Left Jordan $\sigma$-Centralizer on Completely Prime $\Gamma$-Ring

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Abstract

Let $M$ be a $\Gamma$-ring this research introduce the concepts of left (resp. right) $\sigma$-centralizer, left (resp. right) Jordan $\sigma$-centralizer, left (resp. right) Jordan triple $\sigma$-centralizer of $\Gamma$-ring as well as prove that every left (resp. right) Jordan $\sigma$-centralizer of completely prime $\Gamma$-ring $M$ is left (resp. right) $\sigma$-centralizer of $M$.

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1) Introduction

Nobusawa [6] introduced the notion of a $\Gamma$-ring, more general than a ring. Barnes [1] weaken slightly the conditions in the definition of $\Gamma$-ring in the sense of Nobusawa.

Let $M$ and $\Gamma$ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$ (the image of $(a, \alpha, b)$ being denoted by $a\alpha b$, $a, b \in M$ and $\alpha, \beta \in \Gamma$) satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:

i) $(a + b)\alpha c = a\alpha c + b\alpha c$
   $a(\alpha + \beta)c = a\alpha c + a\beta c$
   $a\alpha(b + c) = a\alpha b + a\alpha c$

ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

then $M$ is called a $\Gamma$-ring. This definition is due to Barnes[1]. If the following condition holds for a $\Gamma$-ring $M$ then $M$ is called a prime $\Gamma$-ring [2], $a\Gamma M \Gamma b = 0$ then $a=0$ or $b=0$, $a, b \in M$, $M$ is called a semiprime $\Gamma$-ring if $a\Gamma M \Gamma a = 0$ then $a=0$ and $M$ is called a completely prime $\Gamma$-ring if $a\Gamma b = 0$ then $a=0$ or $b=0$, $M$ is called 2-torsion free if $2a=0$ implies $a=0$ for all $a \in M$. Jing [5] defined a derivation on $\Gamma$-ring as followings an additive map $d: M \rightarrow M$ is called derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$. Sapani and Nakajima defined a Jordan derivation on $\Gamma$-ring as follows $d$ is called Jordan derivation if $d(x\alpha x) = d(x)\alpha x + x\alpha d(x)$.

Zalar [7] defined left (resp. right ) centralizer and left (resp. right) Jordan centralizer of ring $R$ and proved that any left (resp. right) Jordan centralizer on a 2-torsion free semiprime ring is a left (resp. right) centralizer. Hoque and Paul [4] defined centralizer and Jordan centralizer on Jordan ideal of $\Gamma$-ring $M$ and
Left Jordan $\sigma$-Centralizer on Completely Prime $\Gamma$-Ring

Dr. Salah Mehdi Salih

proved that every Jordan centralizer on a 2-torsion free semiprime $\Gamma$-ring $M$ such that $x\alpha yz = x\beta yz$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ is centralizer on $M$. Cortes and Haetinger [3] defined left (resp. right) $\sigma$-centralizer, left (resp. right) Jordan $\sigma$-centralizer of Lie ideal $U$ onto ring $R$ and proved that if $R$ is 2-torsion free ring then every left (resp. right) Jordan $\sigma$-centralizer of Lie ideal $U$ onto ring $R$ is left (resp. right) $\sigma$-centralizer.

This research generalization the results of Cortes and Haetinger by present the concepts of left (resp. right) $\sigma$-centralizer, left (resp. right) Jordan $\sigma$-centralizer, left (resp. right) Jordan triple $\sigma$-centralizer of $\Gamma$-ring as well as the research prove that:

i) Every left (resp. right) Jordan $\sigma$-centralizer of $\Gamma$-ring $M$ into completely prime $\Gamma$-ring is left (resp. right)$\sigma$-centralizer of $M$.

ii) Every left (resp. right) Jordan triple $\sigma$-centralizer of 2-torsion free completely prime $\Gamma$-ring $M$ such that $x\alpha yz = x\beta yz$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ is left (resp. right) $\sigma$-centralizer of $M$.

2) Left Jordan $\sigma$-Centralizer on $\Gamma$-Ring:

In this section the research present the concepts of left $\sigma$-centralizer, left Jordan $\sigma$-centralizer, left Jordan triple $\sigma$-centralizer of $\Gamma$-ring and the research study the relation among them. We begin by the following definition:

**Definition 2.1:**

Let $M$ be $\Gamma$-ring and $\sigma : M \rightarrow M$ be endomorphism an additive mapping $T : M \rightarrow M$ is called left $\sigma$-centralizer (right $\sigma$-centralizer) if for all $x, y \in M$ and $\alpha \in \Gamma$ then $T(x\alpha y) = T(x)\alpha \sigma(y)$ (respectively $T(x\alpha y) = \sigma(x)\alpha T(y)$). $T$ is called left Jordan $\sigma$-centralizer (right Jordan $\sigma$-centralizer) if $T(x\alpha x) = T(x)\alpha \sigma(x)$ (respectively $T(x\alpha x) = \sigma(x)\alpha T(x)$) for all $x \in M$ and $\alpha \in \Gamma$. $T$ is called left Jordan triple centralizer (right Jordan triple centralizer) if $T(x\alpha y\beta x) = T(x)\alpha \sigma(y)\beta \sigma(x)$ (respectively $T(x\alpha y\beta x) = \sigma(x)\alpha \sigma(y)\beta T(x)$).

