Using Bernstein Polynomials for Solving Systems of Volterra Integral Equations of the Second Kind

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Abstract:

The aim of this work is to use Bernstein polynomials for deriving some modified quadrature methods for solving systems of the one-dimensional Volterra linear integral equations of the second kind. These are modified Trapezoidal rule and the modified Simpsons 3/8 rule of the first order.

For each method, some numerical examples are solved and computer are written in (Excel), And the results are presented in tabulated forms.

Key words:

Bernstein Polynomials, Trapezoidal Rule, Simpson Rule , Volterra Integral Equation .

1-Introduction:

Many researchers concerned with the system of integral equations say, Hacia and Kaezmarek in 1999, [1], presented bounds of the solutions of one-dimensional Volterra integral equations.

Biazar J., Babolian, E and Islam, R in 2003, [2] gave the solutions of system of one-dimensional Volterra integral equations of the first kind by using Adomian method, Maleknejad K and Shahrezaee M in 2004, [3] used Rungekubba method for finding numerical solution of systems of one-dimensional Volterra integral equations of the second kind, Al-Sa'dawy in 2008 used some modified Quadrature methods for solving systems of Volterra integral equations[4], Maleknajad K, and Rabbani M, in 2006, [5] applied Taylor expansion method to find solution of system of Fredholm integral eqution of the second kind, Ibrahim in 2006, [6] used the numerical method for solving system of one-dimensional Fredholm linear integral equations. Ali Y in 2007, [7] used this method to solve the multidimensional Fredholm and Volterra linear integral equations of the second kind and Vahidi A. and Mokhtari M. in 2008, [8] denoted the decomposition methods for finding the numerical solutions of systems of the one-dimensional Fredholm linear integral equations of the second kind the aim of this work is to use Bernstein polynomials for deriving some modified quadrature methods for solving systems of the one-dimensional Volterra linear integral equations of the second kind.

2. The Main Results:

2.1 Derivation of Modified Quadrature Methods

2.1.1 Modified Trapezoidal Rule:

Consider Bernstein polynomials given by the following equation:-

$$\boldsymbol{P}(\boldsymbol{x}) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

Where f is a function, $k = 0,1,\dots,n$

Then:-

$$P(x) = f\left(\frac{0}{n}\right)\binom{n}{0} x^0 (1-x)^{n-0} + f\left(\frac{1}{n}\right)\binom{n}{1} x(1-x)^{n-1}$$
$$+ f\left(\frac{2}{n}\right)\binom{n}{2} x^2 (1-x)^{n-2} + f\left(\frac{3}{n}\right)\binom{n}{3} x^3 (1-x)^{n-3}$$
$$+ \cdots + f\left(\frac{n}{n}\right)\binom{n}{n} x^n (1-x)^{n-n}$$
$$= f(0)(1-x)^n + f\left(\frac{1}{n}\right)\left(\frac{n!}{1!(n-1)!}\right) x(1-x)^{n-1} +$$
$$f\left(\frac{2}{n}\right)\left(\frac{n!}{2!\,(n-2)!}\right) x^2 (1-x)^{n-2} +$$
$$f\left(\frac{3}{n}\right)\left(\frac{n!}{3!\,(n-3)!}\right) x^3 (1-x)^{n-3} + \cdots + f(1)x^n$$
$$= f(0)(1-x)^n + nf\left(\frac{1}{n}\right) x(1-x)^{n-1} +$$
$$\frac{n(n-1)}{2!} f\left(\frac{2}{n}\right) x^2 (1-x)^{n-2} +$$
$$\frac{n(n-1)(n-2)}{3!} \quad f\left(\frac{3}{n}\right) x^3 (1-x)^{n-3} + \cdots + f(1)x^n$$

By substituting $n = 1$. Then

$$p(x) = f(0)(1-x) + f(1)x(1-x)^0$$
$$= f(0)(1-x) + f(1)x$$

Let $y_0 = f(0)\ and \quad y_1 = f(1) \quad then$

$$P(x) = y_0(1-x) + y_1 x \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2.1)$$

By integrating both sides of above equation from (0to1) one can get:-

$$\int_0^1 f(x)dx \simeq \int_0^1 p(x)dx$$
$$= \int_0^1 [y_0 (1-x) + y_1 x]\, dx$$
$$= \left| y_0 \left(x - \frac{x^2}{2}\right) + y_1 \frac{x^2}{2} \right|_0^1$$
$$= y_0 \left(1 - \frac{1}{2}\right) + y_1 \left(\frac{1}{2}\right)$$
$$= \frac{1}{2} y_0 + \frac{1}{2} y_1$$
$$= \frac{1}{2}(y_0 + y_1)$$

Now by using the transformation. $x = a + t(b-a), h = \frac{b-a}{1}$ *then the above equation, one can get*

$$\int_a^b f(x)dx = \frac{h}{2}[f_0 + f_1] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2)$$

This formula is the modified Trapezoidal Rule of first order.

2.1.2 The Composite Modified Trapezoidal rule of first order:-

We can be derived by extending the modified Trapezoidal Rule of first order .this procedure is began by dividing [a,b] into ***n*** subintervals and applying the modified Trapezoidal Rule of first order over each interval then the sum of the results obtained for each interval is the approximate value of integral ,that is

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \cdots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx$$

Where h= b-a

$$= \frac{h}{2}[f(a) + f(h)] + \frac{h}{2}[f(a+h) + f(a+2h)] + \cdots + \frac{h}{2}[f(a+(n-2)h) + f(a+(n-1)h)] + \frac{h}{2}[f(a+(n-1)h) + f(b)] \ldots\ldots$$

$$= \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \cdots + 2f(a+(n-2)h) + 2f(a+(n-1)h) + f(b)]$$
……………………(2.3)

$$= \frac{h}{2}[f(a) + 2\sum_{j=1}^{i-1} f(x_j) + f(b)] \ldots\ldots\ldots\ldots (2.4)$$

This formula is said to be the composite modified Trapezoidal Rule of the first order .

