On Some α-Proper Mapping Types

Dunya M. Hamed
Department of Mathematics, College of Education
Al-Mustansirya University

Abstract:

The main aim of this paper is to study some new classes of proper mapping which are (α- proper , α*- proper, α**- proper)mappings and discussion the relation between them .as well as several properties of these mappings are proved

1- Introduction

Recently, many researchers have been consider able interest in the study of various forms of new types of open sets and their relationships to other classes of set such as semi-open, semi-closed and pre-open sets[Levience,, 1963]. [3]

In 1965, Njastad [Njastad, 1965] defined new class of open set is said to be α–open set in topology, and [Maheshwari, S. N. and Thakur, S. S.] use these sets in the concoring axiom similar to compatness and introduce new types is α–compact space, see [4]. [Hakeem, 2004] studied and discussion new types of α–continuous mapping, see [2]. Also [Nadia, 2004] studied some types of weakly open set (α-open set and semi-α- open sets) she use these two definitions to define new kind of functions and new kind of weakly separation axioms [8].

In this work, new types of proper mapping namely α–proper mapping is introduced for topological space. Also studied some properties of these types with prove several theorems about it is.

2- Preliminaries

In this section we introduce some basic definitions and concepts of (α-open set, α-closed set, some α-continuous mapping types, some type of α-closed mappings, some types of weakly regularity, proper mapping and the restriction).

Definition (2-1): [11]

A sub set A of a topological space X is called an **α-open** set if $A \subseteq \text{int}\, CL\ \text{int}\, A$. The family of all α-open sets is denoted by τ_α.

Definition (2-2): [11]

The complement of α-open sets is called **α-closed** set. The family of all α-closed sets is denoted by αc(x).

<u>Remark</u> (2-3): [11]

Every open set is α-open and every closed set is α-closed set.

<u>Remark</u> (2-4): [6], [11]

In any topological space X

1. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of α-open sets in a topological space X, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is α-open set.
2. The intersection of any α-open sets in a topological space X is α-open set.
3. Clearly $\tau \subset \tau_\alpha$. But if $\tau_\alpha(x) = \tau$ then every α-open set in X is an open set.
4. If $\tau_\alpha = \tau$, then the complement of each α-open sets in X = the complement of an open sets in X. (i.e every α-closed set in X is closed).

Here is a list of some definitions and concepts that shall be needed in this work.

<u>Definition</u> (2-5): [6], [7], [9]

Let X and Y be two topological spaces, a mapping f: X → Y is said to be:

1. **Continuous** if the inverse image of each open set in Y is an open set in X.
2. **α-continuous** if the inverse image of each open set in Y is an α-open in X.
3. **α^*-continuous** if the inverse image if each α-open set in Y is an α-open in X.
4. **α^{**}-continuous** if the inverse image of each α-open in Y is an open set in X.

<u>Definition</u> (2-6): [1], [6]

A mapping f: (X, τ) → (Y, τ) is said to be:

1. **Closed** if the image of every closed set in X is closed set in Y.
2. **α-closed** if the image of every closed set in X in an α-closed set in Y.
3. **α^*-closed** if the image of every α-closed set in X is an α-closed in Y.
4. **α^{**}- closed** if the image of every closed set in X is an closed in Y.

<u>Definition</u> (2-7) :[1], [10]

Let X be a topological space, then X is said to be:

1. **α-Regular** if every $x \in X$ and every F closed set such that $x \notin F$, there exist two α-open sets A and B such that $x \in A$, $F \subseteq B$ and $A \cap B = \phi$.
2. **α^*-Regular** if every $x \in X$ and every F α-closed set such that $x \notin F$, there exist two α-open sets A and B such that $x \in A$, $F \subseteq B$ and $A \cap B = \phi$.
3. **α^{**}-Regular** if every $x \in X$ and every F α-closed set such that $x \notin F$, there exist two open sets A and B such that $x \in A$, $F \subseteq B$ and $A \cap B = \phi$.

<u>Proposition</u> (2-8): [8]

If X × Y is α^{**}-regular space, then X and Y are α^{**}-regular spaces.

<u>Theorem</u> (2-9) :[8]

If X is α^*-regular space, then $\tau = \tau_\alpha(x)$.

Corollary (2-10): [8]

If X is α^{**}-regular space, then $\tau = \tau_\alpha(x)$.

Definition (2-11): [1]

Let X and Y be two topological spaces. A mapping f: X → Y is said to be **proper** mapping if f is continuous mapping and, for any topological space Z a mapping $f \times I_z$: X × Z → Y × Z is closed.

Definition (2-12) :[5]

Let f: X → Y be a napping and let A be a subset of a space X, then a mapping f | A: A → Y is called the **restriction** of f on A.

Theorem (2-13): [5]

If f: X → Y is an α-continuous mapping and A is and α-closed (i.e a closed α-set) in X, then the restriction f | A: A → Y is α-continuous.

