The Subjected Corrected Method for Solving the Linear Fredholm Integral Equations of the Second Kind

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Abstract

The aim of this paper is to use some new numerical method to solve linear Fredholm integral equations of the second kind. This method namely is the subjected corrected method. To illustrated the accuracy of this method, we give a numerical example.

Keyword: Fredholm integral equation, composite Simpson's method, composite Trapezoidal rule, Hermite interpolating polynomial

1-Introduction

The integral equations have received considerable interest in the mathematical literature, because of their many field of application in different areas of sciences (see, for example [1-4]). Many authors given numerical solutions for different types of Fredholm integral equations (see, for example [3-8]).

In this paper, we derive similarity methods namely the composite subjected corrected formula to solve the linear Fredholm integral equation of the second kind, we recall that the linear Fredholm integral equation of the second kind is the form:

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(y)dy, a \le x \le b \qquad (1)$$

Where λ is real numbers, f(x), k(x, y) are given continuous function, b is given function and u(x) is the unknown function to be determined.

2- The derivative subjected corrected method by using Hermite interpolating polynomial

The general form of the Hermite interpolating polynomials is given by

$$p(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_i - 1} y_i^{(k)} L_{i,k}(x) \qquad (2)$$

where $y_i = y(x_i), i = 0,1,...,n$

$L_{i,n_{i-1}}(x) = \ell_{i,m_{i-1}}(x), \ i = 0,1,\ldots,n,$

$$\ell_{i,k}(x)=\frac{(x-\zeta_i)^k}{k!}\prod_{\substack{j=0\\ i\neq j}}^{n}\left(\frac{x-\zeta_j}{\zeta_i-\zeta_j}\right)^{m_j},\quad i=0,1,\ldots,n,\quad k=0,1,\ldots,m_j,\quad m_j=m_i-1,\quad \zeta_i=x_i,$$

$i=0,1,\ldots,n$. (2a)

and recursively for $k = m_{i-2}, m_{i-3},\ldots, 0$,

$$L_{i,k}(x)=\ell_{i,k}(x)-\sum_{v=k+1}^{m_i-1}\ell^{(v)}{}_{i,k}(\zeta_i)L_{i,v}(x), \qquad [8]$$

(2b)

Now, by substituting $n = 1$, $m_j=0, 1$, if $m_j=0$, then $m_i=1, i=1$ and, if $m_j=1$, then $m_i=2$, $i=0$ in equation (2), one can have:

$$p(x) = y_0L_{0,0}(x)+y_1L_{1,0}(x)+y'_0\, L_{0,1}(x)$$

where $y_0 = f(0)$, $y_1 = f(1)$, and $y'_0 = f'(0)$.

Now, from equation (2a) and (2b), we have

$$L_{0,0}(x)= 2x^3 - 3x^2 + 1,$$

$$L_{1,0}(x)= x$$

and

$$L_{0,1}(x) = x^3 - 2x^2 + x..$$

Hence

$$p(x) = y_0(2x^3 - 3x^2 + 1)+ y_1(x)+y'_0(x^3 - 2x^2 + x)$$

By integrating both sides of the above equation from 0 to 1, one can have:

$$\int_0^1 f(x)dx \approx \int_0^1 p(x)\,dx = \frac{1}{2}[f(a)+f(b)]+\frac{1}{12}f'(a). \qquad (3)$$

Now, by using the transformation $x = a + t (b – a)$ and equation (3), one can have:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a)+f(b)]+\frac{h^2}{12}[f'(a)]-\frac{h^5}{720}f^{(4)}(\zeta_i), a<\zeta<b. \qquad (4)$$

This formula is called the subjected corrected method.

And its composite formula is as follows:

$$\int_a^b f(x)dx \approx \frac{\lambda h}{2}f(a)+\lambda h\sum_{j=1}^{n-1}f(x_j)+\frac{\lambda h}{2}f(b)+\frac{\lambda h^2}{12}[f'(a)] \qquad (5)$$

3- Expresses the method

Consider the one-dimensional Fredholm linear integral equation of the second kind given by equation (1). First, we divide the interval [a, b] into n subinterval $[x_i, x_{i+1}]$, $i = 0,1,\ldots,n-1$ such that $x_i = a+ih$, $i = 0,1,\ldots,n$, where $h=\frac{b-a}{n}$. So, the problem here is to find the solution of equation (1) at x_i, $i = 0,1,\ldots,n$. Let

u_i denote the numerical solution of the integral equation (1) at each $x_i, i=0,1,...,n$. Then, we approximate the integral that appeared in the right hand side of equation (1) at $x=x_i$, i=0,1,…,n, to get

$$u_i = f_i + \lambda\int_a^b k(x_i,y)u(y)dy, \quad i=0,1,...,n \tag{6}$$

Then, we approximate the integral that appeared in the right hand side of equation (6) with the composite subjected corrected method to get

$$u_i = f_i + \frac{\lambda h}{2}k(x_i,x_0)u_0 + \lambda h\sum_{j=1}^{n-1}k(x_i,x_j)u_j + \frac{\lambda h}{2}k(x_i,x_n)u_n + \frac{\lambda h^2}{12}\left[\frac{\partial}{\partial y}\left(k(x_i,y)u(y)\right)\Big|_{y=x_0}\right]$$