2.1.3 The Modified Simpson's 3/8 Rule:-

By the Bernstein polynomials:-

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

where f is a function, $k = 0,1,\dots,n$

Then:-

$$P(x) = f\left(\frac{0}{n}\right)\binom{n}{0}x^0(1-x)^{n-0} + f\left(\frac{1}{n}\right)\binom{n}{1}x(1-x)^{n-1}$$
$$+f\left(\frac{2}{n}\right)\binom{n}{2}x^2(1-x)^{n-2} + f\left(\frac{3}{n}\right)\binom{n}{3}x^3(1-x)^{n-3}$$
$$+\cdots+f\left(\frac{n}{n}\right)\binom{n}{n}x^n(1-x)^{n-n}$$
$$= f(0)(1-x)^n + f\left(\frac{1}{n}\right)\left(\frac{n!}{1!\,(n-1)!}\right)x(1-x)^{n-1} +$$
$$f\left(\frac{2}{n}\right)\left(\frac{n!}{2!\,(n-2)!}\right)x^2(1-x)^{n-2} +$$
$$f\left(\frac{3}{n}\right)\left(\frac{n!}{3!\,(n-3)!}\right)x^3(1-x)^{n-3} + \cdots + f(1)x^n$$
$$= f(0)(1-x)^n + nf\left(\frac{1}{n}\right)x(1-x)^{n-1} +$$
$$\frac{n(n-1)}{2!}f\left(\frac{2}{n}\right)x^2(1-x)^{n-2} +$$
$$\frac{n(n-1)(n-2)}{3!}\quad f\left(\frac{3}{n}\right)x^3(1-x)^{n-3} + \cdots + f(1)x^n$$

By substituting $n = 3$. Then

$$P(x) = f(0)(1-x)^3 + 3f\left(\frac{1}{3}\right)x(1-x)^2 + 3f\left(\frac{2}{n}\right)x^2(1-x) + 3f\left(\frac{3}{3}\right)x^3(1-x)^0$$

Let

$$f(0) = y_0, f\left(\frac{1}{3}\right) = y_1, f\left(\frac{2}{3}\right) = y_2, f(1) = y_3$$

$$P(x) = y_0(1-x)^3 + 3y_1x(1-x)^2 + 3y_2x^2(1-x) + y_3x^3 \quad \dots (2.5)$$

By integrating both sides of equation (2.5)
From 0 to 1 one can have:-

$$\int_0^1 f(x)dx \simeq \int_0^1 P(x)dx$$

$$= \int_0^1 [y_0(1-x)^3 + 3y_1x(1-x)^2 + 3y_2x^2(1-x) + y_3x^3]\,dx$$

$$= \int_0^1 [y_0(1-3x+3x^2-x^3) + 3y_1(x-2x^2+x^3) + 3y_2(x^2-x^3) + y_3x^3]\,dx$$

$$= y_0\left(x - \frac{3}{2}x^2 + x^3 - \frac{1}{4}x^4\right) + 3y_1\left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right) + 3y_2\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right) + \frac{1}{4}y_3x^4\Big|_0^1$$

$$= y_0\left(1 - \frac{3}{2} + 1 - \frac{1}{4}\right) + 3y_1\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + 3y_2\left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{4}y_3$$

$$= \frac{1}{4}y_0 + \frac{3}{12}y_1 + \frac{3}{12}y_2 + \frac{1}{4}y_3$$

$$= \frac{1}{4}y_0 + \frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3$$

$$= \frac{1}{4}(y_0 + y_1 + y_2 + y_3)$$

$$= \frac{1}{4}(f_0 + f_1 + f_2 + f_3)$$

Now by using the transformation.

$x = a + t(b-a), h = \frac{b-a}{3}$ and above equation, one can get:-

$$\int_a^b f(x)dx = \frac{3h}{4}[f_0 + f_1 + f_2 + f_3] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.6)$$

This formula is said to be modified Simpson's 3/8 rule of first order.

2.1.4 The composite modified Simpson's 3/8 rule of first order:

The composite modified Simpson's 3/8 rule of first order can be derived by extending the modified Simpson's 3/8 rule of first order.

This procedure is begin by dividing [a, b] into **n** subintervals (n is multiple of three) and applying the modified Simpson's 3/8 rule of first order over each interval then the sum of the results obtained for each interval is the approximate value of integral, that is

$$\int_b^a f(x)dx = \int_a^{a+3h} f(x)dx + \int_{a+3h}^{a+6h} f(x)dx + \ldots\ldots + \int_{a+(n-6)h}^{a+(n-3)h} f(x)dx + \int_{a+(n-3)h}^{b} f(x)dx \qquad \text{where } h = \frac{b-a}{n}$$

$$\int_a^b f(x)dx = \frac{3h}{4}[f(a) + f(a+h) + f(a+2h) + f(a+3h)]$$

$$+\frac{3h}{4}[f(a+3h)+f(a+4h)+f(a+5h)+f(a+\ 6h)]\ +\cdots+\ \frac{3h}{4}[f(a+(n-6)h)+ f(a+(n-5)h)+f(a+(n-4)h)+f(a+(n-3)h)]$$

$$+\frac{3h}{4}[f(a+(n-3)h)+f(a+(n-2)h)+f(a+(n-1)h)+f(b)]$$

$$=\frac{3h}{4}\ [\,f(a)+f(a+h)+f(a+2h)+2f(a+3h)\,f(a+4h)+f(a+5h)+\cdots + 2f(a+(n-3)h)+\ f(a+(n-2)h)+f(a+(n-1)h) + \ f(b)]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots..(2.7)$$

$$=\frac{3h}{4}\left[f(a)+\sum_{j=1,4,7,\ldots}^{n-2}[f(x_j)+f(x_{j+1})]+2\sum_{j=3,6,9\ldots}^{n-3}f(x_j)+f(b)\right]\ldots\ldots\ldots\ldots\ldots.\ldots..(2.8)$$

