Reduction of Order Method for Solving Some Time-Fractional Differential Equations

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Abstract
The fractional derivative is considered in modified Riemann-Liouville derivative sense. A reduction of order method is used for constructing exact solution of some time fractional differential equations. More new soliton solution is obtained for time-fractional Klein-Gordon equation, time-fractional Burgers equation and time-fractional Hirota-Satsuma coupled KdV system. This method can be applied to many other nonlinear fractional partial differential equations in mathematical physics.
Keywords: Modified Riman-Liouville derivative, time-fractional Klein-Gordon equation, time-fractional Burgers equation and time-fractional Hirota-Satsuma coupled KdV system.

1-Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. The nonlinear fractional differential equations in mathematical physics have played a major role in various areas. These equations appear in a great variety of contexts, such as physics, biology, engineering, fluid flow, signal processing, control theory and fractional dynamics. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations so far. For example, these methods include the first integral method [1], the (G'/G)-expansion method [2], the functional variable method [3],],the invariant subspace method [4], the exp-function method [5], the integral transform method [6], the fractional sub-equation method [7], the generalized Exp-function method[8], the variational iteration method [9], the improved (G'/G)-expansion method [10], the generalized (G'/G)-expansion method [11],the homotopy analysis method[12], the expanding perturbation approach [13], the extended trial equation method[14]. Recently, M.Saravi [15] introduced a new method called pseudo-first integral method to look for traveling wave solutions of non linear partial differential equations in this method the author use part of the first integral method [16], and then by reduction of order method [17].In this paper, we propose a reduction of order method to establish exact solutions for time-fractional partial differential equations in the sense modified Riemann-Liouville derivative by Jumarie [18,19].To illustrate the validity and advantages of proposed method , we apply it to the time-fractional Klein-Gordon equation, the time-fractional Burgers equation and the time-fractional generalized Hirota-Satsuma coupled KdV system.

2- The modified Riemann-Liouville derivative and reduction of order method

The Jumarie's modified Rieman-Liouville derivative of order α is defined by the expression [19]

$$D_x^\alpha f(x) = \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}(f(\xi)-f(0))d\xi, \quad \alpha < 0 \right. \tag{1}$$

$$D_x^\alpha f(x) = \left\{ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha-1}(f(\xi)-f(0))d\xi, \quad 0 < \alpha < 1, \right. \tag{2}$$

$$D_x^\alpha f(x) = \left\{ [f^{(\alpha-n)}(x)]^{(n)} \quad , \ n \le \alpha < n+1, n \ge 1. \right. \tag{3}$$

Some properties for the proposed modified Riemann-Liouville derivative are listed in [19] as follows:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha} \ , \ \gamma > 0, \tag{4}$$

$$D_x^\alpha (f(x)g(x)) = g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x) \tag{5}$$

$$D_x^\alpha f[g(x)] = f'_g[g(x)]D_x^\alpha g(x) = D_g^\alpha f[g(x)](g'(x))^\alpha \tag{6}$$

Where $f : R \to R, \ x \to f(x)$ denote a continuous (but not necessarily differentiable) function. We now describe the proposed method for finding exact solutions of nonlinear time-fractional differential equations as follows. Let us consider the time fractional differential equation with independent variables $x = (x_1, x_2, \ldots x_m, t)$ and a dependent variable u,

$$F(u, D_t^\alpha u, u_{x_1}, u_{x_2}, \ldots, D_t^{2\alpha} u, u_{x_1 x_2}, \ldots) = 0 \tag{7}$$

Using the variable transformation [1]:

$$u = (x_1, x_2, \ldots x_m, t) = U(\xi), \xi = x_1 + l_1 x_2 + \ldots + l_{m-1} x_m - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \tag{8}$$

Where $k, l,$ and λ are constants. The fractional differential Eq. (7) is reduced to a nonlinear ordinary differential equation

$$Q = (U(\xi), U'(\xi), U''(\xi), \ldots), \tag{9}$$

Where " ' " $= \frac{d}{d\xi}$ By introducing a new independent variable $p = U'(\xi)$ then $U''(\xi) = p\frac{dp}{du}$, which change Eq.(9) to an ordinary differential equation of the form

$$N(u,p,p\frac{dp}{du})=0 \qquad (10)$$

Now, we seek a solution for Eq.(10) by a desired method that been considered in the ordinary differential equation textbooks [20].

