

# An Application of Differential Transform Method for Optimal Control Problems

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## ABSTRACT

In this paper, a numerical method is presented for finding the solution of some optimal control problems. Finding the solution of these problems need to solve the corresponding ordinary differential equations which are generally nonlinear. Indirect method is used in this work with the aid of differential transform method (DTM) for solving optimal control problems (OCPs). Since this method gives a closed form solution of the problem and avoids the round off errors, it can be considered as an efficient method for solving various kind of problems. Some examples are presented to show the efficiency of the proposed technique.

**Keywords:** differential transform method; optimal control problems; indirect method.

## 1. Introduction

Optimal control is an important branch of mathematics and applications for it abound in engineering, science and economics, and it is one of the most useful systematic methods for controller design [1]. It is now widely used in multi-discipline applications such as biological systems, communication networks [2]. Optimal control is well established in some areas, like trajectory planning in the aerospace field and robotics, or model predictive control in chemical industry [3]. As a result, more people will benefit greatly by learning to solve the optimal control problems numerically.

The essential elements of a control problem are:

- The system which is to be controlled;
- A system objective;
- A set of admissible controls (inputs);
- A performance function which measures the effectiveness of a given control action [1].

Several methods had been used to solve variational problems. For example, In [4] a way of estimating adjoint variables of OCPs by direct collection method had been described. In [5] a computational method was obtained for solving the OCP with time variables in the objective

function. The method is based on the combination of the control enhancing transform and the control parameterization technique. The method is efficient and supported by rigorous mathematical analysis. In [6] they developed an efficient and accurate method for solving a class of fractional OCPs by utilizing the Legendre basis and the operational matrices of fractional integration and multiplication and the Lagrange multiplier method for constrained optimization. They reduce the main problem to the problem of solving system of algebraic equations. In [7] he proposed framework to solve OCP for switched input affine nonlinear system. One specific feature of this class is that singular controls have to be considered to obtain the optimal solution. As singular controls are determined by an inclusion depending on the state and the co-state, he has reformulated the optimization problem using the augmented ( in case of path constraints) Hamiltonian system and a set of slack variables defining a set of complementarily constraints. This augmented optimization problem is then solved with a direct method dedicated to optimal control. In [8] a new method was introduced first the given problem is transformed into an equivalent problem, then the variational problem is solved approximately by utilizing the Legendre orthonormal basis, operational matrix of Riemann-Liouville fractional integration, Gauss quadrature formula and Newton's iterative formula for solving non-linear system of equations. The convergence of the method is extensively discussed. In [9] numerical algorithms were presented to solve the underlying stochastic partial differential equations and obtained rigorous error estimates. The problem of optimal control was posed and designed optimization of a caustic linear under uncertainty to reduce the radiated engine noise given the source. Other works can be found in [10-16].

## 2. Differential Transform Method (DTM)

Differential Transform Method can easily be applied to linear or nonlinear problems and reduces the size of computational work. Without this method exact solutions may be obtained without any need of further difficult computation and it is a useful tool for analytical and numerical solution [17].

**Definition [17]:** The differential transform of the function  $y(x)$  for the  $k$ th derivative is defined as follows:

$$Y(k) = \frac{1}{k!} \left[ \frac{d^k y(x)}{dx^k} \right]_{x=x_0} \quad (1)$$

where  $y(x)$  is the original function and  $Y(k)$  is the transformed function.

The inverse differential transform of  $Y(k)$  is defined as

$$y(x) = \sum_{k=0}^{\infty} (x - x_0)^k Y(k) \quad (2)$$

Note that, the substitution of (1) into (2) gives:

$$y(x) = \sum_{k=0}^{\infty} (x - x_0)^k \frac{1}{k!} \left[ \frac{d^k y(x)}{dx^k} \right]_{x=x_0} \quad (3)$$

Which is Taylor's series for  $y(x)$  at  $x = x_0$ .