This formula is said to be the composite modified Simpson's 3/8 rule of first order

2.2 Numerical Solution of Systems of Volterra Integral Equations of the 2'nd Kind:

2.2.1 The Composite Modified Trapezoidal Rule:

In this section, we use the composite modified trapezoidal rule to solve systems of Volterra linear integral equations of the second kind. To do this consider the system of Volterra linear integral equations of the second kind is:-

$$u_r(x)=f_r(x)+\sum_{s=1}^{m}\lambda_{rs}\int_a^x k_{rs}(x,y)u(y)dy \qquad a\le x\le b,\ldots\ldots.(3.1)$$

$$r=1,2,\ldots,m$$

By dividing the interval [a, b] into $\boldsymbol{n}$ subintervals $[\boldsymbol{x_i},\boldsymbol{x_{i+1}}]$, $\boldsymbol{i=0,1,2,..,n-1}$, such that $x_i=a+ih,\ i=0,1,..,n,$ where $h=\frac{b-a}{n}$ and by setting $x=x_i$, $i=1,2,\ldots,n$,in equation (3.1)

one can have:-

$$u_r(x_i)=f_r(x_i)+\sum_{s=1}^{m}\lambda_{rs}\int_a^{x_i} k_{rs}(x_i,y)u_s(y)dy$$

Now, we approximate the integral term that appeared in the right hand side of the above equation by the composite modified Trapezoidal rule, one can get:-

$u_{r,0}=f_r(x_0),\quad r=1,2,\ldots.,m$ and

$$u_{r,i} = f_r(x_i) + \sum_{s=1}^{m} \lambda_{rs} \, [\frac{h}{2} k_{rs}(x_i, x_0) u_{s,0} \; + \; h \sum_{j=1}^{i-1} k_{rs}(x_i, x_j) u_{s,j} + \frac{h}{2} h k_{rs}(x_i, x_i) u_{s,i}] \ldots\ldots\ldots\ldots\ldots (3.2)$$

Where $u_{r,i}$ denote the numerical solution of u_r at x_i, $i = 1,2,\ldots,n$ for $i = 1,2,\ldots,n$. One must evaluate equation (3.2) for each $r = 1,2,..,m$ to get a system of m linear equation with *m* unknown only $\{u_{r,i}\}_{r=1}^{m}$

This system can be solved by two ways,

i. By writing it as $A_i u_i = F_i$ where, A_i the coefficient matrix, u_i the column of numerical solutions and *F* is the column of non homogenous part.

ii. By substitution.

Example (3-1)

Consider the following system of Volterra linear equations of the second kind [27]

$$u_1(x) = \left[-\tfrac{1}{2}\, x^2 + \tfrac{1}{4}\, x + 1\right] e^{2x} + \left(x + \tfrac{1}{4}\right) e^{-2x} - \tfrac{3}{4} x - \tfrac{1}{4} + \int_0^x xy\, u_1\ (y) dy + \int_0^x (x+y)\, u_2(y) dy, where\ 0 \le x \le 1 \ldots\ldots\ldots (3.3)$$

$$u_2(x) = \left[2x^2 + x + \frac{5}{4}\right] e^{-2x} - \frac{1}{4} e^{2x} - \frac{1}{2} x^2 + \int_0^x (x-y) u_1(y) dy + \int_0^x (x+y)^2 u_2(y) dy \quad \text{where } 0 \le x \le 1 \ldots\ldots\ldots (3.4)$$

This example is constructed such that the exact solution is $u_i(x) = e^{2x}$, and $u_2(x) = e^{-2x}$. We solve this example numerically by using the composite modified Trapezoidal rule. To do this, first we divide the interval [0, 1] into 9 subintervals such that $x_i = \frac{i}{9}$, $i = 0,1,..,9$. Then $u_{1,0} = f_1(0) = 1,\ f_2(0) = 1$ in this case, for $r = 1,2$ equation (3-2) becomes:-

$$u_{1,i} = \left[-\frac{1}{2} x_i^2 + \frac{1}{4} x_i + 1\right] e^{2x_i} + \left[x_i + \frac{1}{4}\right] e^{-2x_i} - \frac{3}{4} x_i - \frac{1}{4}$$

$$+\frac{1}{9}\sum_{j=1}^{i-1} x_i x_j u_{1,j} + \frac{1}{18}\, x_i^2\, u_{1,i} + \frac{1}{18} x^2 u_{2,0}$$

$$+\frac{1}{9}\sum_{j=1}^{i-1} (x_i + x_j) u_{2,j} + \frac{1}{9} x_i u_{2,i} \dots\dots\dots\dots\dots\dots (3.5)$$

$$u_{2,i} = \left[2x_i^2 + x_i + \frac{5}{4}\right] e^{-2x_i} - \frac{1}{4} e^{2x_i} - \frac{1}{2} x_i^2 + \frac{1}{18} x_i u_{1,0}$$

$$+\frac{1}{9}\sum_{j=1}^{i-1} (x_i - x_j) u_{1,j} + \frac{1}{18} x_i^2 u_{2,0}$$

$$+\frac{1}{9}\sum_{j=1}^{i-1} (x_i + x_j)^2 u_{2,j} + \frac{2}{9} x_i^2 u_{2,i} \dots\dots\dots\dots\dots\dots (3.6)$$

for $i = 1$, we evaluating the above equation to get a system of two linear equation with two unknown $\{u_{r,i}\}_{r=1}^{2}$. We solve this system directly by substitutions. We get ones $u_{2,1}$ and substitutions $u_{2,1}$ in $u_{1,1}$ and get $u_{1,1}$and in the same way we solve when $i = 2,3,\dots,9$

By substituting i=1 , r=1,2 in equation (3.5) , (3.6) to get $u_{1,1}$, $u_{2,1}$ to get the following system see the result of example (3,1) in the appendix .By continuing in this manner one can get with results that are tabulated down with comparison with exact solutions .

