Left Jordan σ -Centralizer on Completely Prime Γ-Ring

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Abstract

Let M be Γ-ring this research introduce the concepts of left (resp. right)σ -centralizer, left (resp. right) Jordan σ -centralizer, left (resp. right) Jordan triple σ -centralizer of Γ-ring as well as prove that every left (resp. right) Jordan σ -centralizer of completely prime Γ-ring M is left (resp. right)σ -centralizer of M.

Mathematics Subject Classification: 16W25,16A12, 16A78.
Keywords: gamma ring, left σ -centralizer, left Jordan σ -centralizer.

1) Introduction

Nobusawa [6] introduced the notion of a Γ-ring, more general than a ring. Barnes [1] weakend slightly the conditions in the definition of Γ-ring in the sense of Nobusawa.

Let M and Γ be two additive abelian groups. Suppose that there is a mapping from M×Γ×M→M (the image of (a,α,b) being denoted by aαb, a,b∈M and α∈Γ) satisfying for all a,b,c∈M and α,β∈Γ:

i) (a+b)αc = aαc + bαc
a(α+β)c = aαc + aβc
aα(b+c) = aαb +aαc

ii) (aαb)βc = aα(bβc)

then M is called a Γ-ring. This definition is due to Barnes[1].If the following condition holds for a Γ-ring M then M is called a prime Γ-ring [2], aΓMΓb =0 then a=0 or b=0, a,b∈M, M is called a semiprime Γ-ring if aΓMΓa =0 then a=0 and M is called a completely prime Γ-ring if aΓb =0 then a=0 or b=0, M is called 2-torsion free if 2a=0 implies a=0 for all a∈M. Jing [5] defined a derivation on Γ-ring as followings an additive map d:M→M is called derivation if d(xαy)= d(x)αy + xαd(y). Sapanci and Nakajima defined a Jordan derivation on Γ-ring as follows d is called Jordan derivation if d(xαx)= d(x)αx + xαd(x).

Zalar [7] defined left (resp. right) centralizer and left (resp. right) Jordan centralizer of ring R and proved that any left (resp. right) Jordan centralizer on a 2-torsion free semiprime ring is a left (resp. right) centralizer. Hoque and Paul [4] defined centralizer and Jordan centralizer on Jordan ideal of Γ-ring M and proved that every Jordan centralizer on a 2-torsion free semiprime Γ-ring M such that $x\alpha y\beta z = x\beta y\alpha z$ for all x,y,z∈M and $\alpha,\beta\in\Gamma$ is centralizer on M. Cortes and Haetinger [3] defined left (resp. right) σ-centralizer, left (resp. right) Jordan σ-centralizer of Lie ideal U onto ring R and proved that if R is 2-torsion free ring then every left (resp. right) Jordan σ-centralizer of Lie ideal U onto ring R is left (resp. right) σ- centralizer.

This research generalization the results of Cortes and Haetinger by present the concepts of left (resp. right) σ-centralizer, left (resp. right) Jordan σ-centralizer, left (resp. right) Jordan triple σ-centralizer of Γ-ring as well as the research prove that:

i) Every left (resp. right) Jordan σ-centralizer of Γ-ring M into completely prime Γ-ring is left (resp. right) σ-centralizer of M.

ii) Every left (resp. right) Jordan triple σ-centralizer of 2-tortion free completely prime Γ-ring M such that $x\alpha y\beta z = x\beta y\alpha z$ for all x,y,z∈M and $\alpha,\beta\in\Gamma$ is left (resp. right) σ-centralizer of M.

