-1)/2, 0)$

When j=3 we have

$C_{32}= (v_3^{(1)}, u_2) = \frac{24}{5}$

$V_3^{(2)} = v_3^{(1)} - c_{32}\, v_2^{(2)} = (\frac{1}{2}, 1 , 0 , (-5)/2)$

Therefore

$$V^{(2)} = \begin{pmatrix} 0 & 0 & \frac{-1}{2} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & 1 & 0 & \frac{-5}{2} \end{pmatrix}$$

When k =3 and j =3 we have

$C_{33}= (v_3^{(2)}, u_3) = \frac{15}{2}$, $C_3= \frac{2}{15}$

$v_3^{(3)} = C_3\, v_3^{(2)} =(\frac{1}{15} , \frac{2}{15} , 0, \frac{-5}{15})$.

when j=1 we have

$C_{13}= (v_1^{(2)}, u_3) = \frac{-1}{2}$

$V_1^{(3)} = v_1^{(2)} - c_{13}\, v_3^{(3)} = (\frac{1}{30} , \frac{1}{15} , \frac{-1}{2} , 1/6)$.

When j=2 we have

$C_{23}= (v_2^{(2)}, u_3) = 2$

$V_2^{(3)} = v_2^{(2)} - c_{23}\, v_3^{(3)} = (\frac{1}{30} , \frac{1}{15} , \frac{1}{2} , 5/6)$.

Hence

$$V^{(3)}= \begin{pmatrix} \frac{1}{30} & \frac{1}{15} & \frac{-1}{2} & \frac{1}{6} \\ \frac{1}{30} & \frac{1}{15} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{15} & \frac{2}{15} & 0 & \frac{-5}{15} \end{pmatrix} .$$

The matrix $V^{(3)}$ is generalized inverse of U, Moreover the matrix

$$N=\begin{pmatrix} \frac{1}{30} & \frac{1}{15} \\ \frac{1}{30} & \frac{1}{15} \\ \frac{1}{15} & \frac{2}{15} \end{pmatrix}$$ is inverse of A after deleting the last two columns.

4- Program {computing of generalized inverse of a matrix A by using Biorthogonalization method}

```
a = input ('input matrix');
r = rank (a); {computing the rank of A}
[l, b] = size (a);
if r < b  {inputting the orthogonal rows}
   m = b – r ;
   for I = 1: m
      n = input ('input orthogonal row')
      a = [a;n];
   end
end
c (b, b) = zeros;
v = a'; {computing the inverse}
u = v;
for  k = 1: b
   c ( k, k) = v (k , :) * (u (k ,: ) )';
   d ( k) = 1 / c( k , k) ;
   v ( k, :) = d( k) * v( k , :);
   for  j = 1 : b
      if j ~ = k
         c ( j , k) = v( j, :) * ( u (k,:))';
         v (j,:) = v ( j , :) – c ( j, k) * (v(k,:))
      end
   end
end
```

4.1 In the following section we use the program to find generalized inverse of a matrix A.

<u>Example 4.1</u>: Consider the matrix $A=\begin{pmatrix} 5 & 3 & 8 \\ 2 & 9 & 1 \\ 1 & 5 & 7 \end{pmatrix}$ (3×3)(non singular matrix)

Rank= 3

$$v^{(1)} = \begin{pmatrix} \frac{1}{6} & \frac{1}{15} & \frac{1}{30} \\ -\frac{10}{3} & \frac{97}{15} & \frac{56}{15} \\ 8 & 1 & 7 \end{pmatrix}$$

$$v^{(2)} = \begin{pmatrix} \frac{461}{2006} & -\frac{56}{1003} & -\frac{75}{2006} \\ -\frac{50}{1003} & \frac{97}{1003} & \frac{56}{1003} \\ \frac{259}{2006} & -\frac{1656}{583} & \frac{569}{113} \end{pmatrix}$$

$$v^{(3)} = \begin{pmatrix} \frac{58}{259} & \frac{19}{259} & -\frac{69}{259} \\ -\frac{13}{259} & \frac{27}{259} & \frac{11}{259} \\ \frac{1}{259} & -\frac{22}{259} & \frac{39}{259} \end{pmatrix}$$