Table (3.1) represents the exact and the numerical solution of example (3.1) at specific points for n=9

X	Exact Solution		Numerical Solution N=9	
	1	2	u1	u2
0.111111111	1.248848869	0.8007374029	1.2485618607	0.8003332255
0.222222222	1.559623498	0.6411803884	1.5593690349	0.6400977991
0.333333333	1.947734041	0.5134171190	1.9478223270	0.5114303310
0.444444444	2.432425454	0.4111122905	2.4331971529	0.4079475020
0.555555556	3.037731778	0.3291929878	3.0396058766	0.3244084574
0.666666667	3.793667895	0.2635971381	3.7971863433	0.2563934002
0.777777778	4.73771786	0.2110720878	4.7435579366	0.1999247284
0.888888889	5.916693591	0.1690133154	5.9255601978	0.1508628740

1	7.389056099	0.1353352832	7.4011826534	0.1036171092

Second, we divide [0, 1] into 18 subintervals such that

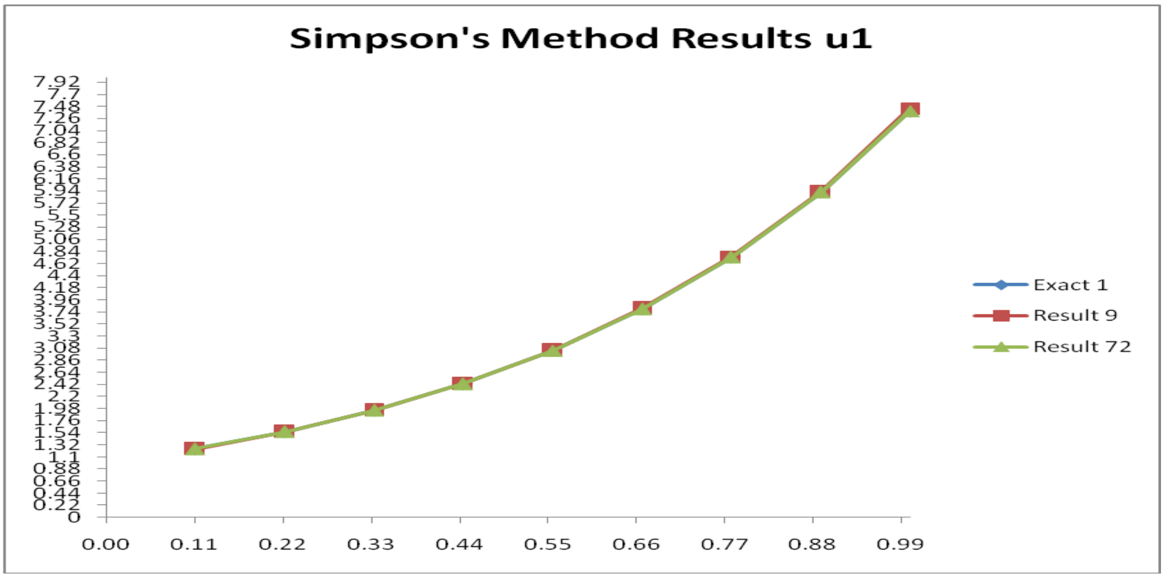
$x_i = \frac{i}{18}$, $i = 0, \dots, 18,$ the equation (3.2) becomes:-

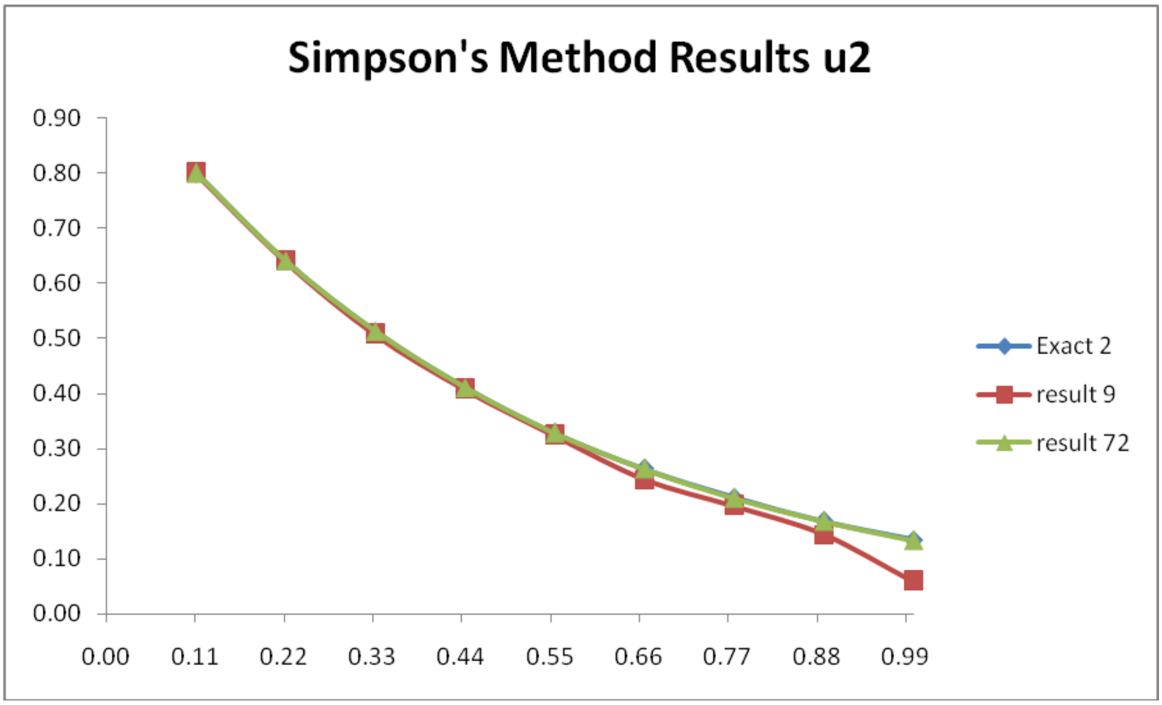
$$u_{1,i} = \left[-\frac{1}{2}x_i^2 + \frac{1}{4}x_i + 1\right]e^{2x_i} + \left[x_i + \frac{1}{4}\right]e^{-2x_i} - \frac{3}{4}x_i - \frac{1}{4} + \frac{1}{18}\sum_{j=1}^{i-1} x_i x_j u_{i,j} + \frac{1}{32}x^2 u_{1,i} + \frac{1}{32}x_i u_{2,0} + \frac{1}{18}\sum_{j=1}^{i-1}(x_i + x_j)u_{2,j} + \frac{1}{18}x_i u_{2,i} \dots \dots \dots \dots \dots \dots (3.7)$$