Remark (2-14) [5]

In theorem (2-13) if A is simply closed then f | A is not always α-continuous as is shown by the following example.

Example(2-15):

Let X = {a, b, c, d} and Y = {x, y} be equipped with the topologies T_x = { ϕ, x, {a}} and T_y = { ϕ, y, {x}} respectively. Take A = {b, c, d}. define f: X → X by f(a) = f(b) = x and f(c) = f(d) = y, then f is an α-continuous mapping but the restriction f | A is not α-continuous.

3- Main Result

In this section we introduce new definitions of proper mapping, which are α-proper, α^*-proper and α^{**}-proper mappings and studying the relations between them.

Also, we prove some propositions about the composition and the restriction of these mapping.

Definition: (3-1)

Let X and Y be two topological spaces. A mapping f: X → Y is said to be **α-proper mapping** if f is α-continuous mapping, and for any topological space Z, then $f \times I_z$: X × Z → Y × Z is α-closed mapping.

Proposition: (3-2)

Every proper mapping is α-proper.

Proof:

Let f: X → Y be a proper mapping.
T-P f is α-proper mapping.
Let A be an open set in Y.
Thus, $f^{-1}(A)$ is an open set in X [since f is proper mapping]. By using Remark (2-3) we get $f^{-1}(A)$ is an α-open set in X. Then, f: X → Y is α-continuous mapping.

And, for any topological space Z.
Let B is α-closed set in X × Z.
Thus, $(f \times I_Z)_{(B)}$ is α-closed set in Y × Z [since f is proper].
By using Remark (2-3) we have $(f \times I_Z)_{(B)}$ is an α-closed set in Y × Z.
Hence, $f \times I_Z: X \times Z \rightarrow Y \times Z$ is α-closed mapping.
Therefore, f: X → Z is α-proper mapping.
But the converse is not true in general, as the following example show:

Example 1: Let X= {1, 2, 3, 4}, τ_x = {X, ϕ, {2}, {3},{2, 3},{1, 2}, {1, 2, 3,}, {2, 3, 4}} and $\tau_\alpha(x) = \tau_x \cup$ {{1}, {1, 3}, {3, 4}, {1, 3, 4}}
Define f: X → X by f(1) = 3, f(2) = 4, f(3) = 2 and f(4) = 1.
We observe that f is α-proper mapping, which is not proper.

Now, the following proposition give the condition to make the converse of proposition (3-2) is true.

Proposition :(3-3)

If f: X → Y is α-proper mapping and X×Z,Y×Z for any topological space Z are α^{**}-regular spaces, then f is proper.

Proof:

Let f: X → Y be an α-proper mapping.
T-P f is proper.
Let A be an open set in Y.
Thus, $f^{-1}(A)$ is an α-open set in X [since f is α-proper].
Since X×Z is α^{**}-regular space and by proposition (2-8), then X is α-regular space and by corollary (2-10) we get $f^{-1}(A)$ is an open set in X.
Hence, f: X → Y is continuous mapping.
And, for any topological space Z.
Let B be a closed set in X×Z.
Thus, $(f\times I_Z)_{(B)}$ is an α-closed set in Y×Z [since f is α-proper].
But Y×Z is α^{**}-regular space then by corollary (2-10) and Remark (2-4) we obtain $(f\times I_Z)_{(B)}$ is a closed set in Y×Z.
Then, $f\times I_Z: X\times Z \rightarrow Y\times Z$ is closed mapping.
Therefore, f: X → Z is proper mapping

Definition: (3-4)

Let X, Y be two topological space, a mapping f: X → Y is said to be **α^*-proper mapping** if f is α^*-continuous mapping, and for any topological space Z, then the mapping $f\times I_Z: X\times Z \rightarrow Y\times Z$ is α^*-closed mapping.

Proposition: (3-5)

Every α^*-proper mapping is α-proper.

Proof:

Let f: X → Y is α^*-proper mapping.
T-P f is α-proper.

Let A be an open set in Y and by using Remark (2-3) we get A is an α-open set in Y.
Thus, $f^{-1}(A)$ is an α-open set in X [since f is α^*-proper].
Then, f: X → Y is α-continuous mapping.
And, for any topological space Z.
Let B is a closed set in X×Z and by using Remark (2-3) we get, B is an α-closed set in X×Z.
Thus, $(f\times I_Z)_{(B)}$ is an α-closed set in Y×Z [since f is α^*-proper].
Hence, $f\times I_Z$: X×Z → Y×Z is α-closed mapping.
Therefore, f: X → Y is α-proper mapping.
The following example shows the converse is not necessarily true.