3- Applications

In this section, we present three examples:

Example 1: **Time-fractional Klein-Gordon equation**

The following time-fractional Klein-Gordon equation:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}=\frac{\partial^2 u(x,t)}{\partial x^2}+a\,u(x,t)+c\,u^3(x,t), \quad t>0\,,0<\alpha\le 1 \qquad (11)$$

We introduce as in [1] the following transformations

$$u(x,t)=U(\xi)\ \ ,\ \xi=lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \qquad (12)$$

Where l,λ is constant. Substituting Eq.(12) into Eq.(11), then Eq.(11) is reduced to an ordinary differential equation:

$$\lambda^2\,\frac{\partial^2 U}{\partial \xi^2}=l^2\,\frac{\partial^2 U}{\partial \xi^2}+a\,U+c\,U^3 \qquad (13)$$

This is a non linear second order ordinary differential equation free of ξ .

The substitution $U'=\frac{dU}{d\xi}=p$ and $U''=\frac{d^2U}{d\xi^2}=p\frac{dp}{dU}$, reduces Eq.(13) to

$$\lambda^2\,p\frac{dp}{dU}=l^2 p\,\frac{dp}{dU}+a\,U+c\,U^3 \qquad (14)$$

This can be written as

$$(\lambda^2-l^2)pdp=aU+cU^3 \qquad (15)$$

Integrating Eq.(15) once and let the constants of integration to be zero, then

$$p^2=\frac{a}{\lambda^2-l^2}\,U^2+c\frac{U^4}{2} \qquad (16)$$

That is,

$$p=\sqrt{\frac{c}{2}}\ \ U\,\sqrt{U^2+\frac{2a}{c(\lambda^2-l^2)}} \qquad (17)$$

Since $p=U'=\frac{dU}{d\xi}$, then

$$\frac{dU}{U\,\sqrt{U^2+\frac{2a}{c(\lambda^2-l^2)}}}=\sqrt{\frac{c}{2}}\ \ d\xi \qquad (18)$$

The solution of Eq.(18) is

$$-\sqrt{\frac{c(\lambda^2-l^2)}{2a}}\,\mathrm{csch}^{-1}\left|U\sqrt{\frac{c(\lambda^2-l^2)}{2a}}\right|=\sqrt{\frac{c}{2}}\,\xi+\xi_0 \qquad (19)$$

By setting $\xi_0=0$, $U(\xi)=u(x,t)$ and $\xi=lx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}$, then Eq.(19)

$$u(x,t)=\sqrt{\frac{2a}{c(\lambda^2-l^2)}}\,\mathrm{csch}\left(\sqrt{\frac{a}{\lambda^2-l^2}}\left(lx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\right) \qquad (20)$$

This is a solution for Eq.(11).

Remark 1: In [1], the author solved Eq.(11) by first integral method and obtained four exact solutions different from our solution Eq.(20), while in [3], the authors solved Eq.(11) by functional variable method and obtained four exact solutions, one of them is agreement with our solution Eq.(20).

Example 2: Time – fractional Burgers equation

The following time-fractional Burgers equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\varepsilon\, u\frac{\partial u}{\partial x}-v\frac{\partial^2 u}{\partial x^2}=0, \quad t>0,\ 0<\alpha\le 1 \qquad (21)$$

Where α is a parameter describing the order of the fractional time derivative.

For our purpose, we introduce the following transformations [1]:

$$u(x,t)=U(\xi)\ ;\ \xi=lx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \qquad (22)$$

Where l,λ is constant. Using Eq.(22) carries Eq.(21) into an ordinary differential equation as follows :

$$-\lambda U'+l\,\varepsilon\, UU'-l^2vU''=0 \qquad (23)$$

Where $U'=\frac{dU}{d\xi}$. By integrating Eq.(23) once and considering the constants of integration to be zero, then we obtain the ordinary differential equation

$$-\lambda U+l\,\varepsilon\frac{U^2}{2}-l^2\ v\ U'=0 \qquad (24)$$

The substitution $U'=p$, reduces Eq.(24) ,to

$$-\lambda U+l\,\varepsilon\frac{U^2}{2}-l^2\ v\ p=0 \qquad (25)$$

Eq.(25) can be written as

$$-\lambda U+l\,\varepsilon\frac{U^2}{2}=l^2\ v\frac{dU}{d\xi} \qquad (26)$$

The variables are separated, then integrating both sides of Eq.(26) and let the constant of integration equal to zero, this leads to

$$U(\xi)=\frac{2\lambda}{l\,\varepsilon}\left(\frac{1}{1-e^{\frac{\lambda\xi}{vl^2}}}\right) \qquad (27)$$

Now we substitute $U(\xi)=u(x,t)$ and $\xi=lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}$, then Eq. (27) becomes

$$u(x,t)=\frac{2\lambda}{l\,\varepsilon}\left[\frac{1}{1-e^{\frac{\lambda}{vl^2}\left(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\right] \qquad (28)$$

Remark 2: In [2], the authors solved Eq.(21) by (G'/G)- expansion method and obtained three exact solutions different from our solution.Eq.(28).