From this definition we can get the following table [11]

**Table 1: Fundamental operations**

Original function	Transformed function
$y(x) = u_1(x) \pm u_2(x)$	$Y(k) = U_1(k) \pm U_2(k)$
$y(x) = \alpha u(x), \alpha = \text{constant}$	$Y(k) = \alpha U(k)$
$y(x) = \frac{du(x)}{dx}$	$Y(k) = (k + 1)U(k)$
$y(x) = \frac{d^m u(x)}{dx^m}$	$Y(k) = \frac{(k + m)!}{k!} U(k + m)$
$y(x) = u_1(x) \cdot u_2(x)$	$Y(k) = U_1(k) \times U_2(k) = \sum_{L=0}^k U_1(L)U_2(k - L)$
$y(x) = 1$	$Y(k) = \delta(k), \delta(k) = \begin{cases} 1, k = 0 \\ 0, k \neq 0 \end{cases}$
$y(x) = x^n$	$Y(k) = \delta(k - n), \delta(k - n) = \begin{cases} 1, k = n \\ 0, k \neq n \end{cases}$
$y(x) = \int_{x_0}^x u(t) dt$	$Y(k) = \frac{U(k - 1)}{k}, k \geq 1$
$y(x) = \int_{x_0}^x u_1(t)u_2(t) dt$	$Y(k) = \frac{1}{k} \left[ \sum_{k_1=0}^k U_1(k_1)U_2(k - k_1 - 1), k \geq 1 \right]$

### 3. Solution of OCPs With Fixed Final Time Using Indirect Method

In this free-final-time problem that being talked first to drive the necessary conditions for optimal control. Fixed-final-time problems were treated as an equivalent variation with one more state for time. However, for numerical methods, fixed-final-time problems are the general form and we solve the free-final-time problem by converting it into a fixed-final-time problem. The reason is simple: when dealing with OCPs, it is inevitable to do numerical integration (either by indirect method ). Therefore, a time interval must be specified for these methods [2]. In an indirect method, the OCPs is used to determine the first-order optimality conditions of the OCP. The indirect, approach leads to a multiple-point boundary-value problem that is solved to determine candidate optimal trajectories called extremals. Each of the computed extremals is than examined to see if it is a local minimum, maximum, or a saddle point. In an indirect method the optimal solution is found by solving a system of

differential equations that satisfies endpoint and/or interior point conditions. On the other hand, in an indirect method the optimal solution is found by transcribing an infinite-dimensional optimization problem to a finite-dimensional optimization problem [18].

In this section the performance of the proposed method discussed in the previous section will be compared using several examples.

**Example 4.1** [1] The first example contains one state variable and one control variable. The performance index to be minimized is

$$J = \int_0^1 (x^2 + u^2) dt \tag{1}$$

and the linear system is given by  $\dot{x} = u, x(0) = 1$

the exact solution to this problem is given by

$$x(t) = \frac{\cosh(t-r)}{\cosh 1} \quad \text{and} \quad u(t) = \frac{\sinh(t-r)}{\cosh 1}$$

The implementation of the proposed method requires the following information:

- The Hamiltonian function is defined as  $H = x^2 + u^2 + \lambda u$
- The sufficient condition for optimality  $\frac{\partial H}{\partial u} = 0 \Rightarrow u = -\frac{1}{2}\lambda$
- The necessary optimality conditions

$$\dot{x} = -\frac{1}{2}\lambda, \quad \dot{\lambda} = -2x \tag{2}$$

With the initial condition  $x(0) = 1$  and the final condition  $\lambda(1) = 0$

by using Euler-Lagrange equation:  $F_x - \frac{d}{dt} F_x = 0$  for problem (1) we have

$$\begin{cases} \dot{x} = -\frac{\lambda}{2} \\ \dot{\lambda} = -2x \end{cases} \Rightarrow \begin{cases} 2x + \lambda = 0 \\ \dot{\lambda} + 2x = 0 \end{cases} \tag{3}$$

By using (2) in (3) we get

$$\begin{cases} \dot{x} - x = 0 \\ \dot{\lambda} - \lambda = 0 \end{cases} \tag{4}$$

Then using DTM we have:

$$\begin{cases} ((k+1)(k+2)X(k+2) - X(k)) = 0 \\ ((k+1)(k+2)\lambda(k+2) - \lambda(k)) = 0 \end{cases} \tag{5}$$

$$\begin{cases} X(0) = 1, X(1) = \alpha \\ \lambda(0) = \beta, \lambda(1) = 0 \end{cases} \tag{6}$$