2) Left Jordan σ-Centralizer on Γ-Ring:

In this section the research present the concepts of left σ-centralizer, left Jordan σ-centralizer, left Jordan triple σ-centralizer of Γ-ring and the research study the relation among them. We begin by the following definition:

Definition 2.1:

Let M be Γ-ring and σ: M→M be endomorphism an additive mapping T: M→M is called left σ- centralizer (right σ-centralizer) if for all x,y∈M and $\alpha\in\Gamma$ then $T(x\alpha y)=T(x)\alpha\sigma(y)$ (respectively $T(x\alpha y)=\sigma(x)\alpha T(y)$) .T is called left Jordan σ-centralizer (right Jordan σ-centralizer) if $T(x\alpha x)=T(x)\alpha\sigma(x)$ (respectively $T(x\alpha x)=\sigma(x)\alpha T(x)$) for all x∈M and $\alpha\in\Gamma$.T is called left Jordan triple centralizer (right Jordan triple centralizer) if $T(x\alpha y\beta x) = T(x)\alpha\sigma(y)\beta\sigma(x)$ (respectively $T(x\alpha y\beta x) = \sigma(x)\alpha\sigma(y)\beta T(x)$).

It's clear that every left σ-centralizer (right σ-centralizer) is left Jordan σ-centralizer (right Jordan σ-centralizer).The research gives an example of Jordan left σ-centralizer which is not left σ-centralizer.

Example 2.2:

Let M be a Γ-ring. Define $M_1 = \{(x,x) : x\in M\}$ and $\Gamma_1 = \{(\alpha,\alpha) : \alpha\in\Gamma\}$. Let the operations of addition and multiplication on M_1 be defined by :

$$(x_1,x_1) + (x_2,x_2) = (x_1+x_2, x_1+x_2)$$

$$(x_1,x_1)(\alpha,\alpha)(x_2,x_2) = (x_1\alpha x_2, x_1\alpha x_2)$$

for every $x_1, x_2\in M$ and $\alpha\in\Gamma$.

Then it can easily seen that M_1 is Γ_1-ring. Let σ: M→ M be endomorphism , d_1:M→M be a left σ-centralizer mapping and d_2 :M→M be a right σ-

centralizer and commutating mapping , σ_1: $M_1 \rightarrow M_1$ be endomorphism defined by $\sigma_1(x,x) = (\sigma(x), \sigma(x))$ for all $x \in M$, we define the additive mapping T: $M_1 \rightarrow M_1$ by $T(x,x) = (d_1(x), d_2(x))$ for all $x \in M$.
Then T is a left Jordan σ_1-centralizer, which is not a left σ_1-centralizer of M_1.
Now, the research present the properties of left σ-centralizer.

Lemma 1:

If T is left Jordan σ-centralizer of Γ-ring M then for all $x,y,z \in M$ and $\alpha, \beta \in \Gamma$:

i) $T(x\alpha y + y\beta x) = T(x)\alpha\sigma(y) + T(y)\beta\sigma(x)$

ii) $T(x\alpha y + y\alpha x) = T(x)\alpha\sigma(y) + T(y)\alpha\sigma(x)$

iii) $T(x\alpha y\alpha x) = T(x)\alpha\sigma(y)\alpha\sigma(x)$

iv) $T(x\alpha y\beta z + z\alpha y\beta x) = T(x)\alpha\sigma(y)\beta\sigma(z) + T(z)\alpha\sigma(y)\beta\sigma(x)$

v) $T(x\alpha y\alpha z + z\alpha y\alpha x) = T(x)\alpha\sigma(y)\alpha\sigma(z) + T(z)\alpha\sigma(y)\alpha\sigma(x)$

Proof:

i) $T((x+y)\alpha(y+x)) = T(x+y)\alpha\sigma(y+x)$
$= T(x)\alpha\sigma(y) + T(x)\alpha\sigma(x) + T(y)\alpha\sigma(y) + T(y)\alpha\sigma(x)$

On the other hand

$T((x+y)\alpha(y+x)) = T(x\alpha y + x\alpha x + y\alpha y + y\alpha x)$
$= T(x\alpha x + y\alpha y) + T(x\alpha y + y\alpha x)$
$= T(x)\alpha\sigma(x) + T(y)\alpha\sigma(y) + T(x\alpha y + y\alpha x)$

