On Quasi rgα-open and Quasi rgα -closed Functions In Topological Spaces

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Abstract:

In this paper, we introduce and study new types of open and closed functions called **(quasi rgα-open functions**) and (**quasi rgα –closed functions**) . Also, we study the Characterizations and basic properties of quasi **rgα**-open functions and quasi **rgα**-closed functions .

Key words:
rgα -open set, rgα -closed set, quasi rgα -open function, quasi rgα -closed function, rgα -irresolute function and contra rgα-irresolute function.

Introduction

The concept of rgα-open sets was first introduced and studied by A.vadivel [1].Also, The concept of rgα-open (resp. rgα-closed) functions was introduced by [5] .The purpose of this paper is to give a new type of open and closed functions called quasi rgα-open functions and quasi rgα-closed functions . Also , we study the relation between the quasi rgα-open (resp. quasi rgα-closed) functions and each of open (resp . closed) functions , rgα-open (resp. rgα-closed) functions and rgα*-open (resp. rgα*-closed) functions . Moreover, we study the characterizations and basic properties of quasi rgα- open (resp. quasi rgα- closed) functions .

1. Preliminaries

(1.1)Definition:[1].

A subset A of a topological space X is said to be

1) Regular α-open set (rα-open) if there is a regular open set U such that $U \subset A \subset \alpha cl(U)$
2) **rgα-closed** if $\alpha cl(A) \subset U$ whenever $A \subset U$ and U is regular α-open in X.

The complement of an **rgα-closed** set is said to be **rgα-open**.

The class of all **rgα-closed** (resp. **rgα-open**) subsets of X is denoted by **RGαC(X,τ)** (resp. **RGαO(X,τ)**) .

(1.2)Definition [2]:

A subset A of a topological space X is said to be an **rgα**-neighborhood of a point x in X if there exists an **rgα** -open set U in X such that $x \in U \subset A$.

(1.3)Definition [3]:

Let X be a topological space and $A \subset X$ .Then:-

1) The **rgα**-closure of A ,denoted by **rgα**–cl(A) is the intersection of all **rgα** -closed sets in X which contains A .

2) The **rgα** -interior of A, denoted by **rgα** – int(A)is the union of all **rgα**–open sets in X which are contained in A .

(1.4)Remarks [2]:

Let X be a topological space and $A \subset X$ .Then:-

1)A ⊂ rgα-cl(A)

2)If A is rgα-closed in X then A= rgα-cl(A).

3) rgα-int(A) ⊂A.

4)If A is rgα -open then A= rgα-int(A).

(1.5)Definition [3]:

A function $f: X \to Y$ from a topological space X into a topological space Y is said to be ***rgα* -continuous** if $f^{-1}(V)$ is ***rgα*** -open set in X for every open set V in Y .

(1.6)Theorem [3]:

A function $f: X \to Y$ from a topological space X into a topological space Y is ***rgα*--continuous** iff . $f^{-1}(V)$ is ***rgα*** -closed set in X for every closed set V in Y

(1.7)Definition [3]:

A function $f: X \to Y$ from a topological space X into a topological space Y is said to be ***rgα*-irresolute** if $f^{-1}(V)$ is ***rgα*** -open set in X for every ***rgα***-open set V in Y .

(1.8)Theorem:

A function $f: X \to Y$ from a topological space X into a topological space Y is ***rgα*-irresolute** iff. $f^{-1}(V)$ is ***rgα***-closed set in X for every ***rgα***-closed set V in Y .

Proof: It is obvious.

(1.9)Definition [4]:

A function $f: X \to Y$ from a topological space X into a topological space Y is said to be **contra *rgα*-irresolute** if $f^{-1}(V)$ is ***rgα***-closed set in X for every ***rgα***-open set V in Y .

(1.10)Definition [5]:

A function $f:X\rightarrow Y$ from a topological space X into a topological space Y is said to be ***rgα*-open** (resp. ***rgα*-closed**) if the image of every open (resp. closed) subset of X is an ***rgα***-open (resp. ***rgα***-closed) set in Y.