Substituting (6) into (5) and by an iterative procedure, we achieve

$$X(2) = 0.5, X(3) = \alpha/6, X(4) = 1/24, X(5) = \alpha/120, X(6) = 1/720, X(7) = \alpha/5040,$$

$$X(8) = 1/40320, X(9) = \alpha/362880, X(10) = 1/3628800$$

$$\lambda(2) = \beta/2, \quad \lambda(3) = 0, \quad \lambda(4) = \beta/24, \quad \lambda(5) = 0, \quad \lambda(6) = \beta/720,$$

$$\lambda(7) = 0, \quad \lambda(8) = \beta/40320,$$

$$\lambda(9) = 0, \quad \lambda(10) = \beta/3628800$$

Substituting all of  $X(k)$  and  $\lambda(k)$  into

$$x(t) = \sum_{k=0}^{\infty} X(k).t^k$$

$$\lambda(t) = \sum_{k=0}^{\infty} \lambda(k).t^k$$

The 11-term of DTM series solution of  $x(t)$  and  $\lambda(t)$  can be given by

$$x(t) = 1 + \alpha t + 0.5t^2 + \alpha t^3/6 + t^4/24 + \alpha t^5/120 + t^6/720 + \alpha t^7/5040 + t^8/40320 + \alpha t^9/362880 + t^{10}/3628800$$

(7)

$$\lambda(t) = \beta + \beta t^2/2 + \beta t^4/24 + \beta t^6/720 + \beta t^8/40320 + \beta t^{10}/3628800 \quad (8)$$

This gives the approximation of the  $x(t)$  and  $\lambda(t)$  in a series form. Now to find the constants  $\alpha$  and  $\beta$ , the boundary conditions in (6) is imposed on the approximation solutions in (7) and (8). We have

$$\alpha = -0.7615964170509217, \beta = 0.227079732213287 \quad (9)$$

Replacing (9) into (7) and (8), an approximation solution is obtained for  $x(t)$  and  $\lambda(t)$  by the following tables:

**Table 2: an approximation solutions of  $x(t)$**

$t$	$x(t) = \sum_{k=0}^{10} X(k).t^k$	$x(t) = \frac{\cosh(1-t)}{\cosh 1}$
0	1	1
0.1	0.928717755161836	0.928717756619608
0.2	0.866730429770150	0.866730432700284
0.3	0.813417633837793	0.813417638269582
0.4	0.768245794984694	0.768245800961792
0.5	0.730762818271444	0.730762825846359
0.6	0.700593561509183	0.700593570709864
0.7	0.677436080816131	0.677436091506664
0.8	0.661058608979083	0.661058620401396
0.9	0.651297236645535	0.651297246158581
1	0.648054273663885	0.648054273663885

**Table 3: an approximation solutions of  $\lambda(t)$**

$t$	$\lambda(t) = \sum_{k=0}^{10} \lambda(k).t^k$	$\lambda(t) = \frac{-2t \sinh(t-1)}{\cosh 1}$
0	0.227079732213287	1.523188311911530
0.1	0.227882577315661	1.330477102165283
0.2	0.228964480623469	1.151081754446303
0.3	0.228334602966420	0.983206819817760
0.4	0.223984973096724	0.825172149509178
0.5	0.213870391252708	0.675396079422822

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0.6	0.195887960191300	0.532379600307790
0.7	0.167856038989289	0.394691355171603
0.8	0.127492409373288	0.260953313778318
0.9	0.072391436900565	0.129826980858764
1.0	0	0

We notice here diverge the approximation solution for  $\lambda(t)$  from the exact solution for that we'll calculate  $\beta$  for 0.1, 0.2, ..., 0.9 as shown in the following table:

**Table 4: an approximation solutions of  $\lambda(t)$**

$t$	$\beta$	$\lambda(t) = \sum_{k=0}^{10} \lambda(k).t^k$	$\lambda(t) = \frac{-2\sinh(t-1)}{\cosh 1}$
0	0.227079732213287	0.227079732213287	1.52318831191153 0
0.1	26.58737850784724	1.33047710216528 3	1.33047710216528 3
0.2	57.36262384888915	1.15108175444630 3	1.15108175444630 3
0.3	21.68590741688903 4	1.98320681981776 0	1.98320681981776 0
0.4	10.17821646014966 3	0.82517214950917 8	0.82517214950917 2
0.5	5.291995859411184	0.67539607942282 2	0.67539607942282 2
0.6	2.870509119534365	0.53237960030779 0	0.53237960030779 0
0.7	1.546784078420837	0.39469135517160 3	0.39469135517160 3
0.8	0.773344067492957	0.26095331377831 8	0.26095331377831 8
0.9	0.299771559204696	0.12982698085876 4	0.12982698085876 4
1	0	0	0

Higher accuracy is also obtained using more components of  $x(t)$ , for example if  $n=15$  we get  $\alpha = -0.761594155955726$  and for  $n=20$  we get  $\alpha = -0.761594155955766$  as shown in the following table:

**Table 5: an approximation solutions of  $x(t)$  for  $n=15,20$**

$t$	$x(t) = \sum_{k=0}^{15} X(k).t^k$	$x(t) = \sum_{k=0}^{20} X(k).t^k$	$x(t) = \frac{\cosh(1-t)}{\cosh 1}$
0	1	1	1

0.	0.92871775661961	0.92871775661960	0.92871775661960
1	2	8	8
0.	0.86673043270029	0.86673043270028	0.86673043270028
2	1	3	4
0.	0.81341763826959	0.81341763826958	0.81341763826958
3	4	2	2
0.	0.76824580096180	0.76824580096179	0.76824580096179
4	8	2	2
0.	0.73076282584637	0.73076282584635	0.73076282584635
5	9	8	9
0.	0.70059357070988	0.70059357070986	0.70059357070986
6	8	3	4
0.	0.67743609150669	0.67743609150666	0.67743609150666
7	3	3	4
0.	0.66105862040142	0.66105862040139	0.66105862040139
8	9	5	6
0.	0.65129724615861	0.65129724615858	0.65129724615858
9	3	0	1
1	0.64805427366388	0.64805427366388	0.64805427366388
	5	4	5

**Example 4.2 [15]** In this example studied is a finite time linear quadrature problem containing one state variable and one control variable. The problem is given by

Minimize  $J = \int_0^1 u^2 dt$  subject to  $\dot{x} = ax + u, x(0) = 1$ ,  $a$ -constant and the

exact solution is  $x = e^{at}, u = \frac{-2ae^{-at}}{1-e^{-2a}}$

To start with, the following should be found

- The Hamiltonian:  $H = u^2 + \lambda(x + u)$
- The adjoint equation:  $\dot{\lambda} = -a\lambda$
- The sufficient condition for optimality:  $\frac{\partial H}{\partial u} = 0 \Rightarrow 2u + \lambda = 0$

Therefore  $u = -\frac{1}{2}\lambda$

The final system is: 
$$\begin{cases} \dot{x} = ax - \frac{\lambda}{2} \\ \dot{\lambda} = -a\lambda \end{cases} \quad (1)$$

With the condition:  $x(0) = 1$

By using Euler-Lagrange equation for (1) we have:

$$\begin{cases} \dot{x} - a^2x = 0 \\ \dot{\lambda} - a^2\lambda = 0 \end{cases} \quad (2)$$

Using DTM we get: 
$$\begin{cases} (k+2)(k+1)X(k+2) - \alpha^2 X(k) = 0 \\ (k+2)(k+1)\lambda(k+2) - \alpha^2 \lambda(k) = 0 \end{cases} \quad (3)$$

$$\begin{cases} X(0) = 1, X(1) = \alpha \\ \lambda(0) = \beta, \lambda(1) = 0 \end{cases} \quad (4)$$

Substituting (4) into (3) and by iterative procedure we achieve:

$$X(2) = \alpha^2/2, X(3) = \alpha^2 \alpha/6, X(4) = \alpha^2/24, X(5) = \alpha^2 \alpha/120, X(6) = \alpha^6/720, \\ X(7) = \alpha^2 \alpha/5040, X(8) = \alpha^6/40320, X(9) = \alpha^2 \alpha/362880, X(10) = \alpha^{10}/3628800$$

$$\lambda(3) = \lambda(5) = \lambda(7) = \lambda(9) = 0, \lambda(2) = \alpha^2 \beta/2, \lambda(4) = \alpha^4 \beta/24, \lambda(6) = \alpha^6 \beta/720, \\ \lambda(8) = \alpha^8 \beta/40320, \lambda(10) = \alpha^{10} \beta/3628800$$