Compare (1) and (2) we get:

$T(x\alpha y + y\alpha x) = T(x)\alpha\sigma(x) + T(y)\alpha\sigma(y)$

ii) Replace α for β in (i) we get the require result.

iii) Replace α for β in definition (2.1) we get the require result.

iv) Replace x+z for x in (iii) we get

$T((x+z)\alpha y\beta(x+z)) = T(x+z)\alpha\sigma(y)\beta\sigma(x+z)$
$= (T(x) + T(z))\alpha\sigma(y)\beta\sigma(x+z)$
$= T(x)\alpha\sigma(y)\beta\sigma(x) + T(x)\alpha\sigma(y)\beta\sigma(z) + T(z)\alpha\sigma(y)\beta\sigma(x) + T(z)\alpha\sigma(y)\beta\sigma(z)$

On the other hand

$T((x+z)\alpha y\beta(x+z)) = T(x\alpha y\beta x + x\alpha y\beta z + z\alpha y\beta x + z\alpha y\beta z)$
$= T(x\alpha y\beta x + z\alpha y\beta z) + T(x\alpha y\beta z + z\alpha y\beta x)$
$= T(x)\alpha\sigma(y)\beta\sigma(x) + T(z)\alpha\sigma(y)\beta\sigma(z) + T(x\alpha y\beta z + z\alpha y\beta x)$

Compare (1) and (2) we get:

$T(x\alpha y\beta z + z\alpha y\beta x) = T(x)\alpha\sigma(y)\beta\sigma(z) + T(z)\alpha\sigma(y)\beta\sigma(x)$.

v) Replace α for β in (iv) we get the require result.

Definition 2.3:

Let T be left Jordan σ-centralizer of Γ-ring M. Then for every $x,y \in M$ and $\alpha \in \Gamma$ we define $\Phi_\alpha(x,y) = T(x\alpha y) - T(x)\alpha\sigma(y)$.

Now, the research introduce the property of $\Phi_\alpha(x,y)$.

Lemma 2:

Let T be left Jordan σ-centralizer of Γ-ring M, then for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$:

i) $\Phi_\alpha(x,y) = -\Phi_\alpha(y,x)$

ii) $\Phi_\alpha(x+y,z) = \Phi_\alpha(x,z) + \Phi_\alpha(y,z)$

iii) $\Phi_\alpha(x,y+z) = \Phi_\alpha(x,y) + \Phi_\alpha(x,z)$

iv) $\Phi_{\alpha+\beta}(x,y) = \Phi_\alpha(x,y) + \Phi_\beta(x,y)$

Proof:

i) $T(x\alpha y + y\beta x) = T(x)\alpha\sigma(y) + T(y)\beta\sigma(x)$

$T(x\alpha y) + T(y\beta x) = T(x)\alpha\sigma(y) + T(y)\beta\sigma(x)$

$T(x\alpha y) - T(x)\alpha\sigma(y) = -(T(y\beta x) - T(y)\beta\sigma(x))$

$\Phi_\alpha(x,y) = -\Phi_\alpha(y,x)$

ii) $\Phi_\alpha(x+y,z) = T(x+y)\alpha\sigma(z) - T(x+y)\alpha\sigma(z)$

$= T(x)\alpha\sigma(z) + T(y)\alpha\sigma(z) - (T(x)\alpha\sigma(z) + T(y)\alpha\sigma(z))$

$= T(x)\alpha\sigma(z) - T(x)\alpha\sigma(z) + T(y)\alpha\sigma(z) - T(y)\alpha\sigma(z)$

$= \Phi_\alpha(x,z) + \Phi_\alpha(y,z)$

iii) $\Phi_\alpha(x,y+z) = T(x\alpha(y+z)) - T(x)\alpha\sigma(y+z)$

$= T(x\alpha y + x\alpha z) - (T(x)\alpha\sigma(y) + T(x)\alpha\sigma(z))$

$= T(x)\alpha\sigma(y) - T(x)\alpha\sigma(y) + T(x)\alpha\sigma(z) - T(x)\alpha\sigma(z)$