Example :2

Let X = {1, 2, 3, 4}, τ_x = {X, ϕ, {1}, {2}, {1, 2, 3,}} and $\tau_\alpha(x) = \tau_x \cup$ {{1, 2, 4}}.
Define f: X → X by f(1) = 1, f(2) = f(3) = 3 and f(4) = 4.
It is easily seen that f is α-proper but is not α^*-proper mapping.

Here in the following proposition addition the necessarily condition in order to the converse of proposition (3-5) is true.

<u>Proposition</u> :(3-6)

If f: X → Y is α-proper mapping and X×Z, Y×Z for any topological space Z are α^{**}-regular space, then f is α^*-proper mapping.

Proof:

Let f: X → Y is α-proper mapping.
T-P f is α^*-proper.
Let A be an α-open set in Y.
Since Y×Z is α^{**}-regular space and by Proposition (2-8), then Y is α^{**}-regular space and by using Corollary (2-10) and Remark (2-4) we get A is an open set in Y.
Thus, $f^{-1}(A)$ is an α-open set in X [since f is α-proper].
Hence, f: X → Y is α^*-continuous mapping.
And, for any topological space Z.
Let B be a α-closed set in X×Z.
But X×Z is α^{**}-regular space, then by Corollary (2-10) and Remark (2-4), we obtain B is a closed set in X×Z.
Thus, $(f\times I_Z)_{(B)}$ is an α-closed set in Y×Z [since f is α-proper].
Then, $f\times I_Z$: X×Z → Y×Z is α^*-closed mapping.
Therefore, f: X → Z is α^*-proper mapping.

type of α-proper Now, we give another is called α^{**}-proper mapping.

Definition :(3-7)

Let X, Y be two topological spaces. A mapping f: X → Y is said to be **α^{**}-proper mapping** if f is α^{**}-continuous mapping, and for any topological space Z, then a mapping $f\times I_Z$: X×Z → Y×Z is α^{**}-closed.

Proposition: (3-8)

Every α^{**}-proper mapping is proper.

Proof:

Let f: X → Y be α^{**}-proper mapping.
T-P f is proper.
Let A be an open set in Y and by using Remark (2-3) we get, A is an α-open set in Y.
Thus, $f^{-1}(A)$ is an open set in X [since f is α^{**}-proper].
Then, f: X → Y is continuous mapping.
And, for any topological space Z.
Let B a closed set in X× Z and by using Remark (2-3) we have B is an α-closed set in X×Z.
Thus, $(f\times I_Z)_{(B)}$ is a closed set in Y×Z [since f is α^{**}-proper].
Hence, $f\times I_Z$: X×Z → Y×Z is closed mapping.
Therefore, f: X → Y is proper mapping.
But the converse is not true, as the following example show.

Example: 3

Let X = {1, 2, 3}, τ_x = {X, ϕ, {1}, {2}, {1, 2}} and $\tau_\alpha(x) = \tau_x \cup \{\{3\}\}$
Define f: X → X by f(1) = f(2) = 1 and f(3) = 3.
It is easily seen that f is proper but is not α^{**}-proper mapping.

From Proposition (3-8) and (3-2) we get the following Corollary and it is prove easy. Thus we omitted it is.

Corollary: (3-9)

Every α^{**}-proper mapping is α-proper.

The following example show the converse may be false.

Example: 4

Let X = {1, 2, 3, 4}, τ_x = {X, ϕ,{1}, {2}, {1, 2}, {1, 2, 3}} and $\tau_\alpha(x) = \tau_x \cup \{\{1, 2, 4\}\}$
Define f: X → X by f(1) = f(2) = 1, f(3) = 3 and f(4) = 4.
It is observe that f is α-proper but is not α^{**}-proper mapping.

In Proposition (3-8) and Corollary (3-9) we have showed that proper and α-proper mapping need to be α^{**}-proper mapping. However, when we assume X×Z and Y×Z are α^{**}-regular space, then we get the following proposition.

<u>Proposition:</u> (3-10)

If f: X → Y is α-proper mapping and X×Z, Y×Z for any topological space Z are α^{**}-regular space, then f is α^{**}-proper mapping.

Proof:

Let f: X → Y is α-proper mapping.
T-P f is α^{**}-proper mapping.
Let A be an α-open set in Y.
Since Y×Z is α^{**}-regular space and by Proposition (2-8), then Y is α^{**}-regular space and by using Corollary (2-10) we get A is an open set in Y.
Thus, $f^{-1}(A)$ is an α-open set in X [since f is α-proper].
But X×Z is α^{**}-regular space and by Proposition (2-8), then X is α^{**}-regular space and by using Corollary (2-10) we get $f^{-1}(A)$ is an open set in X
Hence, f: X → Y is α^{**}-continuous mapping.
And, for any topological space Z.
Let B be a α-closed set in X×Z.
But X×Z is α^{**}-regular space, then by Corollary (2-10) and Remark (2-4), we obtain B is a closed set in X×Z.
Thus, $(f\times I_Z)_{(B)}$ is an α-closed set in Y×Z [since f is α-proper].
But Y×Z is α^{**}-regular space, then by Corollary (2-10) and Remarl (2-4) we get $(f\times I_Z)_{(B)}$ is a closed set in Y×Z.
Then, $f\times I_Z$: X×Z → Y×Z is α^{**}-closed mapping.
Therefore, f: X → Z is α^{**}-proper mapping.