Substituting all of  $X(k)$  and  $\lambda(k)$  into

$$x(t) = \sum_{k=0}^{\infty} X(k).t^k \quad (5)$$

$$\lambda(t) = \sum_{k=0}^{\infty} \lambda(k).t^k \quad (6)$$

This gives the approximation of the  $x(t)$  and  $\lambda(t)$  in a series form. Now to find the constants  $\alpha$  and  $\beta$ , the boundary conditions at  $x=1, \lambda=\beta$  is imposed on the approximation solution of  $x(t)$  and  $\lambda(t)$ . At first I'll take  $\alpha=1$ , we get  $\alpha=1.000000023240839$  an approximation solution is obtained for  $x(t)$  by the following table:



**Table 6: an approximation solutions of  $x(t)$**

$t$	$x(t) = \sum_{k=0}^{10} X(k).t^k$	$x(t) = e^t$
0	1	1
0.1	1.105170920403607	1.105170918075648
0.2	1.221402762839387	1.221402758160170
0.3	1.349858814653265	1.349858807576003
0.4	1.491824707186412	1.491824697641270
0.5	1.648721282798058	1.648721270700128
0.6	1.822118815091221	1.822118800390509
0.7	2.013752724574703	2.013752707470477
0.8	2.225540946828011	2.225540928492468
0.9	2.459603126519210	2.459603111156950
1	2.718281828459046	2.718281828459046

We notice here diverge the approximation solution for  $\lambda(t)$  from the exact solution for that we'll calculate  $\beta$  for 0.1, 0.2, ..., 0.9 as shown in the following table:

**Table 7: an approximation solutions of  $\lambda(t)$**

$t$	$\beta$	$\lambda(t) = \sum_{k=0}^{10} \lambda(k).t^k$	$\lambda(t) = \frac{4e^{-t}}{1-e^{-2}}$
0	4.62607057099866 3	4.62607057099866 3	4.62607057099866 3
0.1	4.16501251608172 0	4.18584175111456 9	4.18584175111456 7
0.2	3.71317573080689 3	3.78750624238562 5	3.78750624238562 5
0.3	3.27917372598141 6	3.42707736915529 1	3.42707736915528 8
0.4	2.87024518167364 6	3.10094783811593 9	3.10094783811594 0
0.5	2.49172902452046 6	2.80585363530501 7	2.80585363530501 8
0.6	2.14686554573618 8	2.53884235875620 4	2.53884235875620 4
0.7	1.83689293310670 3	2.29723865985986 9	2.29723865985987 0
0.8	1.56135289142438 9	2.07862749759999 4	2.07862749759999 3
0.9	1.31850180210251 7	1.88081993798692 6	1.88081993798692 5

<b>1</b>	<b>1.10573740977021</b>	<b>1.70183625647864</b> 5	<b>1.70183625647864</b> 4
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For the case of  $\lambda(0) = 1, \lambda(1) = \beta$  if we substituting these conditions into (6) and by an iterative procedure, we achieve  
 $\lambda(2) = \alpha^2/2, \lambda(3) = \beta\alpha^2/6, \lambda(4) = \alpha^4/24, \lambda(5) = \beta\alpha^4/120, \lambda(6) = \alpha^6/720,$   
 $\lambda(7) = \beta\alpha^6/5040, \lambda(8) = \alpha^8/40320, \lambda(9) = \beta\alpha^8/362880, \lambda(10) = \alpha^{10}/3628800$

Substituting all of  $\lambda(k)$  into (5) for  $\alpha = 1$ , we notice here diverge the approximation solution for  $\lambda(t)$  from the exact solution for that we'll calculate  $\beta$  for 0.1, 0.2, ..., 0.9 as shown in the following table:

**Table 8: the approximation solution of  $\lambda(t)$**

$t$	$\beta$	$\lambda(t) = \sum_{k=0}^{10} \lambda(k).t^k$	$\lambda(t) = \frac{4e^{-t}}{1-e^{-2}}$
0	-	1	4.62607057099866 3
0.1	31.75542365534079 5	4.18584175111456 7	4.18584175111456 7
0.2	13.74537183326110 8	3.78750624238562 5	3.78750624238562 5
0.3	7.821281229128291	3.42707736915528 8	3.42707736915528 8
0.4	4.917502201198365	3.10094783811594 1	3.10094783811594 0
0.5	3.220577219647861	2.80585363530501 7	2.80585363530501 8
0.6	2.125765374861297	2.53884135875620 4	2.53884135875620 4
0.7	1.37304249660979	2.29723865985987 0	2.29723865985987 0
0.8	0.834576692543772	2.07862749759999 3	2.07862749759999 3
0.9	0.436167814283181	1.88081993798692 5	1.88081993798692 5
1	0.135088041117792	1.70183625647864 4	1.70183625647864 3

Now solve all the above but with a-constant, first for  $x(t)$  and  $n = 10$  to get

$$\alpha = \frac{[e^2 - (1 + \alpha^2/2 + \alpha^4/24 + \alpha^6/720 + \alpha^8/40320 + \alpha^{10}/3628800)]}{[1 + \alpha^2/6 + \alpha^4/120 + \alpha^6/5040 + \alpha^8/362880]}$$

$$x(0.1) = 1 + (0.1) \alpha + (0.1)^2 a^2 / 2 + (0.1)^3 a^2 \alpha / 6 + (0.1)^4 a^4 / 24 + (0.1)^5 a^4 \alpha / 120 + (0.1)^6 a^6 / 720 + (0.1)^7 a^6 \alpha / 5040 + (0.1)^8 a^8 / 40320 + (0.1)^9 a^8 \alpha / 362880 + (0.1)^{10} a^{10} / 3628800$$

$$x(0.2) = 1 + (0.2) \alpha + (0.2)^2 a^2 / 2 + (0.2)^3 a^2 \alpha / 6 + (0.2)^4 a^4 / 24 + (0.2)^5 a^4 \alpha / 120 + (0.2)^6 a^6 / 720 + (0.2)^7 a^6 \alpha / 5040 + (0.2)^8 a^8 / 40320 + (0.2)^9 a^8 \alpha / 362880 + (0.2)^{10} a^{10} / 3628800$$

$$x(0.3) = 1 + (0.3) \alpha + (0.3)^2 a^2 / 2 + (0.3)^3 a^2 \alpha / 6 + (0.3)^4 a^4 / 24 + (0.3)^5 a^4 \alpha / 120 + (0.3)^6 a^6 / 720 + (0.3)^7 a^6 \alpha / 5040 + (0.3)^8 a^8 / 40320 + (0.3)^9 a^8 \alpha / 362880 + (0.3)^{10} a^{10} / 3628800$$

$$x(0.4) = 1 + (0.4) \alpha + (0.4)^2 a^2 / 2 + (0.4)^3 a^2 \alpha / 6 + (0.4)^4 a^4 / 24 + (0.4)^5 a^4 \alpha / 120 + (0.4)^6 a^6 / 720 + (0.4)^7 a^6 \alpha / 5040 + (0.4)^8 a^8 / 40320 + (0.4)^9 a^8 \alpha / 362880 + (0.4)^{10} a^{10} / 3628800$$

$$x(0.5) = 1 + (0.5) \alpha + (0.5)^2 a^2 / 2 + (0.5)^3 a^2 \alpha / 6 + (0.5)^4 a^4 / 24 + (0.5)^5 a^4 \alpha / 120 + (0.5)^6 a^6 / 720 + (0.5)^7 a^6 \alpha / 5040 + (0.5)^8 a^8 / 40320 + (0.5)^9 a^8 \alpha / 362880 + (0.5)^{10} a^{10} / 3628800$$