$= \Phi_\alpha(x,y) + \Phi_\alpha(x,z)$

iv) $\Phi_{\alpha+\beta}(x,y) = T(x(\alpha+\beta)y) - T(x)(\alpha+\beta)\sigma(y)$

$= T(x\alpha y + x\beta y) - (T(x)\alpha\sigma(y) + T(x)\beta\sigma(y))$

$= T(x\alpha y) - T(x)\alpha\sigma(y) + T(x\beta y) - T(x)\beta\sigma(y)$

$= \Phi_\alpha(x,y) + \Phi_\beta(x,y)$

Remark 2.4:

Note that T is left σ-centralizer of Γ-ring M if and only if $\Phi_\alpha(x,y)=0$ for all $x,y \in M$ and $\alpha \in \Gamma$.

3) The Main Results

In this section the research present the main results of this paper.

Theorem 3:

Let T be left (resp. right) Jordan σ-centralizer of completely prime Γ-ring M, then $\Phi_\alpha(x,y)=0$ for all $x,y \in M$ and $\alpha \in \Gamma$.

Proof:

Replace z by $x\alpha y$ in lemma 1 (v) , we get :

$T(x\alpha y\alpha x\alpha y + x\alpha y\alpha y\alpha x)= T((x\alpha y)\alpha (x\alpha y)) + T((x\alpha y)\alpha (y\alpha x))$

$= T(x\alpha y)\alpha \sigma(x\alpha y) + T(x\alpha y)\alpha \sigma(y\alpha x)$

$= T(x)\alpha \sigma(y)\alpha \sigma(x\alpha y) + T(x\alpha y)\alpha \sigma(y\alpha x)$ …(1)

$T(x\alpha y\alpha x\alpha y + x\alpha y\alpha y\alpha x)= T(x\alpha y\alpha x\alpha y + x\alpha (y\alpha y)\alpha x)$

$= T(x\alpha y)\alpha \sigma(x\alpha y) + T(x)\alpha \sigma(y\alpha y)\alpha \sigma(x)$

$= T(x\alpha y)\alpha \sigma(x\alpha y) + T(x)\alpha \sigma(y)\alpha \sigma(y\alpha x)$ …(2)

Compare (1) and (2) we get:

$T(x\alpha y)\alpha (x\alpha y) - T(x)\alpha \sigma(y)\alpha \sigma(x\alpha y) - T(x\alpha y)\alpha \sigma(y\alpha x) + T(x)\alpha \sigma(y)\alpha \sigma(y\alpha x)=0$

$[T(x\alpha y) - T(x)\alpha \sigma(y)]\alpha \sigma(x\alpha y) - [T(x\alpha y) - T(x)\alpha \sigma(y)] \alpha \sigma(y\alpha x) =0$

$\Phi_\alpha(x,y)\alpha \sigma [x,y]_\alpha =0$

Since M is completely prime we get

$\Phi_\alpha(x,y)=0$.

Corollary:

Every left (resp. right) Jordan σ-centralizer of completely prime Γ-ring M is left (resp. right) σ-centralizer of M.

Proof:

By using theorem 3 and remark 2.3.

In the following theorem the research present the relation between left Jordan triple σ-centralizer and left σ-centralizer on Γ-ring M.

Theorem 4:

Every left (resp. right) Jordan triple σ-centralizer of 2-tortion free completely prime Γ-ring M such that $x\alpha y\beta z= x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta \in \Gamma$ is left (resp. right) σ-centralizer of M.