(1.11)Definition [5]:

A function $f:X\rightarrow Y$ from a topological space X into a topological space Y is said to be ***rgα**-open** (resp. ***rgα**-closed**) if the image of every ***rgα***-open (resp. ***rgα***-closed) subset of X is an ***rgα***-open (resp. ***rgα***-closed) set in Y.

2. Quasi *rgα*-open Functions

We now introduce the following definition:

(2.1)Definition:

Let X and Y be topological spaces. A function $f:X\rightarrow Y$ is said to be **quasi *rgα*-open** if the image of every ***rgα***-open set in X is open in Y.

(2.2)Theorem:

Every quasi ***rgα***-open function is open as well as ***rgα***-open .

Proof:

Let U is open in X since every open is rgα-open [1] thus U is rgα-open ,since f is quasi rgα- open thus f(U)is open so f is open

To prove f is rgα-open let U is open in X since f is open thus f(U) is open,since every open is rgα-open thus f(U)is rgα-open so f is rgα-open .

(2.3)Remark:

The converse of (2.2) may not be true in general. Consider the following example .

Example:

Let $X = Y = \{a,b,c\}$ & $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$.

$\Rightarrow$ $RG\alpha O(X,\tau) = \{X,\phi,\{a\},\{b\},\{c\},\{a,b\}\}$.

Let $f:(X,\tau)\rightarrow(Y,\tau)$ be a function defined by : $f(a) = a$, $f(b) = b$ & $f(c) = c$.

It is clear that f is rgα - open as well as open, but f is not quasi rgα -open, since $\{c\}$ is rgα-open in (X,τ) ,but $f(\{c\}) = \{c\}$ is not open in (Y,τ) .

(2.4)Theorem:

Every quasi rgα-open function is ***rgα****-open .

Proof: let U is rgα-open ,since f is qusi rgα-open thus f(U) is open ,since every open set is rgα-open [1] thus f is rgα*-open .

(2.5)Remark:

The converse of (2.4) may not be true in general. Consider the following example.

Example:

Let $X = Y = \{a,b,c\}$ & $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$. $RG\alpha O(X,\tau) = \{X,\phi,\{a\},\{c\},\{b\},\{a,b\}\}$

Let $f:(X,\tau) \rightarrow (Y.\tau)$ be a function defined by : $f(a) = b$, $f(b) = a$ & $f(c) = c$.

It is clear that f is rgα*-open ,but f is not quasi rgα-open, since $\{c\}$ is rgα-open in (X,τ), but $f(\{c\})=\{c\}$ is not open in (Y,τ).

Thus we have the following diagram:

quasi *rgα*-open function	$\Rightarrow$	***rgα****-open function**
$\Downarrow$		$\Downarrow$
open function	$\Rightarrow$	***rgα*-open function**

(2.6)Theorem:

Let $f:X\to Y$ be a function from a topological space X to a topological space Y. Then f is quasi ***rgα***-open if for each $x\in X$ and each ***rgα***-neighborhood U of x in X, there exists a neighborhood V of f(X) in Y such that $V\subset f(U)$.

Proof:

Let U be an arbitrary ***rgα*** - open set in X. Then for each $y\in f(U)$ there exists $x\in U$ such that $f(x)=y$. By hypothesis there exists a neighborhood V_y of y in Y such that $V_y\subset f(U)$.

Since V_y is a neighborhood of y ,then there exists an open set W_y in Y such that $y\in W_y\subset V_y$.

Thus $f(U)=\bigcup_{y\in f(U)} W_y$ which is an open set in Y . This implies that f is quasi ***rgα***-open function .

(2.7)Theorem:

Let X and Y be topological spaces. A function $f:X\to Y$ is quasi rgα–open iff. for any subset

B of Y and for any rgα -closed set F of X containing $f^{-1}(B)$, there exists a closed set G of Y containing B such that $f^{-1}(G)\subset F$.

Proof: $\Rightarrow$

Suppose that f is quasi rgα -open. Let $B\subset Y$ and F be an rgα -closed subset of X such that $f^{-1}(B)\subset F$. Now, put $G=Y-f(X-F)$.