Therefore,

$$u_i = f_i + \frac{\lambda h}{2}k_{i0}u_0 + \lambda h\sum_{j=1}^{n-1}k_{ij}u_j + \frac{\lambda h}{2}k_{in}u_n + \frac{\lambda h^2}{12}\left[k_{i0}u_0' + J_{i0}u_0\right] \tag{7}$$

Where,

$u_i = u(x_i), f_i = f(x_i),\ k_{iz} = k(x_i,y_z), z=0,j,n,\ J_{i0} = J(x_i,x_0), i=0,1,...,n, J(x,y) = \frac{\partial k(x,y)}{\partial y}$.

The above system of equations consists of n+1 equations with n+2 unknowns namely, $u_i,\ i=0,1,...,n,\ u_0'$.

$$u_0 = f_0 + \frac{\lambda h}{2}k_{00}u_0 + \lambda h\sum_{j=1}^{n-1}k_{0j}u_j + \frac{\lambda h}{2}k_{0n}u_n + \frac{\lambda h^2}{12}\left[k_{00}u_0' + J_{00}u_0\right],$$

$$u_1 = f_1 + \frac{\lambda h}{2}k_{10}u_0 + \lambda h\sum_{j=1}^{n-1}k_{1j}u_j + \frac{\lambda h}{2}k_{1n}u_n + \frac{\lambda h^2}{12}\left[k_{10}u_0' + J_{10}u_0\right],$$

$$u_2 = f_2 + \frac{\lambda h}{2}k_{20}u_0 + \lambda h\sum_{j=1}^{n-1}k_{2j}u_j + \frac{\lambda h}{2}k_{2n}u_n + \frac{\lambda h^2}{12}\left[k_{20}u_0' + J_{20}u_0\right], \tag{8}$$

$$\vdots$$

$$u_n = f_n + \frac{\lambda h}{2}k_{n0}u_0 + \lambda h\sum_{j=1}^{n-1}k_{nj}u_j + \frac{\lambda h}{2}k_{nn}u_n + \frac{\lambda h^2}{12}\left[k_{n0}u_0' + J_{n0}u_0\right]$$

Now, to find u_0', one must differentiate equation (1) with respect to x to get:

$$u'(x) = f'(x) + \lambda\int_a^b H(x,y)u(y)\,dy \tag{9}$$

where $H(x,y) = \frac{\partial k(x,y)}{\partial x}$.

It is easy checking that the solution of equation (1) is a solution of equation (9)

By evaluating equation (9) at $x = x_i,\ i=0,1,...,n$, one can get

$$u'(x_i)=f'(x_i)+\lambda\int_a^b H(x_i,y)u(y)dy,\ \ i=0,1,...,n \qquad (10)$$

Next, to solve equation (10). We suppose if $\frac{\partial^2 k(x,y)}{\partial x\partial y}$ exists, we approximate the integral that appeared in the right hand side of the integral equation (10) with the subjected corrected method to obtain

$$u'(x)=f'(x)+\frac{\lambda h}{2}H(x,x_0)u_0+\lambda h\sum_{j=1}^{n-1}H(x,x_j)u_j+\frac{\lambda h}{2}H(x,x_n)u_n+\frac{\lambda h^2}{12}\left[\frac{\partial}{\partial y}\left(H(x,y)u(y)\right)\Big|_{y=x_0}\right]$$

Therefore,

$$u'(x)=f'(x)+\frac{\lambda h}{2}H(x,x_0)u_0+\lambda h\sum_{j=1}^{n-1}H(x,x_j)u_j+\frac{\lambda h}{2}H(x,x_n)u_n+\frac{\lambda h^2}{12}\left[H(x,x_0)u'_0+L(x,x_0)u_0\right]$$

where $L(x,y)=\frac{\partial H(x,y)}{\partial y}=\frac{\partial^2 k(x,y)}{\partial x\partial y}$.