Proof:

By theorem 1 (iv)

$T(x\alpha y\beta z+ z\alpha y\beta x)= T(x)\alpha \sigma(y)\beta \sigma(z) +T(z)\alpha \sigma(y)\beta \sigma(x)$

Replace z by $x\alpha y$ we get:

$T(x\alpha y\beta x\alpha y + x\alpha y\alpha y\beta x) = T(x\alpha y)\beta(x\alpha y) +T(x\alpha y)\alpha \sigma(y)\beta \sigma(x)$

$= T(x\alpha y)\beta \sigma(x)\alpha \sigma(y) +T(x)\alpha \sigma(y)\alpha \sigma(y)\beta \sigma(x)$ …(1)

On the other hand

$T(x\alpha y\beta x\alpha y + x\alpha y\alpha y\beta x)= T(x)\alpha \sigma(y)\beta \sigma(x\alpha y) + (x\alpha y)\alpha (y\beta x))$

$=T(x)\alpha \sigma(y)\beta \sigma(x)\alpha \sigma(y)+$

$T(x\alpha y)\alpha \sigma(y)\beta \sigma(x)$…(2)

Compare (1) and (2) we get:

$T(x\alpha y)\alpha \sigma(y)\beta \sigma(x)+ T(x)\alpha \sigma(y)\beta \sigma(x)\alpha \sigma(y)- T(x\alpha y)\beta \sigma(x)\alpha \sigma(y)- T(x)\alpha \sigma(y)\alpha \sigma(y)\beta \sigma(x)=0$

$[T(x\alpha y)- T(x)\alpha \sigma(y)]\alpha \sigma(y)\beta \sigma(x)-[T(x\alpha y)- T(x)\alpha \sigma(y)] \beta \sigma(x)\alpha \sigma(y)=0$

$\Phi_{\alpha}(x,y) \beta \sigma[x,y]_{\alpha}=0$

Since M is completely prime Γ-ring we get:

Either $\Phi_{\alpha}(x,y)=0$ or $\sigma[x,y]_{\alpha}=0$

If $\Phi_{\alpha}(x,y)=0$ then by remark 2.3 we have T is left σ-centralizer.

Now if $\sigma[x,y]_{\alpha}=0$ then M is commutative Γ-ring and by theorem 1(i) we get

$T(2x\alpha y) = 2T(x)\alpha \sigma(y)$

Hence M is 2-tortion free Γ-ring we get T is left σ-centralizer.

References

1) W. E. Barnes " On the Γ-rings of Nobusawa " Pacific J. Math. Vol. 18, pp. 411-422, 1966.
2) Y. Ceven " Jordan left derivations on completely prime gamma rings " C.U. Fen-Edebiyat Fakultesi Fen Bilimleri Dergisi Vol. 23, No. 2, pp. 39-43, 2002.
3) W. Cortes and C. Haetinger " On Lie ideals and left Jordan σ-centralizers of 2-torsion-free rings " Math. J. Okayama Univ. Vol. 51, pp. 111-119, 2009.
4) M. F. Hoque and A. C. Paul " On centralizers of semiprime gamma rings " International Math. Forum, Vol. 6, No. 13, pp. 627-638, 2011.
5) F. Jing " On derivations of Γ-ring " Qu Fu Shi Fan Daxue Xuebeo Ziran Kexue Ban, Vol. 13, No. 4, pp. 159-161, 1987.
6) N. Nobusawa " On a generalization of the ring theory " Osaka J. Math., Vol. 1, pp 81-89, 1964.
7) B. Zalar " On centralizers of semiprime rings " Comment. Math. Univ. Corolinae 32, pp. 609-614, 1991.

تمركزات-σ جوردان اليسارية على الحلقات- الاولية المتكاملة

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الخلاصة

لتكنM حلقة- Γ البحث قدم المفاهيم التالية تمركز-σ اليساري (اليميني على التوالي)، تمركز-σ جوردان اليساري (اليميني على التوالي) وتمركز-σ جوردان الثلاثي اليساري (اليميني على التوالي) على حلقة-Γ كما برهن البحث كل تمركز-σ جوردان اليساري (اليميني على التوالي) على الحلقة-Γ الاولية المتكاملةM هو تمركز-σ اليساري (اليميني على التوالي) علىM .