Since $f^{-1}(B)\subset F\Rightarrow X-F\subset f^{-1}(B^c)\Rightarrow f(X-F)\subset f(f^{-1}(B^c))\subset B^c$

$\Rightarrow B\subset Y-f(X-F)\ \Rightarrow B\subset G$. Since f is quasi rgα –open , then G is a closed subset of Y. Moreover, we have $f^{-1}(G)\subset F$.

Conversely, let U be an rgα -open set in X .To prove that $f(U)$ is an open set in Y Put $B=Y-f(U)$, then $X-U$ is an rgα -closed set in X such that $f^{-1}(B)\subset X-U$.

By hypothesis, there exists a closed subset F of Y such that $B\subset F$ and $f^{-1}(F)\subset X-U$.

Hence, we obtain $f(U)\subset Y-F$. On the other hand ,since $B\subset F\Rightarrow Y-F\subset Y-B=f(U)\ \Rightarrow Y-F\subset f(U)$. Thus $f(U)=Y-F$ which is open and hence f is a quasi rgα -open function .

(2.8)Theorem:

Let X and Y be two topological spaces. A function $f: X \to Y$ is quasi rgα - open iff $f^{-1}(cl(B)) \subset rg\alpha\text{-}cl(f^{-1}B))$ for every subset B of Y .

Proof: $\Rightarrow$

Suppose that f is quasi rgα-open. To prove that $f^{-1}(cl(B)) \subset rg\alpha - cl(f^{-1}B))$ for every subset B of Y .Since $f^{-1}(B) \subset rg\alpha - cl(f^{-1}B))$ for any subset B of Y ,then by (2.7) there exists a closed set F in Y such that $B \subset F$ and $f^{-1}(F) \subset rg\alpha - cl(f^{-1}B))$,Since $B \subset F \Rightarrow cl(B) \subset cl(F) = F$.

Therefore, we obtain $f^{-1}(cl(B)) \subset f^{-1}(F) \subset rg\alpha - cl(f^{-1}B))$.

Thus $f^{-1}(cl(B)) \subset rg\alpha - cl(f^{-1}B))$ for every subset B of Y .

Conversely, let $B \subset Y$ and F be an rgα-closed subset of X such that $f^{-1}(B) \subset F$. then we have $B \subset cl(B)$ and $f^{-1}(cl(B))=f^{-1}(cl(B)) \subset rg\alpha -cl(f^{-1}B)) \subset rg\alpha -cl(F)=F$ (By(1.4)no.2)

Then by theorem (2.7) f is a quasi rgα-open function .

However the following theorem holds. The proof is easy and hence omitted.

(2.9)Theorem:

Let X, Y, Z be three topological spaces ,and $f: X \to Y, g: Y \to Z$ be two functions. Then:-

1) If f and g are quasi rgα-open ,then $g \circ f$ is quasi rgα -open .

2) If f and g are quasi rgα -open ,then $g \circ f$ is rgα *-open .

3) If f is quasi rgα -open and g is open , then $g \circ f$ is quasi rgα-open .

4) If f is quasi rgα -open and g is rgα -open , then $g \circ f$ is rgα*-open .

5) If f is quasi rgα -open and g is rgα *-open , then $g \circ f$ is rgα *-open .

6) If f is rgα -open and g is quasi rgα -open , then $g \circ f$ is open .

7) If f is rgα *-open and g is quasi rgα -open , then $g \circ f$ is quasi rgα -open .

8) If f is open and g is quasi rgα -open , then $g \circ f$ is open .

(2.10)Theorem:

Let X, Y, Z be three topological spaces ,and $f: X \to Y, g: Y \to Z$ be two functions. Then:-

1) If $g \circ f$ is quasi rgα -open and g is continuous and one-to-one, then f is quasi rgα-open .

2) If $g \circ f$ is quasi rgα -open and g is rgα -continuous and one-to-one, then f is rgα *-open .

3) If $g \circ f$ is contra rgα -irresolute and g is quasi rgα -open and one-to-one, then f is contra rgα -irresolute .

4) If $g \circ f$ is quasi rgα -open and f is rgα -irresolute and onto, then g is open .

Proof:

1) To prove that $f:X\to Y$ is quasi rgα -open .