$$u_{2,i} = \left[2x_i^2 + x_i + \frac{5}{4}\right]e^{-2x_i} - \frac{1}{4}e^{2x_i} - \frac{1}{2}x_i^2 + \frac{1}{32}x_i u_{1,0} + \frac{1}{18}\sum_{j=1}^{i-1}(x_i - x_j)u_{1,j} + \frac{1}{32}x_i^2 u_{2,0} + \frac{1}{18}\sum_{j=1}^{i-1}(x_i + x_j)^2 u_{2,j} + \frac{1}{9}x_i^2 u_{2,i} \dots \dots \dots \dots \dots \dots (3.8)$$

By following the same previous steps one can get $u_{1,i}, u_{2,i}$ By substituting i=1 , r=1,2 in equation (3.7) , (3.8) to get $u_{1,1}$, $u_{2,1}$ to get the following system see the result of example (3.1) .

Third, we divide [0, 1] into 36 subintervals such that $x_i = \frac{i}{32}$, $i = 0, \dots, 36$ and we divide [0, 1] into 72 subintervals such that $x_i = \frac{i}{72}$, $i = 0, 1, \dots, 72$ respectively .





2.2.2 The Composite Modified Simpson 3/8 Rule of the 1'st order:

In this section, we use the composite modified Simpson's 3/8 rule of first order for solve systems of volterra linear integral equations of the second kind given in (3.1) by dividing the interval [a, b] into n subintervals $[x_i, x_{i+1}]$, $i = 0,1, \dots, n-1$ such that $x_i = a + ih$, $i = 0,1, \dots, n$ (n is multiple of three) where $h = \frac{b-a}{n}$.

Then we approximate the integral term that appeared in the right hand side of the integral equation (3.1) by the composite modified Simpson's 3/8 rule of first order get:-

$$u_{r,i} = f_r(x_i) + \sum_{s=1}^{m} \lambda_{rs} \left\{ \frac{3h}{4} [\, k_{r,s}(x_i,x_0)u_{s,0} + \sum_{j=1,4,7,..}^{i-2} k_{r,s}(x_i,x_j)u_{s,j} + k_{r,s}(x_i,x_{j+1})u_{s,j+1}] + 2\sum_{j=3,6,9,..}^{i-3} k_{rs}(x_i,x_j)u_{s,j} + k_{rs}(x_i,x_i)u_{s,i} \right. \quad \ldots\ldots\ldots (3.9)$$

$$i = 3,6,\ldots,n \quad and \quad r = 1,2,..,m$$

And by equation (3.2) we get:-

$$u_{r,i} = f_r(x_i) + \sum_{s=1}^{m} \lambda_{rs} \left\{ \frac{h}{2} k_{rs}(x_i,x_0)u_{s,0} + h\sum_{j=1}^{i-1} k_{rs}(x_i,x_j)u_{s,j} + \frac{h}{2} k_{r,s}(x_i,x_i)u_{s,i} \right. \quad \ldots\ldots\ldots (3.10)$$

Where $i \neq 3,6,..,n \quad and \quad r = 1,2,..,m,$

Example (3.2):-

Consider example (3.1) we solve this example numerically by using the composite modified Simpson's 3/8 rule of first order.

To do this first, we divide the interval [0, 1] into 9 subintervals such that $x_i = \frac{i}{9}$, $i = 0,1,\ldots,9$. Then $u_{1,0} = f_1(0) = 1$ and

$u_{2,0} = f_2(0) = 1$, $k_{11}(x,y) = xy$, $k_{12}(x,y) = x + y$, $k_{21}(x,y) = x - y$, $k_{22}(x,y) = (x+y)^2$ .In this case, for $r = 1,2$ equation (3.3), (3.10) becomes:-

$$u_{1,i} = \left[-\frac{1}{2}x_i^2 + \frac{1}{4}x_i + 1\right]e^{2x_i} + \left[x_i + \frac{1}{4}\right]e^{-2x_i} - \frac{3}{4}x_i - \frac{1}{4}$$
$$+\frac{1}{12}\sum_{j=1,4,7}^{i-2}\left[(x_i x_j)u_{1,j} + (x_i x_{j+1})u_i, u_{j+1}\right]$$
$$+\frac{1}{6}\sum_{j=3,6,9,..}^{i-3}(x_i x_j)u_{1,j} + \frac{1}{12}x_i{}^2 u_{1,i} + \frac{1}{12}(x_i + x_0)u_{2,0}$$
$$+\frac{1}{12}\sum_{j=1,4,7,..}^{i-2}\left[(x_i + x_j)\,u_{2,j} + (x_i + x_{j+1})u_{2,j+1}\right]$$
$$+\frac{1}{6}\sum_{j=3,6,9,..}^{i-3}(x_i + x_j)u_{2,j} + \frac{1}{6}x_i u_{2,i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.11)$$