Example 3: Time – fractional generalized Hirota-Satsuma coupled KdV system.

Let us apply our method to the generalized Hirota-Satsuma coupled KdV system which is of the form

$$D_t^{\alpha}u=\frac{1}{4}u_{xxx}+3uu_x+3(-v^2+w)_x,$$

$$D_t^{\alpha}v=\frac{-1}{2}v_{xxx}-3uv_x, \qquad (29)$$

$$D_t^{\alpha}w=\frac{-1}{2}w_{xxx}-3uw_x, \qquad 0<\alpha\le 1,$$

Where $u=u(x,t)$, $v=v(x,t)$ and $w=w(x,t)$.For our purpose, we introduce the following transformations as in [1]:

$$u(x,t)=\frac{1}{\lambda}U(\xi)^2,\ v(x,t)=-\lambda+U(\xi),\ w(x,t)=2\lambda^2-2\lambda U(\xi),\ \xi=x-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \qquad (30)$$

Where λ is a constant. Substituting Eq.(30) into Eq.(29), we can see that Eq.(29) are reduced into an ordinary differential equation

$$\lambda\frac{\partial^2 U}{\partial\xi^2}+2U^3-2\lambda^2U=0 \qquad (31)$$

If we suppose $\frac{dU}{d\xi}=p$ and $\frac{d^2U}{d\xi^2}=p\frac{dp}{dU}$, then Eq.(31) converts to

$$\lambda p\frac{dp}{dU}+2U^3-2\lambda^2U=0 \qquad (32)$$

This can be written

$$\lambda\, p\, dp=(2\lambda^2U-2U^3)du \qquad (33)$$

The variables are separated, then once integration leads to

$$\mathbf{p^2 = 2\lambda U^2 - \frac{1}{\lambda} U^4 + C} \qquad (34)$$

Let $\mathbf{C = 0}$, then

$$\mathbf{p = U\sqrt{2\lambda - \frac{1}{\lambda} U^2}} \qquad (35)$$

Since $\mathbf{p = \frac{dU}{d\xi}}$, then

$$\mathbf{\frac{dU}{U\sqrt{2\lambda^2 - U^2}} = \frac{1}{\sqrt{\lambda}} d\xi} \qquad (36)$$

By integrating both sides of Eq.(36), then

$$\mathbf{\frac{-1}{\lambda\sqrt{2}} sech^{-1}\left(\frac{U}{\lambda\sqrt{2}}\right) = \frac{1}{\sqrt{\lambda}}\xi + B} \qquad (37)$$

By setting B=0, Eq.(37) become

$$\mathbf{U(\xi) = \lambda\sqrt{2}\, sech(-\sqrt{2\lambda}\,\xi)} \qquad (38)$$

Now, we substitute Eq.(38) in Eq.(30), and $\mathbf{U(\xi) = u(x,t)}$,then

$$\mathbf{u(x,t) = 2\lambda sech^2[\sqrt{2\lambda}(\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} - x)]}$$

$$\mathbf{v(x,t) = -\lambda + \lambda\sqrt{2}\, sech[\sqrt{2\lambda}(\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} - x)]} \qquad (39)$$

$$\mathbf{w(x,t) = 2\lambda^2 - 2\lambda^2\sqrt{2}\, sech[\sqrt{2\lambda}(\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} - x)]}$$

Eq.(39) is the solution of Eq.(29).

Remark 3: In [1], the author solved Eq.(29) by using first integral method and obtained two exact solutions different from our solution, while in [3], the authors solved by functional variable and obtained two exact solution agreement with our solution Eq.(39).

4- Conclusions

The reduction of order method is applied successfully for solving the time-fractional Klein-Gordon equation and the time-fractional Burgers equation and the time-fractional Hirota-Satsuma coupled KdV system. The performance of this method is reliable, effective and produce easily exact solutions for several families of time-fractional differential equations and on comparing the proposed method in this article with the other methods used in [1, 2, 3], we find that the reduction of order method is simpler than those methods.