<u>Corollary</u> :(3-11)

If f: X → Y is proper mapping and X×Z, Y×Z for any topological space Z are α^{**}-regular space, then f is α^{**}-proper mapping.

Proof:

By Proposition (3-2), then f: X → Y is α-proper mapping and by Proposition (3-10) we get f is α^{**}-proper mapping.

Now the following proposition show the relation between α^{**}-proper and α^{*}-proper mapping.

<u>Proposition</u> :(3-12)

Every α^{**}-proper mapping is α^{*}-proper.

Proof:

Let f: X → Y is α^{**}-proper mapping.
T-P f is α^{*}-proper.
Let A be an α-open set in Y.
Thus, $f^{-1}(A)$ is an open set in X [since f is α^{**}-proper].
By using Remark (2-3) we get $f^{-1}(A)$ is an α-open set in X.
Then, f: X → Y is α^{*}-continuous.
And, for any topological space Z.

Let B is an α-closed set in X×Z

Thus, $(f\times I_Z)_{(B)}$ is a closed set in Y×Z, and by using Remark (2-3) we have $(f\times I_Z)_{(B)}$ is an α-closed set in Y×Z.

Hence, $f\times I_Z$: X×Z → Y×Z is α^*-closed mapping.

Therefore, f: X → Z is α^*-proper mapping.

But the converse is not true, as the following example show.

Example 5:

Let X = {1, 2, 3, 4}, τ_x = {X, ϕ, {1}, {2}, {1, 2}, {1, 2, 3}} and $\tau_\alpha(x) = \tau_x \cup$ {{1, 2, 4}}

Define f: X → X by f(1) = f(2) = 2, f(3) = 4 and f(4) = 3.

It is easily seen that f is α^*-proper but is not α^{**}-proper mapping.

The following result give the condition to make the converse of Proposition (3-12) is true.

<u>Corollary</u> :(3-13)

If f: X → Y is α^*-proper mapping and X×Z, Y×Z for any topological space Z are α^{**}-regular space, then f is α^{**}-proper mapping.

Proof:

By Proposition (3-5). A mapping f: X → Y is α-proper mapping, then by using Proposition (3-10) we get f is α^{**}-proper mapping.

<u>Remark:</u> (3-14)

Every proper mapping is not necessarily α^*-proper [it is easy seen that in Example 3].

Also, every α^*-proper mapping is not need to be proper [it is observe in Example 5].

The following result give the condition to make every proper mapping is α^*-proper and every α^*-proper is proper.

<u>Corollary:</u> (3-15)

If X×Z, Y×Z for any topological space Z are α^{**}-regular spaces, then f is proper iff f is α^*-proper mapping.

Proof:

Let f: X → Y be proper mapping.

T-P f is α^*-proper.

From Proposition (3-2). A mapping f: X → Y is α-proper mapping and by Proposition (3-6) we get, f is α^*-proper mapping.

Suppose that f is α^*-proper mapping.

T-P f is proper.

By Corollary (3-13). A mapping f: X → Y is α^{**}-proper mapping and by Proposition (3-8) we have f is proper.

The following diagram illustrates the relation between the α-proper mapping types (without using condition), where the converse is not necessarily true.

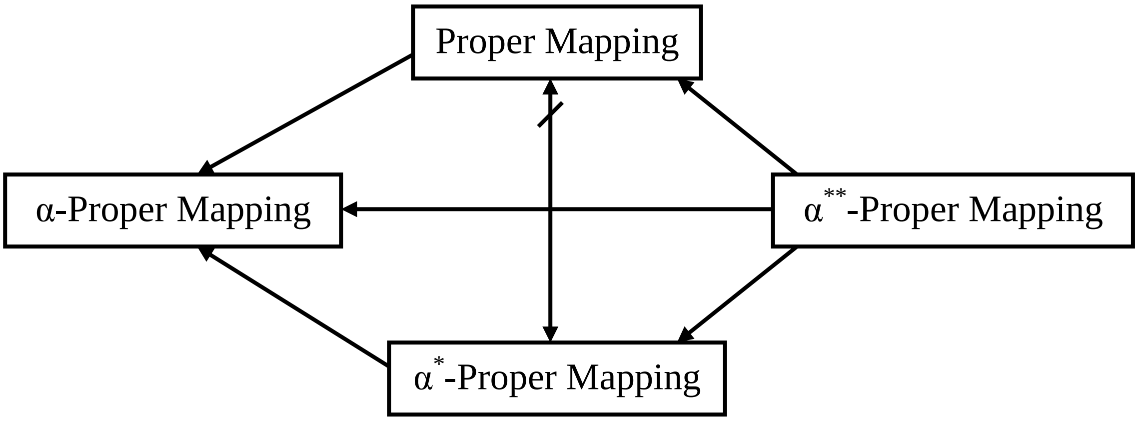

Diagram (1)

Summarized the relations between α-Proper Mapping types.