It’s clear that every left $\sigma$-centralizer (right $\sigma$-centralizer) is left Jordan $\sigma$-centralizer (right Jordan $\sigma$-centralizer). The research gives an example of Jordan left $\sigma$-centralizer which is not left $\sigma$-centralizer.

**Example 2.2:**

Let $M$ be a $\Gamma$-ring. Define $M_1 = \{ (x, x) : x \in M \}$ and $\Gamma_1 = \{ (\alpha, \alpha) : \alpha \in \Gamma \}$. Let the operations of addition and multiplication on $M_1$ be defined by:

$$
(x_1, x_1) + (x_2, x_2) = (x_1 + x_2, x_1 + x_2)
$$

$$
(x_1, x_1) (\alpha, \alpha) (x_2, x_2) = (x_1 \alpha x_2, x_1 \alpha x_2)
$$

for every $x_1, x_2 \in M$ and $\alpha \in \Gamma$.

Then it can easily seen that $M_1$ is $\Gamma_1$-ring. Let $\sigma : M \rightarrow M$ be endomorphism, $d_1 : M \rightarrow M$ be a left $\sigma$-centralizer mapping and $d_2 : M \rightarrow M$ be a right $\sigma$-
Left Jordan $\sigma$-Centralizer on Completely Prime $\Gamma$-Ring

Dr. Salah Mehdi Salih

centralizer and commuting mapping, $\sigma_1: M_1 \rightarrow M_1$ be endomorphism defined by $\sigma_1(x,x) = (\sigma(x), \sigma(x))$ for all $x \in M_1$, we define the additive mapping $T: M_1 \rightarrow M_1$ by $T(x,x) = (d_1(x), d_2(x))$ for all $x \in M_1$.

Then $T$ is a left Jordan $\sigma_1$-centralizer, which is not a left $\sigma$-centralizer of $M_1$.

Now, the research present the properties of left $\sigma$-centralizer.

**Lemma 1:**

If $T$ is left Jordan $\sigma$-centralizer of $\Gamma$-ring $M$ then for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$:

i) $T(x \alpha y + y \beta x) = T(x) \alpha \sigma(y) + T(y) \beta \sigma(x)$

ii) $T(x \alpha y + y \beta x) = T(x) \alpha \sigma(y) + T(y) \alpha \sigma(x)$

iii) $T(x \alpha y \alpha x) = T(x) \alpha \sigma(y) \alpha \sigma(x)$

iv) $T(x \alpha y \beta z + z \alpha y \beta x) = T(x) \alpha \sigma(y) \beta \sigma(z) + T(z) \alpha \sigma(y) \beta \sigma(x)$

v) $T(x \alpha y \alpha z + z \alpha y \alpha x) = T(x) \alpha \sigma(y) \alpha \sigma(z) + T(z) \alpha \sigma(y) \alpha \sigma(x)$

**Proof:**

i) $T((x+y) \alpha (y+x)) = T(x+y) \alpha \sigma(y+x) = T(x) \alpha \sigma(y) + T(x) \alpha \sigma(x) + T(y) \alpha \sigma(y) + T(y) \alpha \sigma(x)$

On the other hand

$T((x+y) \alpha (y+x)) = T(x \alpha y + y \alpha x + y \alpha y + y \alpha x)$

$= T(x \alpha x + y \alpha y) + T(x \alpha y + y \alpha x)$

$= T(x) \alpha \sigma(x) + T(y) \alpha \sigma(y) + T(x \alpha y + y \alpha x)$

Compare (1) and (2) we get:

$T(x \alpha y + y \alpha x) = T(x) \alpha \sigma(y) + T(y) \alpha \sigma(y)$

ii) Replace $\alpha$ for $\beta$ in (i) we get the require result.

iii) Replace $\alpha$ for $\beta$ in definition (2.1) we get the require result.

iv) Replace $x+z$ for $x$ in (iii) we get

$T((x+z) \alpha y \beta (x+z)) = T(x+z) \alpha \sigma(y) \beta \sigma(x+z)$

$= T(x+z) \alpha \sigma(y) \beta \sigma(x) + T(z) \alpha \sigma(y) \beta \sigma(z) + T(z) \alpha \sigma(y) \beta \sigma(x)$