$V^{(3)}$ is the inverse of $A=A^{+}=A^{-1}$

Example 4.2: Consider the matrix $A= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ (4×4)(singular matrix)

Rank= 3

Input orthogonal row [1 1 1 1]

$$V^{(1)}= \begin{pmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ -1 & 1 & 0 & 0 & 1 \\ -\frac{1}{3} & -1 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$V^{(2)}= \begin{pmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ -\frac{1}{3} & -1 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$V^{(3)}=\begin{pmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ -\frac{1}{3} & -1 & 1 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} & -1 & 1 & \frac{2}{3} \end{pmatrix}$$

$$V^{(3)}=\begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{3} & -\frac{1}{3} & -1 & 1 & \frac{2}{3} \end{pmatrix}$$

$$V^{(4)}=\begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

$$V^{(5)}=\begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \\ -\frac{3}{8} & \frac{3}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

The following matrix is the inverse of A by deleting the last column from $V^{(5)}$

$$\begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \end{pmatrix} = A^+$$

Example 4.3: [7] Consider the matrix $F_7 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \\ 7 & 8 & 9 \end{pmatrix}$ (7×3 rectangular matrix)

Rank = 2

Input orthogonal row [-1 2 -1]

$V^{(1)}=$

$-\frac{1}{141}$@ $\frac{448}{141}$@$\frac{56}{\text{[illegible]}}$

$V^{(2)}=$

(■(−125/1631 @116/1631@306/233)■(−12/233@13/233@ 228/233)■(−43/1631@66/1631 @ 150/233)■

$$V^{(3)}= \begin{pmatrix} -\frac{3}{8} & -\frac{23}{84} & -\frac{29}{168} & -\frac{1}{14} & \frac{5}{168} & \frac{11}{84} & \frac{13}{56} & -\frac{1}{6} \\ -\frac{1}{28} & -\frac{1}{42} & -\frac{1}{84} & 0 & \frac{1}{84} & \frac{1}{42} & \frac{1}{28} & \frac{1}{3} \\ \frac{17}{56} & \frac{19}{84} & \frac{25}{168} & \frac{1}{14} & -\frac{1}{168} & -\frac{1}{12} & -\frac{9}{56} & -\frac{1}{6} \end{pmatrix}$$

$$F_7^{+}= \begin{pmatrix} -\frac{3}{8} & -\frac{23}{84} & -\frac{29}{168} & -\frac{1}{14} & \frac{5}{168} & \frac{11}{84} & \frac{13}{56} \\ -\frac{1}{28} & -\frac{1}{42} & -\frac{1}{84} & 0 & \frac{1}{84} & \frac{1}{42} & \frac{1}{28} \\ \frac{17}{56} & \frac{19}{84} & \frac{25}{168} & \frac{1}{14} & -\frac{1}{168} & -\frac{1}{12} & -\frac{9}{56} \end{pmatrix}$$

$V^{(3)}$ is the inverse of F_7^{+} by deleting the last column from $V^{(3)}$

5- Conclusions

In this paper , we give a new program to find the inverse to any real matrix (singular, nonsingular and rectangular depending on Biorthogonalization method).We have illustrated some examples by applying the new program, and a comparison with other methods has been made [5,6].The results that obtained are identical with those obtained by other methods.

6- References

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الخلاصة:

الغرض من هذا البحث بناء برنامج جديد يعتمد على طريقة الثنائية التعامد (Biorthogonalization Method) لحساب المعكوس العام للمصفوفات ذات عناصر حقيقية (معكوس بن-روس) لدراسة كفاءة البرنامج فقد اعطيت امثلة توضيحية وبينت دقة النتائج على كفاءة البرنامج الجديد.