Now, we give the composition of some types of α-proper mappings.

<u>Proposition</u> :(3-16)

If f: X → Y is proper and g: Y → Z is α^{**}-proper mapping, then a mapping $g \circ f : X \to Z$ is proper (α-proper) respectively.

Proof:

T-P $g \circ f : X \to Z$ is proper.

Let A is an open set in Z, and by using Remark (2-3) we get, A is an α-open set is Z.

Thus $g^{-1}(A)$ is an open set in Y. [since g is α^{**}-proper]

Also, $f^{-1}(g^{-1}(A))$ is an open set in X. [since f is proper]

But $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$,

Then, $g \circ f : X \to Z$ is continuous mapping.

And, for any topological space W.

Let $f_1 = f \times I_W: X\times W \to Y\times W$, $g_1 = g\times I_W: Y \times W \to Z\times W$

T-P $(g \circ f)_X I_W: X\times W \to Z\times W$ is closed mapping.

Let B is a closed set in X×W

Thus, $f_{1(B)}$ is a closed set in Y×W. [since f is proper]

By using Remark (2-3), we have $f_{1(B)}$ is an α-closed set in Y×W.

Thus, $g_1(f_{1(B)})$ is an closed set in Z×W. [since g is α^{**}-proper]

But $g_1(f_{1(B)}) = (g_1 \circ f_1)_{(B)}$

Then, $(g \circ f)\times I_W: X\times W \to Z\times W$ is closed mapping.

Therefore, $g \circ f : X \to Z$ is proper mapping.

Also, by Proposition (3-2) a mapping $g \circ f : X \to Z$ is α-proper.

Remark :(3-17)

If f is proper and g is α^*-proper (α-proper) mapping, then $g \circ f$ is not necessarily α^{**}-proper.

A mapping $g \circ f$ in Remark (3-17) may be to make α^{**}-proper if we assume X×W, Y×W and Z×W are α^{**}-regular spaces, then we get the following proposition.

Proposition:(3-18)

If f: X → Y is proper and g: Y → Z is α-proper mapping and X×W, Y×W, Z×W for any topological space W are α^{**}-regular spaces, then $g \circ f : X \to Z$ is α^{**}-proper mapping.

Proof:

T-P $g \circ f : X \to Z$ is α^{**}-proper mapping.

Let A is an α-open set in Z.

[since Z×W is α^{**}-regular space, and by Proposition (2-8)].

$\Rightarrow$ Z is α^{**}-regular space and by using Corollary (2-10) we get A is an open set is Z.

Thus g^{-1}(A) is an open set in Y. [since g is α-proper]

[since Y×W is α^{**}-regular space and by Proposition (2-8)]

$\Rightarrow$ Y is α^{**}-regular space and by using Corollary (2-10) we have g^{-1}(A) is an open set in Y.

Thus, $f^{-1}(g^{-1}(A))$ is an open set in X. [since f is proper]

But $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$,

Then, $g \circ f : X \to Z$ is α^{**}-continuous.

And, for any topological space W.

Let $f_1 = f \times I_W: X \times W \to Y \times W$, $g_1 = g \times I_W: Y \times W \to Z \times W$

T-P $(g \circ f) \times I_W: X \times W \to Z \times W$ is α^{**}-closed mapping.

Let B be an α-closed set in X×W

[But X×W is α^{**}-regular space, then by using corollary (2-10) and Remark (2-4) we get B is a closed set in X×W]

Then, $f_{1(B)}$ is a closed set in Y×W. [since f is proper]

Thus, $g_1(f_{1(B)})$ is an α-closed set in Z×W. [since g is α-proper]

[But Z×W is α^{**}-regular space, then by using Corollary (2-10) and Remark (2-4) we obtain $g_1(f_{1(B)})$ is a closed set in Z×W]

But $(g_1(f_{1(B)})) = (g_1 \circ f_1)_{(B)}$

Hence, $(g \circ f) \times I_W: X \times W \to Z \times W$ is α^{**}-closed mapping.

Therefore, $g \circ f : X \to Z$ is α^{**}-proper mapping.

<u>Corollary</u>: (3-19)

If f: X → Y is proper, g: Y → Z is α^*-proper and X×W, Y×W, Z×W for any topological space W are α^{**}-regular spaces, then
$g \circ f : X \to Z$ is α^{**}-proper mapping.