When $i = 3, 6, \ldots n$

$$u_{1,i} = \left[-\frac{1}{2}x_i^2 + \frac{1}{4}x_i + 1\right]e^{2x_i} + \left[x_i + \frac{1}{4}\right]e^{-2x_i} - \frac{3}{4}x_i - \frac{1}{4}$$
$$+\frac{1}{9}\sum_{j=1}^{i-1}\left[x_i x_j u_{1,j}\right] + \frac{1}{18}x_i{}^2 u_{1,i} + \frac{1}{18}x_i u_{2,0}$$
$$+\frac{1}{9}\sum_{j=1}^{i-1}\left[(x_i + x_j)u_{2,j}\right] + \frac{1}{9}x_i u_{2,i} \ldots\ldots\ldots\ldots\ldots\ldots (3.12)$$

When $i = 1, 2, 4, 5, 7, 8$

$$u_{2,i} = \left[\frac{1}{2}x_i^2 + x_i + \frac{5}{4}\right]e^{-2x_i} - \frac{1}{4}e^{2x_i} - \frac{1}{2}x_i{}^2 + \frac{1}{12}(x_i - x_0)u_{1,0}$$
$$+\frac{1}{12}\sum_{j=1,4,7}^{i-2}\left[(x_i - x_j)u_{1,j} + (x_i - x_{j+1})u_{1,j+1}\right]$$
$$+\frac{1}{6}\sum_{j=3,6,9}^{i-3}(x_i - x_j)u_{1,j} + \frac{1}{12}(x_i + x_0)^2 u_{2,0}$$
$$+\frac{1}{12}\sum_{j=1,4,7}^{i-2}\left[(x_i + x_j)^2 u_{2,j} + (x_i - x_{j+1})^2 u_{2,j+1}\right]$$
$$+\frac{1}{6}\sum_{j=3,6,9}^{i-3}(x_i + x_j)^2 u_{2,j} + \frac{1}{12}(2x_i)^2 u_{2,i} \ldots\ldots\ldots\ldots (3.13)$$

When $i = 3,6,9$

$$u_{2,i} = \left[2x_i^2 + x_i + \frac{5}{4}\right] e^{-2x_i} - \frac{1}{4}e^{2x_i} - \frac{1}{2}{x_i}^2 + \frac{1}{18}(x_i - x_0)u_{1,0} + \frac{1}{9}\sum_{j=1}^{i-1}\left[(x_i - x_j)u_{1,j}\right]$$

$$+\frac{1}{18}(x_i)^2 u_{2,0} + \frac{1}{9}\sum_{j=1}^{i-1}\left[(x_i + x_j)^2 u_{2j} + \frac{1}{12}(x_i + x_0)^2 u_{2,j}\right]$$

$$+\frac{2}{9}x_i^2 u_{2,i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.14)$$

When $i = 1,2,4,5,7,8$

By substituting $i = 1,\ r = 1,2$ in equation (3.12), (3.14) to get $u_{1,1}$, $u_{2,1}$. By substituting $i = 2, r = 1,2$ in equation (3.12) , (3.14) to get. $u_{1,2}$, $u_{2,2}$. By substituting $i = 3,\ r = 1,2$ in equation (3.11), (3.12), (3.13), (3.14) to get $u_{1,3}$, $u_{2,3}$

By continuing in this manner one can get with results that are tabulated down with comparison with exact solutions.

Table (3.3) represents the exact and the numerical solution of example (3.2) at specific points for n=9

X	**Exact Solution**		**Numerical Solution N=9**	
	1	**2**	**u1**	**u2**
0.1111111111	1.248848869	0.8007374029	1.2485618607	0.8003332255
0.2222222222	1.559623498	0.6411803884	1.5593690349	0.6400977991
0.3333333333	1.947734041	0.5134171190	1.9480993168	0.5075468839
0.4444444444	2.432425454	0.4111122905	2.4328489796	0.4076780673
0.5555555556	3.037731778	0.3291929878	3.0391564190	0.3240130044
0.6666666667	3.793667895	0.2635971381	3.8057678750	0.2439737698
0.7777777778	4.73771786	0.2110720878	4.7410291348	0.1959515199
0.8888888889	5.916693591	0.1690133154	5.9215908866	0.1445662959
1	7.389056099	0.1353352832	7.4348466911	0.0600111748

Second we divide the [0, 1] into 18 subintervals such that $x_i = \frac{1}{18}$, $i = 0,1,2,\ldots,18$, then the equation (3.9), (3.10) becomes:-

$$u_{1,i} = \left[-\frac{1}{2}x_i^2 + \frac{1}{4}x_i + 1\right]e^{2x_i} + \left[x_i + \frac{1}{4}\right]e^{-2x_i} - \frac{3}{4}x_i - \frac{1}{4}$$
$$+\frac{1}{24}\sum_{j=1,4,7,\dots}^{i-2}\left[(x_ix_j)u_{1,j} + (x_ix_{j+1})u_{1,j+1}\right]$$
$$+\frac{1}{12}\sum_{j=3,6,\dots}^{i-3}(x_ix_j)u_{1,j} + \frac{1}{24}{x_i}^2u_{1,i} + \frac{1}{24}(x_i + x_0)u_{2,0}$$
$$+\frac{1}{24}\sum_{j=1,4,7,\dots}^{i-2}\left[(x_i + x_j)u_{2,j} + (x_i + x_{j+1})u_{2,j+1}\right]$$
$$+\frac{1}{12}\sum_{j=3,6,9,\dots}^{i-3}(x_i + x_j)u_{2,j} + \frac{1}{12}{x_i}^2u_{1,i}$$
$$+\frac{1}{24}x_iu_{2,i} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3.15)$$