$$x(0.6) = 1 + (0.6) \alpha + (0.6)^2 a^2 / 2 + (0.6)^3 a^2 \alpha / 6 + (0.6)^4 a^4 / 24 + (0.6)^5 a^4 \alpha / 120 + (0.6)^6 a^6 / 720 + (0.6)^7 a^6 \alpha / 5040 + (0.6)^8 a^8 / 40320 + (0.6)^9 a^8 \alpha / 362880 + (0.6)^{10} a^{10} / 3628800$$

$$x(0.7) = 1 + (0.7) \alpha + (0.7)^2 a^2 / 2 + (0.7)^3 a^2 \alpha / 6 + (0.7)^4 a^4 / 24 + (0.7)^5 a^4 \alpha / 120 + (0.7)^6 a^6 / 720 + (0.7)^7 a^6 \alpha / 5040 + (0.7)^8 a^8 / 40320 + (0.7)^9 a^8 \alpha / 362880 + (0.7)^{10} a^{10} / 3628800$$

$$x(0.8) = 1 + (0.8) \alpha + (0.8)^2 a^2 / 2 + (0.8)^3 a^2 \alpha / 6 + (0.8)^4 a^4 / 24 + (0.8)^5 a^4 \alpha / 120 + (0.8)^6 a^6 / 720 + (0.8)^7 a^6 \alpha / 5040 + (0.8)^8 a^8 / 40320 + (0.8)^9 a^8 \alpha / 362880 + (0.8)^{10} a^{10} / 3628800$$

$$x(0.9) = 1 + (0.9) \alpha + (0.9)^2 a^2 / 2 + (0.9)^3 a^2 \alpha / 6 + (0.9)^4 a^4 / 24 + (0.9)^5 a^4 \alpha / 120 + (0.9)^6 a^6 / 720 + (0.9)^7 a^6 \alpha / 5040 + (0.9)^8 a^8 / 40320 + (0.9)^9 a^8 \alpha / 362880 + (0.9)^{10} a^{10} / 3628800$$

And for  $\lambda(t)$  we have a special  $\beta$  for each one where

$$\beta_{0,1} = \left( \frac{4\alpha e^{-(0.1)\alpha}}{1 - e^{-2\alpha}} \right) / \left( 1 + \frac{(0.1)^2 a^2}{2} + \frac{(0.1)^4 a^4}{24} + \frac{(0.1)^6 a^6}{720} + \frac{(0.1)^8 a^8}{40320} + \frac{(0.1)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.2} = \left( \frac{4ae^{-(0.2)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.2)^2 a^2}{2} + \frac{(0.2)^4 a^4}{24} + \frac{(0.2)^6 a^6}{720} + \frac{(0.2)^8 a^8}{40320} + \frac{(0.2)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.3} = \left( \frac{4ae^{-(0.3)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.3)^2 a^2}{2} + \frac{(0.3)^4 a^4}{24} + \frac{(0.3)^6 a^6}{720} + \frac{(0.3)^8 a^8}{40320} + \frac{(0.3)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.4} = \left( \frac{4ae^{-(0.4)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.4)^2 a^2}{2} + \frac{(0.4)^4 a^4}{24} + \frac{(0.4)^6 a^6}{720} + \frac{(0.4)^8 a^8}{40320} + \frac{(0.4)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.5} = \left( \frac{4ae^{-(0.5)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.5)^2 a^2}{2} + \frac{(0.5)^4 a^4}{24} + \frac{(0.5)^6 a^6}{720} + \frac{(0.5)^8 a^8}{40320} + \frac{(0.5)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.6} = \left( \frac{4ae^{-(0.6)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.6)^2 a^2}{2} + \frac{(0.6)^4 a^4}{24} + \frac{(0.6)^6 a^6}{720} + \frac{(0.6)^8 a^8}{40320} + \frac{(0.6)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.7} = \left( \frac{4ae^{-(0.7)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.7)^2 a^2}{2} + \frac{(0.7)^4 a^4}{24} + \frac{(0.7)^6 a^6}{720} + \frac{(0.7)^8 a^8}{40320} + \frac{(0.7)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.8} = \left( \frac{4ae^{-(0.8)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.8)^2 a^2}{2} + \frac{(0.8)^4 a^4}{24} + \frac{(0.8)^6 a^6}{720} + \frac{(0.8)^8 a^8}{40320} + \frac{(0.8)^{10} a^{10}}{3628800} \right)$$