Proof:

From Proposition (3-5), then a mapping g: Y → Z is α-proper and by Proposition (3-18) we get $g \circ f$ is α^{**}-proper mapping.

<u>Proposition</u>: (3-20)

If f: X → Y is α-proper mapping and g: Y → Z is α^{**}-proper mapping, then $g \circ f : X \to Z$ is α-proper mapping.

Proof:

T-P $g \circ f : X \to Z$ is α-proper mapping.

Let A is an open set in Z, and by using Remark (2-3) we get A is an α-open set in Z.

Thus g^{-1}(A) is an open set in Y. [since g is α^{**}-proper]

Also, $f^{-1}(g^{-1}(A))$ is an α-open set in X. [since f is α-proper]

But $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$,

Then, $g \circ f : X \to Z$ is α-continuous mapping.

And, for any topological space W.

Let $f_1 = f \times I_W$: X×W → Y×W, $g_1 = g \times I_W$: Y ×W → Z×W

T-P $(g \circ f) \times I_W$: X×W → Z×W is α-closed.

Let B is a closed set in X×W

Thus, $f_{1(B)}$ is an α-closed set in Y×W. [since f is α-proper]

Also, $g_1(f_{1(B)})$ is a closed set in Z×W. [since g is α^{**}-proper]

By Remark (2-3) we get $g_1(f_{1(B)})$ is an α-closed set in Z×W

And since $(g_1(f_{1(B)})) = (g_1 \circ f_1)_{(B)}$

Then, $(g \circ f) \times I_W$: X×W → Z×W is α-closed mapping.

Hence, $g \circ f : X \to Z$ is α-proper mapping.

<u>Remark</u> :(3-21)

If f is α-proper mapping and g is α^*-proper (α-proper), then $g \circ f$ is not necessarily α^{**}-proper mapping.

Here in the following proposition and corollary addition the condition in order to Remark (3-21) is true.

<u>Proposition</u>: (3-22)

If f: X → Y and g: Y → Z are α-proper mappings and X×W, Y×W, Z×W for any topological space W are α^{**}-regular spaces, then
$g \circ f : X \to Z$ is α^{**}-proper mapping.

Proof:

T-P $g \circ f : X \to Z$ is α^{**}-proper mapping.

Let A is an α-open set in Z.
Since Z×W is α^{**}-regular space, and by Proposition (2-8).
Then, Z is α^{**}-regular space and by Corollary (2-10) we get A is an open set is Z.
Thus, g^{-1}(A) is an α-open set in Y. [since g is α-proper]
But Y×W is α^{**}-regular space and by Proposition (2-8)
Then, Y is α^{**}-regular space and by Corollary (2-10) we have g^{-1}(A) is an open set in Y.
Thus, $f^{-1}(g^{-1}$(A)) is an α-open set in X. [since f is α-proper]
Since X×Z is α^{**}-regular space and by Proposition (2-8)
$\Rightarrow$ X is α^{**}-regular space and by using Corollary (2-10) we get $f^{-1}(g^{-1}$(A)) is an open set in X.
But $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$,
Hence, $g \circ f : X \to Z$ is α^{**}-continuous mapping.
And, for any topological space W.
Let $f_1 = f\times I_W: X\times W \to Y\times W$, $g_1 = g\times I_W: Y\times W \to Z\times W$

T-P $(g \circ f)\times I_W: X\times W \to Z\times_X W$ is α^{**}-closed mapping.
Let B be an α-closed set in X×W
Since X×W is α^{**}-regular space, then by using Corollary (2-10) and Remark (2-4) we have B is a closed set in X×W
Thus, $f_{1(B)}$ is an α-closed set in Y×W. [since f is α-proper]
Since Y×W is α^{**}-regular space, then by Corollary (2-10) and Remark (2-4) we get $f_{1(B)}$ is a closed set in Y×W.
Thus, $g_1(f_{1(B)})$ is an α-closed set in Z×W. [since g is α-proper]
Since Z×W is α^{**}-regular space, then by Corollary (2-10) and Remark (2-4) we get $g_1(f_{1(B)})$ is a closed set in Z×W.
But $(g_1(f_{1(B)})) = (g_1 \circ f_1)_{(B)}$
Then, $(g \circ f)\times I_W: X\times W \to Z\times W$ is α^{**}-closed mapping.
Therefore, $g \circ f : X \to Z$ is α^{**}-proper mapping.

Corollary: (3-23)

If f: X → Y is α-proper, g: Y → Z is α^{*}-proper mapping and X×W, Y×W, Z×W for any topological space W are α^{**}-regular spaces, then $g \circ f : X \to Z$ is α^{**}-proper mapping.

Proof:

From Proposition (3-5), then a mapping g: Y → Z is α-proper mapping and by Proposition (3-22) we get, $g \circ f : X \to Z$ is α^{**}-proper mapping.