On the other hand

$T((x+z) \alpha y \beta (x+z)) = T(x \alpha y \beta x + x \alpha y \beta z + z \alpha y \beta x + z \alpha y \beta z)$

$= T(x \alpha y \beta x + z \alpha y \beta z) + T(x \alpha y \beta z + z \alpha y \beta x)$

$= T(x) \alpha \sigma(y) \beta \sigma(x) + T(z) \alpha \sigma(y) \beta \sigma(z) + T(x \alpha y \beta z + z \alpha y \beta x)$

Compare (1) and (2) we get:

$T(x \alpha y \beta z + z \alpha y \beta x) = T(x) \alpha \sigma(y) \beta \sigma(z) + T(z) \alpha \sigma(y) \beta \sigma(x)$.

v) Replace $\alpha$ for $\beta$ in (iv) we get the require result.
Definition 2.3:

Let $T$ be left Jordan $\sigma$-centralizer of $\Gamma$-ring $M$. Then for every $x,y \in M$ and $\alpha \in \Gamma$ we define $\Phi_{\alpha}(x,y) = T(x\alpha y - T(x)\alpha \sigma(y)$.

Now, the research introduce the property of $\Phi_{\alpha}(x,y)$.

Lemma 2:

Let $T$ be left Jordan $\sigma$-centralizer of $\Gamma$-ring $M$, then for all $x,y,z \in M$ and $\alpha, \beta \in \Gamma$:

i) $\Phi_{\alpha}(x,y) = \Phi_{\alpha \beta}(y,x)$

ii) $\Phi_{\alpha}(x+y,z) = \Phi_{\alpha}(x,z) + \Phi_{\alpha}(y,z)$

iii) $\Phi_{\alpha}(x,y+z) = \Phi_{\alpha}(x,y) + \Phi_{\alpha}(x,z)$

iv) $\Phi_{\alpha \beta}(x,y) = \Phi_{\alpha}(x,y) + \Phi_{\beta}(x,y)$

Proof:

i) $T(x\alpha y + y \beta x) = T(x)\alpha \sigma(y) + T(y)\beta \sigma(x)$

$T(x\alpha y) + T(y \beta x) = T(x)\alpha \sigma(y) + T(y)\beta \sigma(x)$

$T(x\alpha y) - T(x)\alpha \sigma(y) = -(T(y \beta x) - T(y)\beta \sigma(x))$

$\Phi_{\alpha}(x,y) = \Phi_{\alpha}(y,x)$

ii) $\Phi_{\alpha}(x+y, z) = T(x+y)\alpha \sigma(z) - T(x+y)\alpha \sigma(y)$

$= T(x)\alpha \sigma(z) + T(y)\alpha \sigma(z) - (T(x)\alpha \sigma(y) + T(x)\alpha \sigma(y))$

$= T(x)\alpha \sigma(z) - T(x)\alpha \sigma(y) + T(y)\alpha \sigma(z) - T(y)\alpha \sigma(y)$

$= \Phi_{\alpha}(x,z) + \Phi_{\alpha}(y,z)$

iii) $\Phi_{\alpha}(x,y+z) = T(x\alpha(y+z)) - T(x)\alpha \sigma(y+z)$

$= T(x\alpha y + x\alpha z) - (T(x)\alpha \sigma(y) + T(x)\alpha \sigma(z))$

$= T(x)\alpha \sigma(y) - T(x)\alpha \sigma(y) + T(x)\alpha \sigma(z) - T(x)\alpha \sigma(z)$

$= \Phi_{\alpha}(x,y) + \Phi_{\alpha}(x,z)$

iv) $\Phi_{\alpha \beta}(x, y) = T(x(\alpha + \beta)y - T(x)\alpha + \beta \sigma(y)$

$= T(x\alpha y + x\beta y) - (T(x)\alpha \sigma(y) + T(x)\beta \sigma(y))$

$= T(x\alpha y) - T(x)\alpha \sigma(y) + T(x\beta y) - T(x)\beta \sigma(y)$

$= \Phi_{\alpha}(x,y) + \Phi_{\beta}(x,y)$

Remark 2.4:

Note that $T$ is left $\sigma$-centralizer of $\Gamma$-ring $M$ if and only if $\Phi_{\alpha}(x,y)=0$ for all $x,y \in M$ and $\alpha \in \Gamma$.

3) The Main Results

In this section the research present the main results of this paper.