References

1- Bin.J.U., "The first integral method for some time-fractional equations", Journal of Mathematical Analysis and Applications, 395, 2012, 684-693.

2- Ahmet Bekir and Ozkan Guner,"Exact solutions of Nonlinear Fractional Differential Equations by (G'/G)-Expansion Method", Chin.Phys.B, 22, 11, 2013, 110202-1 – 110202-6.

3- Wenjun Liu and Kewang Chen.,"The Functional Variable Method for Finding Exact Solutions of Some Nonlinear Time-Fractional Differential Eqations", Pramana Journal of Physics, 81, 3, 2013, 377-384.

4- Pietro Artale and Roberto Garra,"Analytic Solution of Nonlinear Fractional Burgers-Type Equation by Invariant Subspace Method",Nlin.Sl, 1, 8, 2013,1-8.

5- Bin Zheng., "Exp-Function Method for Solving Fractional Partial Differential Equations", The Scientific World Journal,Vol.2013, 2013, 1-8.

6- Aghili A. and Motahhari A., "Integral Transform Method for Solving Nonhomogenous Time-Fractional Partial Differential Equations", Electronic Journal, 4, 2011, 194-204.

7- Zhang S,Zhang HQ.,"Fractional Sub-Equation Method and its Applications to Nonlinear Fractional Partial Differential Equations",Phys.Lett.A.,375,7, 2011,1069-1073.

8- Zhang S, Zong QA,Liu D,Gao Q., A., "Generalized Exp-function method for fractional Riccati differential equations", Commun.Fract.Calc. , 1, 1, 2010, 48-51.

9- M.Inc,."The Approximate and Exact Solutions of the Space-and time-Fractional Burgers Equations with Initial Conditions by Variational Iteration Method", Journal of Mathematical Analysis Application, ,345,1, 2008,476-484.

10- Elsayed M.E,Yasser.A. and Reham M.,"Exact Traveling Wave Solutions for Nonlinear Fractional Partial Differential Equations Using the Improved (G'/G)- Expansion Method",4,7, 2014,18-31.

11-Taha Elghareb,S.K.Elagan,Y.S.Hamed and M.Sayed,"An Exact Solutions for the Generalized Fractional Kolmogrove-Petrovskii Piskunov Equation by Using the Generalized (G'/G)-Expansion Method",13,1, 2013,19-22.

12- M.Dehghan,J.Manafian,A.Saadatmandi, "Solving Nonlinear Fractional Partial Differential Equations Using the Homotopy Analysis Method", Numerical Methods for Partial Differential Equations, 26, 2, 2010,448-479.

13- Khosro Sayevand,"Efficient Solution of Fractional Initial Value Problems Using Expanding Perturbation Approach", Journal of Mathematics and Computer Science, 8, 2014, 359-366.

14- Pandir Y., Gurefe Y. and Misirli E. "The extended Trial Equation Method for Some Time-Fractional Differential Equations", Discrete Dynamics in Nature and Society, vol.2013, Article ID 491359, 2013,13-23,
15- M.Saravi, "Pseudo-First Integral Method for Benjamin-Bona-Mahony, Gardner and Foam Drainage Equations", Journal of Basic and Applied Scientific Research, 1, 7, 2011, 521-526,
16- Taghizadeh. N., Mirzazadeh .M and Farahrooz. F.," New Exact Solutions of some Nonlinear Partial Differential Equations by the First Integral Method", Applications and Applied Mathematics, 5, 2010, 543-1553.
17- Frank Ayres,JR. Theory and Problems of Differential Equations,McGraw-Hill,1974.
18- Jumarie G., "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results", Comput.Math.Appl, 51(9- 10), 2006, 1367-1376.
19- Jumari G., "Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution",J.Appl.Math.Comput. 24(1-2), 2007, 31- 48.
20- Erel D.Rainville,Philipe A.Bedint, Elementary Differential Equations,Mac.Millan Publishing Co.,1974.

طريقة تخفيض الرتبة لحل بعض المعادلات التفاضلية الكسورية

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الخلاصة

في هذا البحث استخدمت طريقة تخفيض الرتبة مع استخدام تعريف لاوفيل – ريمان للمشتقة الكسورية وايجاد الحلول الحقيقية لبعض المعادلات الكسورية التفاضلية المعروفة وهي Klein-Gordon equation و Burgers equation و Hirota-Satsuma coupled KdV system. هذه الطريقة ممكن تطبيقها لبقية المعادلات الجزئية الغير خطية الكسورية في مجال التطبيقات الرياضية الفيزيائية.