<u>Proposition</u>: (3-24)

If f: X → Y and g: Y → Z are α^*-proper mappings, then a mapping $g \circ f : X \to Z$ is α^{**}-proper (α-proper) respectively.

Proof:

T-P $g \circ f : X \to Z$ is α^*-proper mapping.

Let A is an α-open set in Z.

Thus, g^{-1}(A) is an α-open set in Y. [since g is α^*-proper]

Also, $f^{-1}(g^{-1}$(A)) is an α-open set in X. [since f is α^*-proper]

But $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$

Then, $g \circ f : X \to Z$ is α^*-continuous mapping.

And, for any topological space W.

Let $f_1 = f \times I_W: X \times W \to Y \times W$, $g_1 = g \times I_W: Y \times W \to Z \times W$

T-P $(g \circ f) \times I_W: X \times W \to Z \times W$ is α^*-closed mapping.

Let B is an α-closed set in X×W

Thus, $f_{1(B)}$ is an α-closed set in Y×W. [since f is α^*-proper]

Also, $g_1(f_{1(B)})$ is an α-closed set in Z×W. [since g is α^*-proper]

And since $(g_1(f_{1(B)})) = (g_1 \circ f_1)_{(B)}$

Hence, $(g \circ f) \times I_W: X \times W \to Z \times W$ is α^*-closed mapping.

Therefore, $g \circ f : X \to Z$ is α^*-proper mapping.

And, by Proposition (3-5) we obtain $g \circ f : X \to Z$ is also α-proper mapping.

<u>Remark</u>: (3-25)

If f is α^*-proper mapping and g is proper then, $g \circ f$ is not necessarily α^{**}-proper mapping.

The following result give the condition to make Remark (3-25) is true.

<u>Corollary</u>: (3-26)

If f: X → Y is α^*-proper mapping, g: Y → Z is proper and X×W, Y×W, Z×W for any topological space W are α^{**}-regular spaces, then $g \circ f : X \to Z$ is α^{**}-proper mapping

Proof:

This follows from Proposition (3-5) and Proposition (3-2) we have f and g are α-proper mappings respectively. Then by Proposition (3-22) a mapping $g \circ f : X \to Z$ is α^{**}-proper mapping.

<u>Proposition</u>: (3-27)

If f: X → Y and g: Y → Z are α^{**}-proper mappings, then $g \circ f : X \to Z$ is α^{**}-proper mapping.

Proof:

T-P $g \circ f : X \to Z$ is α^{**}-proper mapping.

Let A is an α-open set in Z.

Thus, $g^{-1}(A)$ is an open set in Y. [since g is α^{**}-proper]

And by using Remark (2-3) we get $g^{-1}(A)$ is an α-open set in Y, thus $f^{-1}(g^{-1}(A))$ is an open set in X. [since f is α^{**}-proper]

But $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$

Then, $g \circ f : X \to Z$ is α^{**}-continuous mapping.

And, for any topological space W.

Let $f_1 = f \times I_W: X \times W \to X \times W$

, $g_1 = g \times I_W: Y \times W \to Z \times W$

T-P $(g \circ f) \times I_W: X \times W \to Z \times W$ is α^{**}-closed mapping.

Let B is an α-closed set in X×W

Thus, $f_{1(B)}$ is a closed set in Y×W. [since f is α^{**}-proper]

And by using Remark (2-3) we have $f_{1(B)}$ is an α-closed set in Y×W.

Thus, $g_1(f_{1(B)})$ is a closed set in Z×W, but $(g_1(f_{1(B)})) = (g_1 \circ f_1)_{(B)}$

Hence, $(g \circ f) \times I_W: X_X W \to Z \times W$ is α^{**}-closed mapping.

Therefore, $g \circ f : X \to Z$ is α^{**}-proper mapping.

<u>Corollary</u>: (3-28)

If f and g are α^{**}-proper mappings, then $g \circ f$ is proper (α-proper) and α^{*}-proper mapping.

Proof:

This follows from Proposition (3-27) $\Rightarrow$ mapping $g \circ f : X \to Z$ is α^{**}-proper and by using Proposition (3-8) and Corollary (3-9) we get $g \circ f$ is proper and α-proper respectively. Also, by Proposition (3-12) a mapping $g \circ f$ is α^{*}-proper.

<u>Corollary</u>: (3-29)

If f: X → Y is α^{**}-proper mapping and g: Y → Z is α^{*}-proper mapping, then a mapping $g \circ f : X \to Z$ is α^{*}-proper (α-proper).

Proof:

By Proposition (3-12) $\Rightarrow$ f: X → Y is α^{*}-proper mapping.

Since g: Y → Z is α^{*}-proper, then by Proposition (3-24) we obtain,

$g \circ f : X \to Z$ is α^{*}-proper (α-proper) mapping.