$$\beta_{0.9} = \left( \frac{4ae^{-(0.9)a}}{1 - e^{-2a}} \right) / \left( 1 + \frac{(0.9)^2 a^2}{2} + \frac{(0.9)^4 a^4}{24} + \frac{(0.9)^6 a^6}{720} + \frac{(0.9)^8 a^8}{40320} + \frac{(0.9)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.1) = \beta_{0.1} \left( 1 + \frac{(0.1)^2 a^2}{2} + \frac{(0.1)^4 a^4}{24} + \frac{(0.1)^6 a^6}{720} + \frac{(0.1)^8 a^8}{40320} + \frac{(0.1)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.2) = \beta_{0.2} \left( 1 + \frac{(0.2)^2 a^2}{2} + \frac{(0.2)^4 a^4}{24} + \frac{(0.2)^6 a^6}{720} + \frac{(0.2)^8 a^8}{40320} + \frac{(0.2)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.3) = \beta_{0.3} \left( 1 + \frac{(0.3)^2 a^2}{2} + \frac{(0.3)^4 a^4}{24} + \frac{(0.3)^6 a^6}{720} + \frac{(0.3)^8 a^8}{40320} + \frac{(0.3)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.4) = \beta_{0.4} \left( 1 + \frac{(0.4)^2 a^2}{2} + \frac{(0.4)^4 a^4}{24} + \frac{(0.4)^6 a^6}{720} + \frac{(0.4)^8 a^8}{40320} + \frac{(0.4)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.5) = \beta_{0.5} \left( 1 + \frac{(0.5)^2 a^2}{2} + \frac{(0.5)^4 a^4}{24} + \frac{(0.5)^6 a^6}{720} + \frac{(0.5)^8 a^8}{40320} + \frac{(0.5)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.6) = \beta_{0.6} \left( 1 + \frac{(0.6)^2 a^2}{2} + \frac{(0.6)^4 a^4}{24} + \frac{(0.6)^6 a^6}{720} + \frac{(0.6)^8 a^8}{40320} + \frac{(0.6)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.7) = \beta_{0.7} \left( 1 + \frac{(0.7)^2 a^2}{2} + \frac{(0.7)^4 a^4}{24} + \frac{(0.7)^6 a^6}{720} + \frac{(0.7)^8 a^8}{40320} + \frac{(0.7)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.8) = \beta_{0.8} \left( 1 + \frac{(0.8)^2 a^2}{2} + \frac{(0.8)^4 a^4}{24} + \frac{(0.8)^6 a^6}{720} + \frac{(0.8)^8 a^8}{40320} + \frac{(0.8)^{10} a^{10}}{3628800} \right)$$

$$\lambda(0.9) = \beta_{0.9} \left( 1 + \frac{(0.9)^2 a^2}{2} + \frac{(0.9)^4 a^4}{24} + \frac{(0.9)^6 a^6}{720} + \frac{(0.9)^8 a^8}{40320} + \frac{(0.9)^{10} a^{10}}{3628800} \right)$$

#### 4. Conclusion

The Indirect method with DTM was employed for finding the solution of the ordinary differential equations which arise from problems of optimal controls. The present study has confirmed that the Indirect method offers great advantages of straightforward applicability, computational efficiency and high accuracy. The Indirect Method needs

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less work in comparison with the traditional methods. Mathematics and Matlab have been used for computation and simulation in this paper.

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## تطبيق طريقة التحويل التفاضلي لمسائل السيطرة المثلى

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### المستخلص

في هذا البحث، تم تقديم طريقة عددية لإيجاد الحل لبعض مسائل السيطرة المثلى. إيجاد الحل لهذه المسائل يحتاج حل المعادلات التفاضلية الاعتيادية المرافقة لها والتي تكون بصورة عامة لا خطية. استخدمت طريقة غير مباشرة لحل مسائل السيطرة المثلى في هذا البحث. وبما إن الطريقة تعطي شكل قريب لحل هذه المسائل وتتجنب أخطاء التدوير، يمكن اعتبارها كطريقة كفوءة لحل أنواع متعددة من هذه المسائل. تم تقديم بعض المسائل لبيان كفاءة التقنية المقترحة.