Hence for $x=x_0$, one can get the following equations:

$$u'_0=f'_0+\frac{\lambda h}{2}H_{00}u_0+\lambda h\sum_{j=1}^{n-1}H_{0j}u_j+\frac{\lambda h}{2}H_{0n}u_n+\frac{\lambda h^2}{12}\left[H_{00}u'_0+L_{00}u_0\right] \qquad (11)$$

where $u'_0=u'(x_0), f'_0=f'(x_0),\ H_{0z}=H(x_0,y_z), z=0,j,n,\ L_{00}=L(x_0,y_0)$.

This system that we get from equation (8) consist of (n+1) equations , and equation (11), we get (n+2) equations.This system can be solved by using any suitable method to find the n+2 unknowns $u_i,\ i=0,1,...,n$, and u'_0 .

Now, if $\frac{\partial^2 k(x,y)}{\partial x\partial y}$ dose not exists, we approximate the integral that appeared in the right hand side of the integral equation (10) with the composite Trapezoidal method to obtain:

$$u'(x)=f'(x)+\frac{\lambda h}{2}H(x,x_0)u_0+\lambda h\sum_{j=1}^{n-1}H(x,x_j)u_j+\frac{\lambda h}{2}H(x,x_n)u_n$$

Hence, for $x=x_0$, , one can get the following equations

$$u'_0=f'_0+\frac{\lambda h}{2}H_{00}u_0+\lambda h\sum_{j=1}^{n-1}H_{0j}u_j+\frac{\lambda h}{2}H_{0n}u_n \qquad (12)$$

where $u'_0=u'(x_0), f'_0=f'(x_0),\ H_{0z}=H(x_0,y_z), z=0,j,n.$

This system that we get from equation (8) consist of (n+1) equations , and equation (12), we get (n+2) equations. This system can be solved by using any suitable method to find the n+2 unknowns $u_i,\ i=0,1,...,n$, and u'_0 .

To illustrate this method, consider the following examples:

Example (1):

Consider the one-dimensional Fredholm linear integral equation of the second kind:

$$u(x)=1+\frac{1}{\pi}\int_{-1}^{1}\frac{1}{1+(x-y)^2}u(y)\,dy,\ -1\le x\le 1$$

We can solve this example numerically via the composite subjected corrected method. Here $k(x,y)=\frac{1}{1+(x-y)^2},-1\le x,y\le 1$. It is clear that $\frac{\partial^2 k(x,y)}{\partial x\partial y}=\frac{2\left[1+(x-y)^2\right]-8(x-y)^2}{\left[1+(x-y)^2\right]^3}$ exists for each $x,y\in[-1,1]$, [9].To do this, first we divide the interval $[-1,1]$ into 8 subintervals such that $x_i=-1+\frac{i}{4},\ i=0,1,\ldots,8.$ Then equation(7) becomes:

$$u_i=1+\frac{1}{\pi}\Bigg[\left(\frac{1}{8}\frac{1}{1+(x_i+1)^2}+\frac{1}{192}\frac{2x_i+2}{\left[1+(x_i+1)^2\right]^2}\right)u_0+\frac{1}{4}\sum_{j=1}^{7}\frac{1}{1+(x_i-x_j)^2}u_j+ \left(\frac{1}{8}\frac{1}{1+(x_i-1)^2}\right)u_8+\left(\frac{1}{192}\frac{1}{1+(x_i+1)^2}\right)u_0'\Bigg],i=0,1,\ldots,8 \qquad (13)$$

And equation(11) become:

$$u_0'=\frac{1}{\pi}\left[\frac{1}{96}u_0+\frac{1}{4}\sum_{j=1}^{7}\frac{2+2x_j}{\left[1+(-1-x_j)^2\right]^2}u_j+\frac{1}{50}u_8\right]$$

By evaluating equation (13) at each i=0,1,…,8 together with the above integral equation one can get a linear system of 10 equations with 10 unknowns $\{u_i\}_{i=0}^{8},\ u_0'$, which has the solution:

Table 1: Comparison between the solution via Trapezoidal, Simpson's1/3 and subjected corrected method for example 1.

Nodes	Trapezoidal method	Simpson's 1/3 method	subjected corrected method
$x=\pm 1$	1.6363911	1.6397595	1.6396647
$x=\pm$ 0.75	1.7469543	1.7520710	1.7518962
X=± 0.5	1.8364100	1.8424642	1.8422914
$x=\pm$ 0.25	1.8933281	1.8996630	1.8995966
$x=0$	1.9126894	1.919050	1.9190039

Example (2):

Consider the one-dimensional Fredholm linear integral equation of the second kind:

$$u(x)=x-\frac{2}{7}(x+1)^{\frac{7}{2}}+\frac{2}{5}(x+1)^{\frac{5}{2}}x-\frac{4}{35}x^{\frac{7}{2}}+\int_0^1(x+y)^{\frac{3}{2}}u(y)dy,\ \ 0\le x\le 1$$

This example is constructed such that the exact solution of it is u(x)=x, [10]. We can solve this example numerically via the composite Trapizoidal method.