Theorem 3:

Let $T$ be left (resp. right) Jordan $\sigma$-centralizer of completely prime $\Gamma$-ring $M$, then $\Phi_{\alpha}(x,y)=0$ for all $x,y \in M$ and $\alpha \in \Gamma$. 
Left Jordan $\sigma$ -Centralizer on Completely Prime $\Gamma$-Ring

Dr. Salah Mehdi Salih

Proof:
Replace $z$ by $x\alpha y$ in lemma 1 (v), we get:
\[
T(x\alpha y \alpha x\alpha y + x\alpha y \alpha y\alpha x) = T((x\alpha y)\alpha (x\alpha y)) + T((x\alpha y)\alpha (y\alpha x)) \\
= T(x\alpha y)\alpha \sigma (x\alpha y) + T(x\alpha y)\alpha \sigma (y\alpha x) \\
= T(x\alpha y)\alpha \sigma (y\alpha x) + T(x\alpha y)\alpha \sigma (y\alpha x) \\
(1)
\]
\[
T(x\alpha y \alpha x\alpha y + x\alpha y \alpha y\alpha x) = T(x\alpha y \alpha x\alpha y + x\alpha (y\alpha y)\alpha x) \\
= T(x\alpha y)\alpha \sigma (x\alpha y) + T(x\alpha y)\alpha \sigma (y\alpha y)\alpha \sigma (x) \\
= T(x\alpha y)\alpha \sigma (x\alpha y) + T(x\alpha y)\alpha \sigma (y\alpha x) \\
(2)
\]

Compare (1) and (2) we get:
\[
T(x\alpha y)\alpha (x\alpha y) - T(x)\alpha \sigma (y)\alpha \sigma (x\alpha y) - T(x)\alpha \sigma (y\alpha x) + T(x)\alpha \sigma (y)\alpha \sigma (y\alpha x) = 0 \\
\Phi_\alpha (x,y)\alpha \sigma [x,y]_\alpha = 0
\]

Since $M$ is completely prime we get
\[
\Phi_\alpha (x,y) = 0.
\]

Corollary:
Every left (resp. right) Jordan $\sigma$-centralizer of completely prime $\Gamma$-ring $M$ is left (resp. right)$\sigma$-centralizer of $M$.

Proof:
By using theorem 3 and remark 2.3.
In the following theorem the research present the relation between left Jordan triple $\sigma$-centralizer and left $\sigma$-centralizer on $\Gamma$-ring $M$.

Theorem 4:
Every left (resp. right) Jordan triple $\sigma$-centralizer of 2-tortion free completely prime $\Gamma$-ring $M$ such that $x\alpha y \beta z = x\beta y \alpha z$ for all $x,y,z \in M$ and $\alpha, \beta \in \Gamma$ is left (resp. right) $\sigma$-centralizer of $M$.

Proof:
By theorem 1 (iv)
\[
T(x\alpha y \beta z + z\alpha y \beta x) = T(x\alpha y \beta z + z\alpha y \beta x) \\
= T(x\alpha y \beta z + z\alpha y \beta x) + T(z)\alpha \sigma (y)\beta \sigma (x) \\
\]
Replace $z$ by $x\alpha y$ we get:
\[
T(x\alpha y \beta x\alpha y + x\alpha y \alpha y \beta x) = T(x\alpha y \beta x\alpha y + x\alpha y \alpha y \beta x) \\
= T(x\alpha y \beta x\alpha y + x\alpha y \alpha y \beta x) + T(x\alpha y)\alpha \sigma (y)\beta \sigma (x) \\
(1)
\]
On the other hand
\[
T(x\alpha y \beta x\alpha y + x\alpha y \alpha y \beta x) = T(x\alpha y \beta x\alpha y + x\alpha y \alpha y \beta x) \\
= T(x\alpha y)\alpha \sigma (y)\beta \sigma (x) \alpha \sigma (y) \\
\]
Compare (1) and (2) we get:
\[
T(x\alpha y)\alpha \sigma (y)\beta \sigma (x) + T(x)\alpha \sigma (y)\beta \sigma (x) - T(x)\alpha \sigma (y)\beta \sigma (x) - T(x)\alpha \sigma (y)\beta \sigma (x) = 0
\]
Left Jordan $\sigma$-Centralizer on Completely Prime $\Gamma$-Ring

Dr. Salah Mehdi Salih

\[ \Phi_{\sigma}(x,y) = 0 \quad \text{and} \quad \sigma(x,y) = 0 \]

Since M is completely prime $\Gamma$-ring we get:

Either $\Phi_{\sigma}(x,y) = 0$ or $\sigma(x,y) = 0$

If $\Phi_{\sigma}(x,y) = 0$ then by remark 2.3 we have T is left $\sigma$-centralizer.

Now if $\sigma(x,y) = 0$ then M is commutative $\Gamma$-ring and by theorem 1(i) we get

$T(2x, y) = 2T(x, \sigma(y))$

Hence M is 2-tortion free $\Gamma$-ring we get T is left $\sigma$-centralizer.

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