when $i = 3,6,9,\dots\dots,$

$$u_{1,i} = \left[-\frac{1}{2}x_i^2 + \frac{1}{4}x_i + 1\right]e^{2x_i} + \left[x_i + \frac{1}{4}\right]e^{-2x_i} - \frac{3}{4}x_i - \frac{1}{4}$$
$$+\frac{1}{18}\sum_{j=1}^{i-1}\left[x_ix_ju_{1,j}\right] + \frac{1}{36}{x_i}^2u_{1,i} + \frac{1}{36}x_iu_{2,0}$$
$$+\frac{1}{18}\sum_{j=1}^{i-1}(x_i + x_j)u_{2,j} + \frac{1}{18}x_iu_{2,i} \quad \dots\dots\dots\dots (3.16)$$

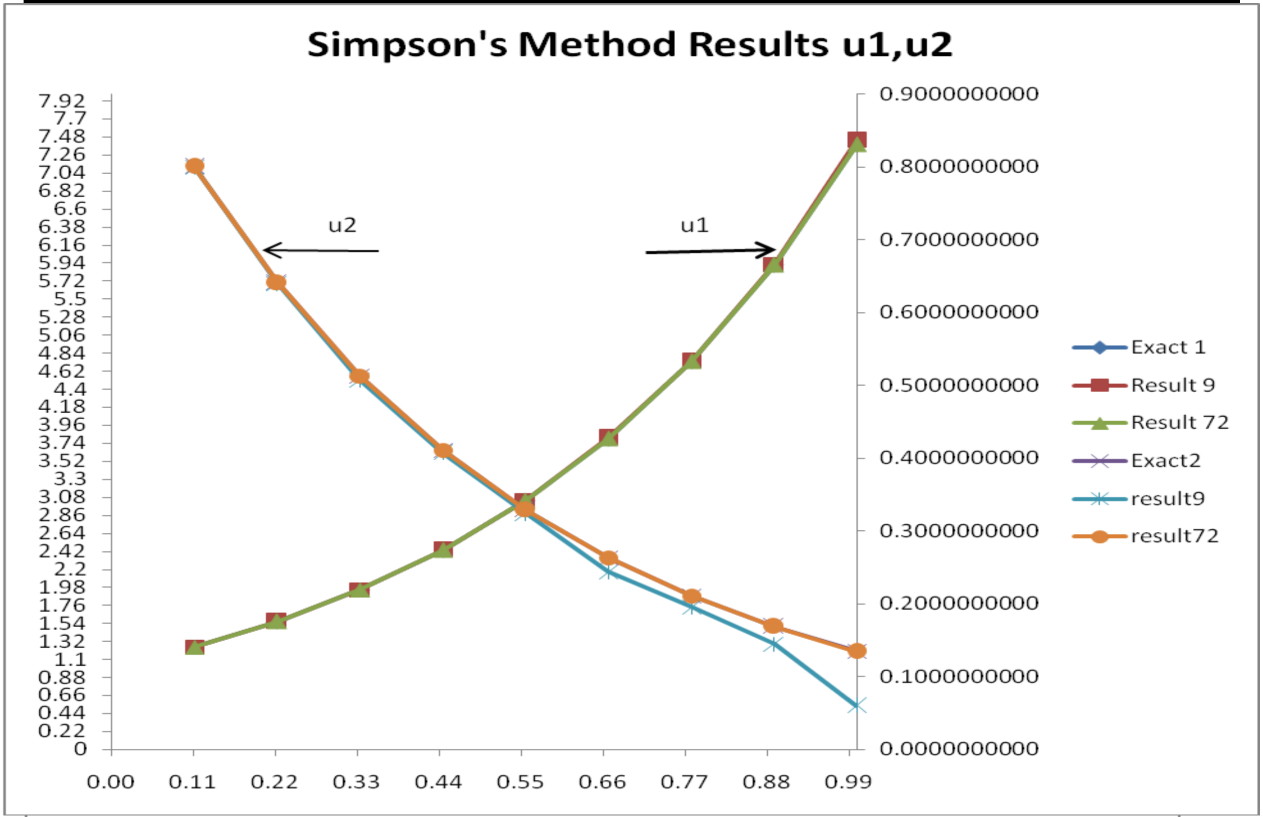
when $i \neq 3,6,9,\dots\dots,$

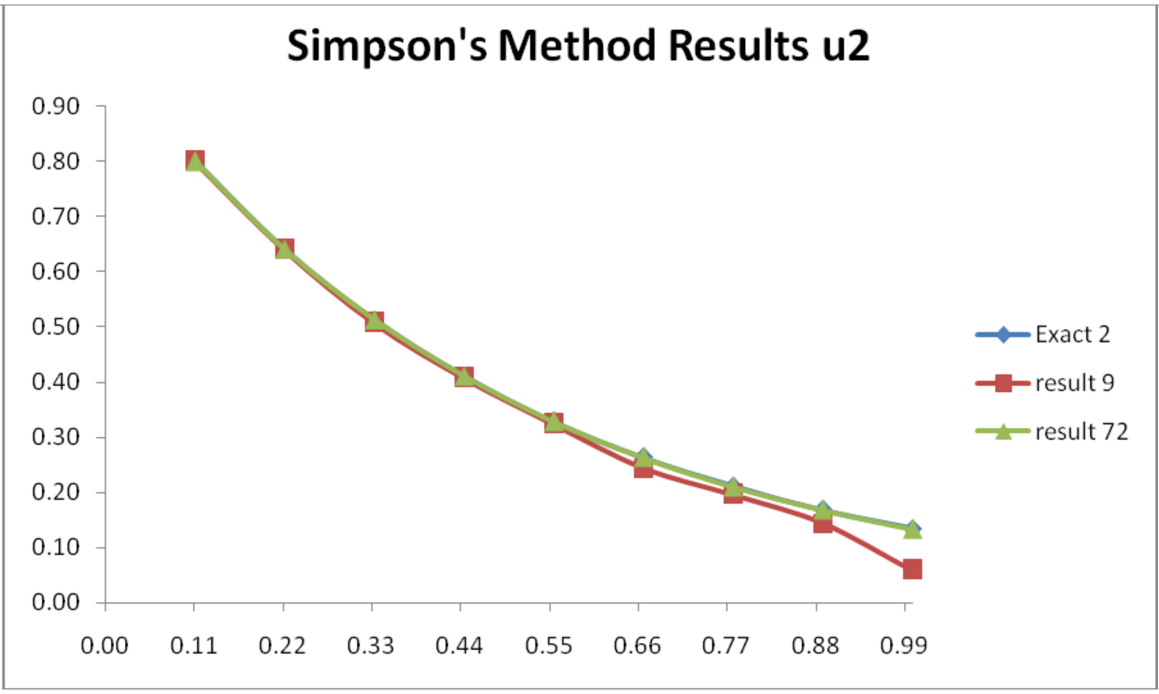
$$u_{2,i} = \left[2x_i^2 + x_i + \frac{5}{4}\right]e^{-2x_i} - \frac{1}{4}e^{2x_i} - \frac{1}{2}{x_i}^2 + \frac{1}{24}(x_i - x_0)u_{1,0}$$
$$+ \frac{1}{24}\sum_{j=1,4,7,..}^{i-2}\left[(x_i - x_j)u_{1,j} + (x_i - x_{j+1})u_{1,j+1}\right]$$
$$+ \frac{1}{12}\sum_{j=3,6,9,..}^{i-3}(x_i - x_j)u_{1,j} + \frac{1}{24}(x_i - x_0)^2 u_{2,0}$$
$$+ \frac{1}{24}\sum_{j=1,4,7,..}^{i-2}\left[(x_i + x_j)^2 u_{2,j} + (x_i + x_{j+1})^2 u_{2,j+1}\right]$$
$$+ \frac{1}{12}\sum_{j=3,6,9,..}^{i-3}(x_i + x_j)^2 u_{2,j} + \frac{1}{24}(2x_i)^2 u_{2,i} \ldots\ldots\ldots (3.17)$$

when $i = 3,6,9, \ldots,$

$$u_{2,i} = \left[2x_i^2 + x_i + \frac{5}{4}\right]e^{-2x_i} - \frac{1}{4}e^{2x_i} - \frac{1}{2}{x_i}^2 + \frac{1}{36}(x_i - x_0)u_{1,0} + \frac{1}{18}\sum_{j=1}^{i-1}(x_i - x_j)u_{1,j}$$
$$+ \frac{1}{36}(x_i)^2 u_{2,0} + \frac{1}{18}\sum_{j=1}^{i-1}(x_i + x_j)^2 u_{2,j} + \frac{1}{9}x_i^2 u_{2,i} \ldots\ldots\ldots\ldots\ldots\ldots (3.18)$$

When $i \neq 3,6,9, \ldots.,$
By substituting $i = 1,\ r = 1{,}2$ in equation (3.16), (3.18) to get $u_{1,1},\ \ u_{2,1}$. By substituting $i = 1,\ r = 1{,}2$ in equation (3.16), (3.18) to get . $u_{1,2}$, $u_{2,2}$. By substituting $i = 3,\ r = 1{,}2$ in equation (3.15), (3.16), (3.17), (3.18) to get $u_{1,3}$, $u_{2,3}$. Third we divide the interval [1, 0] into 36 subintervals such that