Let U be an rgα -open subset of X,since $g\circ f$ is quasi rgα -open, then $(g\circ f)(U)$ is open in Z. Since g is continuous, then $g^{-1}(g\circ f(U))=(g^{-1}\circ g)(f(U))$ is open in Y.

Since g is one-to-one, then $f(U)$ is open in Y. Thus $f:X\to Y$ is a quasi rgα -open function .

2) To prove that $f:X\to Y$ is rgα *-open .

Let U be an rgα -open subset of X, since $g\circ f$ is quasi rgα-open, then $(g\circ f)(U)$ is open in Z

Since g is rgα-continuous, then $g^{-1}(g\circ f(U))=(g^{-1}\circ g)(f(U))$ is rgα-open in Y.Since g is one-to-one, then $f(U)$ is rgα-open in Y. Thus $f:X\to Y$ is an rgα *-open functin .

3) To prove that $f:X\to Y$ is contra rgα-irresolute .

Let U be an rgα -open subset of Y ,since g is quasi rgα-open, then $g(U)$ is open in Z .

Since every open set is rgα -open , then $g(U)$ is rgα-open in Z . Since $g\circ f$ is contra rgα-irresolute, then $(g\circ f)^{-1}(g(U))$ is rgα-closed in X, since g is one-one , then $(g\circ f)^{-1}(g(U))=f^{-1}((g^{-1}\circ g)(U))=f^{-1}(U)$ is an rgα-closed set in X , hence $f^{-1}(U)$ is rgα -closed in X . Thus $f:X\to Y$ is contra rgα -irresolute .

4) To prove that $g:Y\to Z$ is open .

Let U be an open subset of Y,then U is an rgα-open subset of Y,since f is rgα-irresolute

then $f^{-1}(U)$ is an rgα-open set in X, since $g\circ f$ is quasi rgα-open,

then $(g\circ f)(f^{-1}(U))=g(f\circ f^{-1}(U))$ is open in Z.

Since f is onto ,then $g(U)$ is open in Z.

Thus $g:Y\to Z$ is an open function .

(2.11)Theorem:

Let $f:X\to Y$ be a quasi rgα -open function and A be an open subset of X, then the restriction function $f\setminus A:A\to Y$ is a quasi rgα - open function.

Proof:

Let U be an rgα-open subset of A, Since A is an open subset of X ,then by [1], U is rgα-open in X, since $f:X\to Y$ is a quasi rgα -open function , then $(f\setminus A)(U)=f(U)$ is open in Y .Thus the restriction function $f\setminus A:A\to Y$ is a quasi rgα - open function .

3. Quasi rgα-closed Functions

(3.1)Definition:

Let X and Y be two topological spaces. A function $f: X \rightarrow Y$ is said to be **quasi** rgα **-closed** if the image of every rgα-closed set in X is closed in Y.

(3.2)Theorem:

Every quasi rgα-closed function is closed as well as rgα-closed .

Proof: It is obvious.

(3.3)Remark:

The converse of (3.2) may not be true in general. Consider the following example .

Example: Let $X = Y = \{a,b,c\}$ & $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$.

$\tau^c = \{X, \phi, \{b,c\}, \{a,c\}, \{c\}\}$

$\Rightarrow RG\alpha O(X,\tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}\}$.

$RG\alpha C(X,\tau) = \{X, \phi, \{b,c\}, \{a,c\}, \{a,b\}, \{c\}\}$

Let $f:(X,\tau) \rightarrow (Y,\tau)$ be a function defined by : $f(a) = a$, $f(b) = b$ & $f(c) = c$.

It is clear that f is rgα-closed as well as closed , but f is not quasi rgα -closed , Since {a,b} is rgα - closed in (X,τ) , but $f(\{a,b\}) = \{a,b\}$ is not closed in (Y,τ) .

(3.4)Theorem:

Every quasi rgα-closed function is rgα*-closed .

Proof: It is obvious.

(3.5)Remark:

The converse of (3.4) may not be true in general. In (2.5) , f is rgα *- closed , but f is not quasi rgα-closed ,since {a,b} is rgα-closed in (X,τ) ,but $f(\{a,b\}) = \{a,b\}$ is not closed in (Y,τ') .