Here $k(x,y)=(x+y)^{\frac{3}{2}},\ \ 0\le x,y\le 1$, then $\frac{\partial^2 k(x,y)}{\partial x\partial y}=\frac{3}{4\sqrt{(x+y)}}$. It is clear that $\frac{\partial^2 k(x,y)}{\partial x\partial y}$ does not exists at x=y=0. To do this, first, we divide the interval [0,1] into 8 subintervals such that $x_i=\frac{i}{8},\ i=0,1,\ldots,8$. Then equation(7) becomes

$$u_i=x_i-\frac{2}{7}(x_i+1)^{\frac{7}{2}}+\frac{2}{5}(x_i+1)^{\frac{5}{2}}x_i-\frac{4}{35}x_i^{\frac{7}{2}}+\left(\frac{1}{16}x_i^{\frac{3}{2}}+\frac{1}{512}x_i^{\frac{1}{2}}\right)u_0+\frac{1}{8}\sum_{j=1}^{7}(x_i+x_j)^{\frac{3}{2}}u_j+$$
$$\left(\frac{1}{16}(x_i+1)^{\frac{3}{2}}\right)u_8+\left(\frac{1}{768}x_i^{\frac{3}{2}}\right)u'_0,i=0,1,\ldots,8 \qquad (14)$$

And equation(12) become:

$$u'_0=\frac{2}{5}+\frac{3}{16}\sum_{j=1}^{7}(x_j)^{\frac{1}{2}}u_j+\frac{3}{32}u_8$$

By evaluating equation (14) at each i=0,1,…,8 together with the above integral equation one can get a linear system of 10 equations with 10 unknowns $\{u_i\}_{i=0}^{8}$ and u'_0, which has the solution:

Nodes	Exact Solution	Trapezoidal Method	Simpson's 1/3 method	subjected corrected method
X = 0	0	-0.0057843	-0.0000068	0.0000062
X = 0.12	0.1250000	0.1169490	0.1250008	0.1250030
X = 0.25	0.2500000	0.2393950	0.2500041	0.2500032
X = 0.37	0.3750000	0.3615941	0.3750066	0.3750040
X = 0.50	0.5000000	0.4835698	0.5000088	0.5000052
X = 0.62	0.6250000	0.6053392	0.6250109	0.6250061
X = 0.75	0.7500000	0.7269154	0.7500130	0.7500073
X = 0.87	0.8750000	0.8483093	0.8750151	0.8750090
X=1	1	0.9695301	1.0000172	1.0000108
LSE	-------	0.00322065138041	1.003241169767148e-009	3. 0217324357882916e-012
AE	-------	0.15418195845504	8.371347436351637e-005	9.127936600095921e-007

Table 2: Comparison between the solution via Trapezoidal, Simpson's 1/3 and subjected corrected method for example 2.

Where LSE is the Lees Square Error and AE is the Absolute Error

Table 3: Comparison between the error via Trapezoidal, Simpson's 1/3 and subjected corrected method and method for example 2.

Nodes	Exact Solution	Error in Trapezoidal method	Error in Simpson's 1/3 method	Error in subjected corrected method
X = 0	0	5.7×10^{-3}	6.8×10^{-6}	6.2×10^{-6}
X = 0.12	0.125	8.1×10^{-3}	8.3×10^{-6}	3.0×10^{-6}
X = 0.25	0.250	1.1×10^{-2}	4.1×10^{-6}	3.2×10^{-6}
X = 0.37	0.375	1.3×10^{-2}	6.6×10^{-6}	4.0×10^{-6}
X = 0.50	0.500	1.6×10^{-2}	8.8×10^{-6}	5.2×10^{-6}
X = 0.62	0.625	1.9×10^{-2}	1.1×10^{-5}	6.1×10^{-6}
X = 0.75	0.750	2.3×10^{-2}	1.3×10^{-5}	7.3×10^{-6}
X = 0.87	0.875	2.6×10^{-2}	1.5×10^{-5}	9.0×10^{-6}
X=1	1	3.0×10^{-2}	1.7×10^{-5}	1.08×10^{-5}

4- Conclusions

The Fredholm integral equations are usually difficult to solve analytical. In many cases, it is required to obtain the approximate solutions, for this purpose the presented method can be proposed. From numerical examples it can be seen that the proposed numerical methods are efficient and accurate to estimate the solution of these equations and this method is more accurately than the repeated Trapezoidal method and the repeated Simpson's 1/3 method.

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الطريقة المعدلة المقترحة لحل المعادلات التكاملية الخطية من نوع فريدهولم من النوع الثاني

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