$x_i = \frac{1}{36}$, $i = 0,1,\ldots,36$ and we divide the interval [1, 0] into 72 subintervals such that $x_i = \frac{1}{72}$, $i = 0,1,2,\ldots,72$.
Respectively by continuing tabulated down with the comparison with the exact solutions.


Simpson's Method Results u1,u2
u2
u1
7.92
7.7
7.48
7.26
7.04
6.82
6.6
6.38
6.16
5.94
5.72
5.5
5.28
5.06
4.84
4.62
4.4
4.18
3.96
3.74
3.52
3.3
3.08
2.86
2.64
2.42
2.2
1.98
1.76
1.54
1.32
1.1
0.88
0.66
0.44
0.22
0
0.9000000000
0.8000000000
0.7000000000
0.6000000000
0.5000000000
0.4000000000
0.3000000000
0.2000000000
0.1000000000
0.0000000000
0.00
0.11
0.22
0.33
0.44
0.55
0.66
0.77
0.88
0.99
Exact 1
Result 9
Result 72
Exact2
result9
result72


Simpson's Method Results u2
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.00
0.11
0.22
0.33
0.44
0.55
0.66
0.77
0.88
0.99
Exact 2
result 9
result 72

3. Conclusion:

From the present study, we conclude the following:

1. The classification of the one-dimensional integral equations can be extended to include systems of the one-dimensional integral equations.
2. The modified Simpson's 3/8 rule of first order for solving system of the one-dimensional Volterra linear integral equations and systems of Volterra linear integral equations gave more accurate results than the modified.
3. The modified Simpson's Rule of first order can be used for Fredholm linear integral equations.

For future work the following problems could be recommended.

A. Devote another types of the composite modified Simpson's 3/8 rule of first order.
B. Using the modified quadrature methods to solve system of the one-dimensional non-linear integral equation.
C. Solving systems of the multi-dimensional integral equations by using the modified quadrature methods.
D. Appling the method to fractional integral different equations and system of different equations.

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