Thus we have the following diagram:

$$\begin{array}{ccc} \textbf{quasi rg}\alpha\textbf{-closed function} & \Rightarrow & \textbf{rg}\alpha\textbf{*-closed function} \\ \Downarrow & & \Downarrow \\ \textbf{closed function} & \Rightarrow & \textbf{rg}\alpha\textbf{-closed function} \end{array}$$

(3.6)Theorem:

Let X and Y be two topological spaces . A bijective function $f: X \rightarrow Y$ is quasi rgα – closed iff. it is quasi rgα-open .

Proof:

Let f be a quasi rgα-closed function . To prove that f is a quasi rgα-open function .

Let U be an rgα-open set in $X \Rightarrow U^c$ is rgα-closed in X . Since f is quasi rgα-closed , then $f(U^c)$ is closed in Y .Therefore $(f(U^c))^c$ is open in Y . Since f is a bijective function , then $(f(U^c))^c = f(U) \Rightarrow f(U)$ is open in Y

Thus $f: X \rightarrow Y$ is a quasi rgα-open function .

Conversely, Suppose that $f: X \rightarrow Y$ is quasi rgα -open . To prove that f is quasi rgα-closed .

Let F be an rgα-closed set in X $\Rightarrow F^c$ is rgα - open in X . Since f is quasi rgα - open , then $f(F^c)$ is open in Y .Therefore $(f(F^c))^c$ is closed in Y . Since f is a bijective function then $(f(F^c))^c = f(F) \Rightarrow f(F)$ is closed in Y . Thus $f: X \rightarrow Y$ is a quasi rgα-closed function .

(3.7)Theorem:

Let X and Y be two topological spaces. A function $f: X \to Y$ is quasi rgα-closed iff. for any subset B of Y and for any rgα-open set G of X containing $f^{-1}(B)$, there exists an open set U of Y containing B such that $f^{-1}(U) \subset G$.

Proof:

This proof is similar to that of theorem (2.7).

However the following theorem holds. The proof is easy and hence omitted

(3.8)Theorem

Let X, Y, Z be three topological spaces ,and $f: X \to Y, g: Y \to Z$ be two functions. Then:-

1) If f and g are quasi rgα-closed ,then $g \circ f$ is quasi rgα-closed .
2) If f and g are quasi rgα-closed ,then $g \circ f$ is rgα*-closed .
3) If f is quasi rgα-closed and g is closed , then $g \circ f$ is quasi rgα-closed .
4) If f is quasi rgα-closed and g is rgα-closed , then $g \circ f$ is rgα*-closed .
5) If f is quasi rgα-closed and g is rgα*-closed , then $g \circ f$ is rgα*-closed .
6) If f is rgα -closed and g is quasi rgα-closed , then $g \circ f$ is closed .
7) If f is rgα* -closed and g is quasi rgα-closed , then $g \circ f$ is quasi rgα-closed .
8) If f is closed and g is quasi rgα-closed , then $g \circ f$ is closed .

(3.10)Theorem:

Let X, Y, Z be three topological spaces ,and $f: X \to Y, g: Y \to Z$ be two functions. Then:-

1) If $g \circ f$ is quasi rgα-closed and g is continuous and one-to-one , then f is quasi rgα-closed .
2) If $g \circ f$ is quasi rgα -closed and g is rgα - continuous and one-to-one, then f is rgα *-closed .
3) If $g \circ f$ is quasi rgα-closed and f is rgα-irresolute and onto, then g is closed
4) If $g \circ f$ is contra rgα- irresolute and f is quasi rgα - closed and onto , then g is contra rgα-irresolute .

Proof:

The proof is similar to that of theorem (2.10).Hence is omitted.

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الخلاصة

في هذا البحث قمنا بدراسة أنواع جديدة من الدوال المفتوحة والدوال المغلقة أسميناها بالدول تقريبا المفتوحــه-**rgα** quasi
(**rgα** -open functions) والدوال تقريبا المغلقة -**rgα** (quasi **rgα**-closed functions) كذلك درسنا المكافئات والخواص
الأساسية للدوال تقريبا المفتوحة- **rgα** و الدوال تقريبا المغلقة -**rgα** .