<u>Corollary</u>: (3-30)

Let $X_1, X_2, \ldots, X_n$ are topological spaces and let

$f_i: X_i \to X_{i+1}$, i = 1, 2, …, n are α^{**}-proper (α^{*}-proper) mapping.

Then, $f_n \circ f_{n-1} \circ \cdots \circ f_1: X_i \to X_{i+1}$ is α^{**}-proper (α^{*}-proper) mapping.

Proof:

Let U be an α-open set in X_n.

[since f_1, f_2, …, f_n are α^*-proper mapping] and by using Remark (2-3), we get f_n^{-1}(U) is an open (α-open) set in X_{n-1}.

Thus, $f_{n-1}^{-1}(f_n^{-1}(U))$ is an open (α-open) set in X_{n-2}, also

$f_{n-2}^{-1}(f_{n-1}^{-1}(f_n^{-1}(U)))$ is an open (α-open) set in X_{n-3}, … and so on.

Then we have $f_1^{-1}(f_2^{-1}(\cdots(f_{n-1}^{-1}(f_n^{-1}(U)))))$ is an open (α-open) set in X_1.

But $f_1^{-1}(f_2^{-1}(\cdots(f_{n-1}^{-1}(f_n^{-1}(U))))) = (f_n \circ f_{n-1} \circ \cdots \circ f_1)^{-1}(U)$

Hence, $f_n \circ f_{n-1} \circ \cdots \circ f_1$: $X_i \rightarrow X_{i+1}$, i = 1, 2, …, n is α^{**}-continuous mappings.

And, for any topological space Z

Let $g_1 = f_1 \times I_Z$, …, $g_n = f_n \times I_Z$ and let V is an α-closed set in $X_1 \times Z$, thus

$g_{1(V)}$ is an closed (α-closed) set in $X_2 \times Z$. Thus,

$g_2(g_{1(V)})$ is an closed (α-closed) set in $X_3 \times Z$. Also,

$g_3(g_2(g_{1(V)}))$ is an closed (α-closed) set in $X_4 \times Z$, and so on.

Then, we have $g_n(g_{n-1}(\ldots g_3(g_2(g_{1(V)}))))$ us an closed set in $X_n \times Z$.

But $g_n(g_{n-1}(\ldots g_3(g_2(g_{1(V)})))) = (g_n \circ g_{n-1} \circ \cdots \circ g_1)_{(V)}$

Then, $(f_n \circ f_{n-1} \circ \cdots \circ f_1) \times I_Z$: $X_i \times Z \rightarrow X_{i+1} \times Z$, i = 1, 2, …, n is α^{**}-closed mapping.

Therefore, $f_n \circ f_{n-1} \circ \cdots \circ f_1$: $X_i \rightarrow X_{i+1}$ is α^{**}-proper mappings. Also, $f_n \circ f_{n-1} \circ \cdots \circ f_1$: $X_i \rightarrow X_{i+1}$ is α^*-proper, this follows from Proposition (3-12).

Remark :(3-31)

From [3] if f: X → Y is proper mapping and A be any subset of X. then f | A: A → Y is not proper in general.

The following Proposition give the necessarily condition in order to the restriction f | A: A → Y is α-proper mapping.

Proposition: (3-32)

If f: X → Y is an α-proper mapping and A be an α-closed subset of X. then f | A: A → Y is α-proper mapping.

Proof:

By Theorem (2-13) the restriction f | A: A → Y is α-continuous mapping.

And, for any topological space Z

Let W is a closed set in A×Z. Then W is a closed set in X×Z.

Thus, $(f \times I_Z)_{(W)}$ is an α-closed set in Y×Z [since f is α-proper]

But $(f \times I_Z)_{(W)} = (f \times I_Z) \mid_{A \times Z}$.

Hence, $(f \times I_Z) \mid_{A \times Z}$ is α-closed mapping.

Therefore, f | A is α-proper mapping.

From Corollary (3-9), Proposition (3-5) and by Proposition (3-32) we get, the following corollary and it is prove easy. Thus we omitted it is:

Corollary: (3-33)
If f: X → Y is α^{**}-proper (α-proper) mapping and A is an α-closed subset of X, then f | A: A → Y is an α^{**}-proper (α^{*}-proper) mapping.

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حول بعض دوال α− الفعليه

مقدم من قبل

دنيا محمد حميد

الجامعة المستنصريه _ كلية التربية _قسم الرياضيات

المستخلص:

الهدف الرئيسي من هذا البحث هو دراسة بعض انواع الدوال جديدة من صفوف الدوال الفعلية (α− الفعلية ،*α − الفعلية،**α − الفعلية)وناقشنا العلاقه فيما بينها ،وأيضا بعض صفات تللك الدوال درست وبرهنت .